LEC 8 Monday Sept 21

LEM 33 Given

- Left Noetherian algebra R
- commuting indets $x_1, x_2, \ldots x_n$
- $I =$ 2-sided ideal in polynomial algebra $R[x_1, \ldots x_n]$

then the algebra

$$R[x_1, \ldots x_n] / I$$

is left-Noetherian.

pf View

$$R[x_1, \ldots, x_n] = R[x_1 \ldots x_{n-1}][x_n]$$

as Ore ext of $R[x_1, \ldots, x_{n-1}]$ with $\alpha = 1$, $\delta = 0$

Now $R[x_1, \ldots, x_n]$ is left Noetherian by LEM 32 and induction on n

The algebra $R[x_1, \ldots, x_n] / I$ is left Noeth by LEM 31 $\square$

Ch II Tensor Products

Def Given vector spaces U, V

$Hom(U, V)$ is the vector space of all

linear maps $U \to V$

So

$End(U) = Hom(U, U)$

For a vector space W

$Hom^2(U, V; W)$ is the vector space of all

bilinear maps $U \times V \to W$

Given vector spaces U, V we now consider the tensor product

$$U \otimes V = U \otimes_k V$$

<u>Motivation</u> (informal view of $\otimes$)

Assume U has a basis X and
V has a basis Y

Define $U \otimes V$ to be the vector space with basis $x \otimes y$ $x \in X, y \in Y$ *

For

$$u = \sum_{x \in X} a_x x \in U,$$

$$v = \sum_{y \in Y} b_y y \in V$$

define

$$u \otimes v = \sum_{\substack{x \in X \\ y \in Y}} a_x b_y \, x \otimes y$$

then

$$\varphi_0 : \begin{array}{ccc} U \times V & \longrightarrow & U \otimes V \\ u \quad v & \longrightarrow & u \otimes v \end{array}$$

is bilinear and image spans $U \otimes V$.

The map $\varphi_0 : U \times V \to U \otimes V$ has the following property:

Given a vector space W and a bilinear map

$$\theta : U \times V \to W,$$

$\exists$ unique lin trans

$$\bar{\theta} : U \otimes V \to W$$

s.t

$$\begin{tikzcd} U \times V \arrow[r, "\varphi_0"] \arrow[dr, "\theta"'] & U \otimes V \arrow[d, "\bar{\theta}"] \\ & W \end{tikzcd}$$ commutes

Just define $\bar{\theta}$ on the basis $*$ by

$$\bar{\theta}(x \otimes y) = \theta(x, y) \quad \forall x \in X, \ y \in Y.$$

Note that θ is recovered from $\bar{\theta}$ by

$$\theta = \bar{\theta} \circ \varphi_0$$

The linear map

$$\begin{aligned} Hom(U \otimes V, W) &\longrightarrow Hom^2(U, V; W) \\ f &\to f \circ \varphi_0 \end{aligned} \qquad **$$

Sends $\bar{\theta} \to \theta$

One checks $**$ is an iso of VS.

Next goal:

Replace above view of $U \otimes V$ by

"basis free" approach

Until further notice fix vs U, V

LEM 1 Given a vector space $U \otimes V$ and a bilin map

$$\varphi_0 : \begin{array}{ccc} U \times V & \longrightarrow & U \otimes V \\ u, v & \longrightarrow & u \otimes v . \end{array}$$

Then $\forall$ vector spaces W TFAE:

(i) $\forall$ bilin maps

$$\theta : U \times V \to W,$$

$\exists$ unique lin map

$$\bar{\theta} : U \otimes V \to W$$

s.t.

$$\begin{tikzcd} U \times V \arrow[r, "\varphi_0"] \arrow[dr, "\theta"'] & U \otimes V \arrow[d, "\bar{\theta}"] \\ & W \end{tikzcd}$$

commutes

(ii) the map

$$\begin{array}{ccc} \mathrm{Hom}(U \otimes V, W) & \longrightarrow & \mathrm{Hom}^2(U, V; W) \\ f & \longrightarrow & f \circ \varphi_0 \end{array} \quad ✩$$

is a vector space iso.

