Lec 9 Wednesday Sept 23

pf of LEM6, cont.

Check $\bar{g} \in Hom(U, Hom(V, W))$

g is linear in second arg so $\forall u \in U$ the map

$$\bar{g}(u) : \begin{array}{l} V \to W \\ v \to g(u,v) \end{array}$$

is linear. So

$$\bar{g}(u) \in Hom(V, W)$$

Also, g is linear in first arg so

$$\bar{g} : \begin{array}{l} U \longrightarrow Hom(V, W) \\ u \longrightarrow \bar{g}(u) \end{array},$$

is linear.

So $\bar{g} \in Hom(U, Hom(V, W))$

We now have a map

$$\begin{array}{ccc} Hom^2(U, V; W) & \longrightarrow & Hom(U, Hom(V, W)) \\ g & \longrightarrow & \bar{g} \end{array} \qquad \star$$

s.t.

$$\bar{g}(u)(v) = g(u,v) \qquad \forall u \in U \quad \forall v \in V.$$

one checks $*$ is linear

show $*$ is bijection.

$*$ is injective:

Assume $g \in$ kernel of $*$ so $\bar{g} = 0$

$\forall u \in U \quad \forall v \in V$

$$0 = \bar{g}(u,v) = g(u,v)$$

so $g = 0$ ✓

$*$ is surj:

Given $G \in Hom(U, Hom(V, W))$

Define map

$$g: \begin{array}{ccc} U \times V & \to & W \\ u, v & \to & G(u)(v) \end{array}$$

obs g is bilin

so $g \in Hom^2(U, V; W)$

By constr $g(u,v) = G(u)(v)$

so $\bar{g} = G$ ✓

$\square$

LEM 7 Given vector spaces U, V, W.

$\exists$ iso of vector spaces

$$(U\otimes V)\otimes W \to U\otimes(V\otimes W)$$

that sends

$$(u\otimes v)\otimes w \longrightarrow u\otimes(v\otimes w)$$ *

$\forall\, u\in U,\ v\in V,\ w\in W.$

pf For any vector space S, and referring to the iso in LEM 6,

$$\begin{aligned} \mathrm{Hom}((U\otimes V)\otimes W, S) &\simeq \mathrm{Hom}(U\otimes V, \mathrm{Hom}(W, S)) \\ &\simeq \mathrm{Hom}(U, \mathrm{Hom}(V, \mathrm{Hom}(W, S))) \end{aligned}$$

Also

$$\begin{aligned} \mathrm{Hom}(U\otimes(V\otimes W), S) &\simeq \mathrm{Hom}(U, \mathrm{Hom}(V\otimes W, S)) \\ &\simeq \mathrm{Hom}(U, \mathrm{Hom}(V, \mathrm{Hom}(W, S))) \end{aligned}$$

This gives vector space iso

$$\mathrm{Hom}((U\otimes V)\otimes W, S) \to \mathrm{Hom}(U\otimes(V\otimes W), S)$$ ☆

Take

$$S = U \otimes (V \otimes W)$$

Under ✩ let

φ = preimage of ident map

linear map

$$\varphi : (U \otimes V) \otimes W \longrightarrow U \otimes (V \otimes W)$$

satisfies *

Next take

$$S = (U \otimes V) W$$

Under ✩ let

ϕ = image of ident map

Linear map

$$\phi : U \otimes (V \otimes W) \to (U \otimes V) \otimes W$$

sends $u \otimes (v \otimes w) \to (u \otimes v) \otimes w$

$\forall\, u, v, w$.

The maps φ, ϕ are inverses by LEM 2 and Constr.

So φ, ϕ are bij. $\square$

LEM 8 Given vectr spaces U, V

$\exists$ iso of vectr spaces

$$U \otimes V \to V \otimes U$$

that sends

$u \otimes v \to v \otimes u$ $\quad \forall u \in U \ \forall v \in V$

pf The map

$$U \times V \to V \otimes U$$
$$u, v \to v \otimes u$$

is bilin. So $\exists$ lin map

$$\varphi : U \otimes V \to V \otimes U$$

that sends

$u \otimes v \to v \otimes u$ $\quad \forall u \in U \ \forall v \in V$.

Similarly $\exists$ lin map

$$\phi : V \otimes U \to U \otimes V$$

that sends

$v \otimes u \to u \otimes v$ $\quad \forall u \in U \ \forall v \in V$.

φ, ϕ are inverses by LEM 2; hence bijections. $\square$

LEM 9 Given a vector space V

$\exists$ vector space isomorphisms

$$k \otimes V \to V \qquad V \otimes k \to V$$

that read

$$\alpha \otimes v \to \alpha v \qquad v \otimes \alpha \to \alpha v$$

$\forall \alpha \in k$ $\forall v \in V$.

Their inverses read

$$V \to k \otimes V \qquad V \to V \otimes k \qquad \forall v \in V$$
$$v \to 1 \otimes v \qquad v \to v \otimes 1$$

pf Consider $k \otimes V$.

The map $k \times V \to V$
$\alpha, v \to \alpha v$

is bilin. So $\exists$ lin map

$$\varphi : k \otimes V \to V$$

that reads $\alpha \otimes v \to \alpha v$ $\forall \alpha \in k$ $\forall v \in V$

obs $\alpha \otimes v = \alpha (1 \otimes v)$

So $k \otimes V = 1 \otimes V$

Also note

$$\varphi : k \otimes V \to V$$
$$1 \otimes v \to v$$

is surj + inj, hence bij. $\square$

Given vectr spaces $\{V_i\}_{i\in I}$

Recall dir product

$$V = \prod_{i\in I} V_i$$

Fr $i \in I$ the projectn

$$\pi_i : \begin{array}{ccc} V & \longrightarrow & V_i \\ \prod_j v_j & \longrightarrow & v_i \end{array}$$

is linear

LEM 10 Given vector spaces $U, \{V_i\}_{i \in I}$

the following is a vector space iso:

$$\operatorname{Hom}\left(U, \prod_{i \in I} V_i\right) \longrightarrow \prod_{i \in I} \operatorname{Hom}(U, V_i) \qquad *$$

$$f \longrightarrow \prod_{i \in I} f_i$$

where $\quad f_i = \pi_i \circ f \qquad \forall i \in I$

pf $\forall i \in I$, we have comp of lin maps

$$f_i : \quad U \xrightarrow{f} \prod_{j \in I} V_j \xrightarrow{\pi_i} V_i$$

so $\quad f_i \in \operatorname{Hom}(U, V_i)$

obs $*$ is linear ✓

Show $*$ is bijection.

$*$ is inj

Given $f \in \ker *$

$f_i = 0 \quad \forall i \in I$

$\forall u \in U$

$f(u)$ has all coords 0 so $f(u) = 0$

$f = 0$

$*$ is surj:

Given $f_i \in Hom(U, V_i) \quad \forall i \in I$

Define map

$$f \quad \begin{array}{ccc} U & \longrightarrow & \prod_{i \in I} V_i \\ u & \longrightarrow & \prod_{i \in I} f_i(u) \end{array}$$

f is linear so

$$f \in Hom\left(U, \prod_{i \in I} V_i\right)$$

obs $*$ sends

$$f \to \prod_{i \in I} f_i$$

$\square$

Given vector spaces $\{U_i\}_{i\in I}$

Recall dir sum

$$\mathcal{U} = \bigoplus_{i\in I} U_i$$

For $i \in I$ the map

$$q_i : \begin{array}{ccc} U_i & \to & U \\ u & \to & (0,\dots,0,u,0\dots) \end{array}$$

is linear.

obs: For $u = \bigoplus_{i\in I} u_i \in U$,

$$u = \sum_{i\in I} q_i(u_i)$$

LEM 11 Given vector spaces

$\{U_i\}_{i\in I}$, V

the following is a vector space iso:

$$\mathrm{Hom}\left(\bigoplus_{i\in I} U_i, V\right) \longrightarrow \prod_{i\in I} \mathrm{Hom}(U_i, V) \qquad *$$

$$f \longrightarrow \prod_{i\in I} f_i$$

where

$$f_i = f \circ q_i \quad \text{for } i \in I$$

pf $\forall i\in I$, we have comp of linear maps,

$$U_i \xrightarrow{q_i} \bigoplus_{j\in I} U_j \xrightarrow{f} V$$

f_i:

so $f_i \in \mathrm{Hom}(U_i, V)$ $\forall i \in I$

By constr $*$ is linear

Show $*$ is bijection.

Show $*$ is inj

Given $f \in \ker *$

So $f_i = 0 \quad \forall i \in I$

Now $\forall u = \bigoplus_{i \in I} u_i \in \bigoplus_{i \in I} U_i$

$$
\begin{aligned}
f(u) &= f\left(\oplus u_i\right) \\
&= f\left(\sum_{i \in I} q_i(u_i)\right) \\
&= \sum_{i \in I} f(q_i(u_i)) \\
&= \sum_{i \in I} \underbrace{f_i(u_i)}_{=0} \\
&= 0
\end{aligned}
$$

So $f = 0$

Show $*$ is surj.