(i) $\to$ (ii)

$\star$ is linear ✓

$\star$ is inj: by uniqueness of $\bar{\theta}$ in (i)

$\star$ is surj: by existence of $\bar{\theta}$ in (i)

(ii) $\to$ (i)

$\bar{\theta}$ exists: since $\star$ is surj

$\bar{\theta}$ is unique: since $\star$ is inj

$\square$

LEM 2 Referring to LEM 1, assume (i), (ii) hold $\forall$ vs W. Then

the image $\varphi_0(U \times V)$ spans $U \otimes V$

pf Let $S = \mathrm{Span}(\varphi_0(U \times V))$

Show $S = U \otimes V$

Suppose not

In LEM 1 (i), take $W = K$ and

$$\theta : \begin{array}{ccc} U \times V & \to & K \\ u, v & \to & 0 \end{array}$$

$\exists$ lin trans

$$\bar{\theta} : U \otimes V \to K$$

that sends $S \longrightarrow 0$

$\bar{\theta}$ not unique

Obs
$$\begin{tikzcd} U \times V \arrow[r, "\varphi_0"] \arrow[dr, "\theta"'] & U \otimes V \arrow[d, "\bar{\theta}"] \\ & K \end{tikzcd}$$
Commutes

Contr.

So $S = U \otimes V$ $\square$

LEM 3 Given

(i) vs $U \otimes V$ and bil map $\varphi_0 : U \times V \to U \otimes V$ that satisfies LEM 1 (i), (ii) $\forall$ vs W

(ii) vs $U \hat{\otimes} V$ and bil map $\hat{\varphi}_0 : U \times V \to U \hat{\otimes} V$

. . . .

then $\exists$ iso of vector spaces

$$U \otimes V \to U \hat{\otimes} V$$

that sends

$$u \otimes v \to u \hat{\otimes} v \qquad \forall u \in U \quad \forall v \in V$$

pf We have

$$\begin{tikzcd} & U \times V \arrow[dl, "\hat{\varphi}_0"'] \arrow[dr, "\varphi_0"] & \\ U \hat{\otimes} V & & U \otimes V \end{tikzcd}$$

$\varphi_0, \hat{\varphi}_0$ bilin.

By Lem 1 with $W = U \hat{\otimes} V$ $\exists$ lin map $U \otimes V \to U \hat{\otimes} V$ that sends $u \otimes v \to u \hat{\otimes} v$ $\forall u \in U$ $\forall v \in V$.

Swapping roles $U \otimes V \leftrightarrow U \hat{\otimes} V$ $\exists$ lin map $U \hat{\otimes} V \to U \otimes V$ that sends $u \hat{\otimes} v \to u \otimes v$ $\forall u \in U$ $\forall v \in V$.

Above 2 maps are inverses by LEM 2; hence bijections. $\square$

DEF 4. By the <u>tensor product</u> of U, V we mean a vector space $U \otimes V$ together with a linear map

$$\varphi_0 : U \times V \longrightarrow U \otimes V$$
$$u, v \longrightarrow u \otimes v$$

that satisfies

LEM 1 (i), (ii) $\forall$ vector spaces W.

LEM 5 Given vector spaces U, V

The tensor product of U, V exists.

Pf Start with Cartesian product $U \times V$

Let $k\,U\times V$ denote the vector space with basis $U \times V$

Let J denote the subspace of $k\,U\times V$ spanned by all vectors of form

$$(\alpha u, v) - \alpha(u, v) \qquad \alpha \in k$$
$$(u+u', v) - (u,v) - (u', v) \qquad u, u' \in U$$
$$(u, \alpha v) - \alpha(u,v) \qquad v, v' \in V$$
$$(u, v+v') - (u,v) - (u, v')$$

Define the quotient vector space

$$U \otimes V = \frac{k\,U \times V}{J}$$

Define $\varphi_0 : U \times V \to U \otimes V$ to be the composition

$$\varphi_0 : \quad U \times V \xrightarrow{\text{incl}} k\,U\times V \longrightarrow U \otimes V$$
$$a \longrightarrow a + J$$

φ_0 is bilinear by def of J.