Given $f_i \in Hom(U_i, V)$ $\quad \forall i \in I$

Define

$$f : \bigoplus_{i \in I} U_i \longrightarrow V$$

$$\bigoplus_{i \in I} u_i \rightarrow \sum_{i \in I} f_i(u_i)$$

f is linear so

$$f \in Hom\left(\bigoplus_{i \in I} U_i, V\right)$$

$*$ sends

$$f \longrightarrow \prod_{i \in I} f_i \qquad \checkmark$$

$\square$

LEM 12 Given vectr spaces $\{U_i\}_{i\in I}$, V.

Then $\exists$ iso of vectr spaces

$$\left(\bigoplus_{i\in I} U_i\right)\otimes V \longrightarrow \bigoplus_{i\in I}(U_i\otimes V)$$

that sends

$$\left(\bigoplus_{i\in I} u_i\right)\otimes v \longrightarrow \bigoplus_{i\in I}(u_i\otimes v) \qquad *$$

$\forall\ u_i\in U_i\ (i\in I)$ and $\forall v\in V$

pf. Given any vectr space W, we have vectr space isomorphisms

$$\mathrm{Hom}\left(\left(\bigoplus_i U_i\right)\otimes V, W\right) \underset{\text{LEM 6}}{\cong} \mathrm{Hom}\left(\bigoplus_i U_i,\ \mathrm{Hom}(V,W)\right)$$

$$\underset{\text{LEM 11}}{\cong} \prod_{i\in I}\mathrm{Hom}\left(U_i,\ \mathrm{Hom}(V,W)\right)$$

$$\underset{\text{LEM 6}}{\cong} \prod_{i\in I}\mathrm{Hom}\left(U_i\otimes V,\ W\right)$$

$$\underset{\text{LEM 11}}{\cong} \mathrm{Hom}\left(\bigoplus_{i\in I}(U_i\otimes V),\ W\right)$$

This gives vectr space iso

$$\mathrm{Hom}\left(\left(\bigoplus_i U_i\right)\otimes V, W\right) \longrightarrow \mathrm{Hom}\left(\bigoplus_i (U_i\otimes V), W\right) \qquad ☆$$

Take
$$W = \bigoplus_i (U_i \otimes V)$$

Under ✩ let
φ = preimage of ident map.

Linear map
$$\varphi : \left(\bigoplus_{i \in I} U_i\right) \otimes V \longrightarrow \bigoplus_{i \in I} (U_i \otimes V)$$

satisfies *

Next take
$$W = \left(\bigoplus_{i \in I} U_i\right) \otimes V$$

Under ✩ let
ϕ = image of ident map,

Linear map
$$\phi : \bigoplus_{i \in I} (U_i \otimes V) \longrightarrow \left(\bigoplus_{i \in I} U_i\right) \otimes V$$

sends
$$\bigoplus_{i \in I} (u_i \otimes v) \longrightarrow \left(\bigoplus_{i \in I} u_i\right) \otimes v$$

$\forall\, u_i \in U_i\ (i \in I)$ and $\forall v \in V$.

φ, ϕ are inverses by LEM 2; hence bijections

$\square$

LEM 13 Given vector spaces U, V

Assume X is basis for U

Y is basis for V

then

$$x \otimes y \qquad x \in X, \; y \in Y$$

is basis for $U \otimes V$

pf Obs

$$U = \sum_{x \in X} kx \qquad \text{dir sum}$$

$$\cong \bigoplus_{x \in X} kx$$

Sim

$$V \cong \bigoplus_{y \in Y} ky$$

Now using LEM 12 + LEM 8

$$U \otimes V \simeq \bigoplus_{\substack{x \in X \\ y \in Y}} \underbrace{kx \otimes ky}_{\substack{\text{has basis} \\ x \otimes y}} \quad \text{by LEM 9}$$

$\square$

LEM 14 Given vector spaces

U, U', V, V'

and lin maps

$f : U \to U'$ $\qquad g : V \to V'$

$\exists$ lin map

$U \otimes V \to U' \otimes V'$

that sends

$u \otimes v \to f(u) \otimes g(v)$ $\qquad \forall u \in U \ \forall v \in V$

pf The map

$$\varphi : \begin{array}{ccc} U \times V & \longrightarrow & U' \otimes V' \\ u, v & \to & f(u) \otimes g(v) \end{array}$$

is bilinear.

So $\exists$ lin map

$\overline{\varphi} : U \otimes V \to U' \otimes V'$

s.t this diag commutes!

$$\begin{tikzcd} U \times V \arrow[r, "u{,}v \,\to\, u \otimes v"] \arrow[dr, "\varphi"'] & U \otimes V \arrow[d, "\overline{\varphi}"] \\ & U' \otimes V' \end{tikzcd}$$

The map $\overline{\varphi}$ meets the requirements. $\square$