Obs: image $\varphi_0(U \times V)$ spans $U \otimes V$.

Given vector space W and bilin map

$$\theta : U \times V \to W$$

Define lin map

$$\hat{\theta} : k\, U \times V \to W$$

that sends basis vectors $(u, v) \longrightarrow \theta(u, v)$ $\quad u \in U, v \in V$

<u>claim</u> $\hat{\theta}(J) = 0$

<u>pf of</u> $\hat{\theta}$ sends

$$(\alpha u, v) - \alpha(u, v) \longrightarrow \underset{\alpha\theta(u,v)}{\overset{\theta(\alpha u, v)}{\shortparallel}} - \alpha\theta(u, v) = 0$$

$$(u + u', v) - (u, v) - (u', v) \longrightarrow \underset{\theta(u,v) + \theta(u',v)}{\overset{\theta(u+u', v)}{\shortparallel}} - \theta(u, v) - \theta(u', v) = 0$$

$$(u, \alpha v) - \alpha(u, v) \longrightarrow \underset{\alpha\theta(u,v)}{\overset{\theta(u, \alpha v)}{\shortparallel}} - \alpha\theta(u, v) = 0$$

$$(u, v + v') - (u, v) - (u, v') \longrightarrow \underset{\theta(u,v) + \theta(u,v')}{\overset{\theta(u, v+v')}{\shortparallel}} - \theta(u, v) - \theta(u, v') = 0$$

claim proved ✓

By the claim, $\hat{\theta}$ induces a linear map

$$\bar{\theta} : \begin{array}{ccc} U \otimes V & \longrightarrow & W \\ a + J & \longrightarrow & \hat{\theta}(a) \end{array}$$

Check this diag commutes:

$$\begin{tikzcd} U \times V \arrow[r, "\varphi_0"] \arrow[dr, "\theta"'] & U \otimes V \arrow[d, "\bar{\theta}"] \\ & W \end{tikzcd} \qquad *$$

$$\begin{tikzcd} (u,v) \arrow[r] \arrow[dr] & (u,v) + J \arrow[d] \\ & \hat{\theta}(u,v) \end{tikzcd} \qquad \text{ok}$$

$$\theta(u,v) = \checkmark$$

Above map $\bar{\theta}$ is unique lin map $U \otimes V \to W$ that makes $*$ commute, since $\varphi_0(U \times V)$ spans $U \otimes V$.

We have shown that the vs $U \otimes V$ together with the bilin map $\varphi_0 : U \times V \to U \otimes V$ is a tensor product of U, V.

$\square$

LEM 6 Given vectr spaces U, V, W.

Then $\exists$ vectr space iso

$$\mathrm{Hom}(U \otimes V, W) \longrightarrow \mathrm{Hom}(U, \mathrm{Hom}(V, W))$$

$$f \longrightarrow F$$

s.t

$$F(u)(v) = f(u \otimes v) \qquad \forall u \in U \quad \forall v \in V$$

pf We obtain the desired iso as a composition

$$Hom(U \otimes V, W) \xrightarrow{iso} Hom^2(U, V; W) \longrightarrow Hom(U, Hom(V, W))$$

$$f \to f \circ \varphi_0 \qquad g \longrightarrow \underset{(\text{Find})}{\bar{g}}$$

For $g \in Hom^2(U, V; W)$ find $\bar{g}$

Require lin map

$\bar{g} : U \to Hom(V, W)$

$\forall u \in U$ require lin map

$\bar{g}(u) : V \to W$

For $v \in V$ require

$\bar{g}(u)(v) \in W$.

Declare

$\bar{g}(u)(v) = g(u, v)$