

• Left Noetherian algebra R

• commuting indets x_1, x_2, \dots, x_n

• $I =$ 2-sided ideal in polynomial algebra $R[x_1, \dots, x_n]$

then the algebra

$$R[x_1, \dots, x_n] / I$$

is left-Noetherian.

pf

View

$$R[x_1, \dots, x_n] = R[x_1, \dots, x_{n-1}][x_n]$$

as Ore ext of $R[x_1, \dots, x_{n-1}]$ with $\delta = 1, \sigma = 0$

Now $R[x_1, \dots, x_n]$ is left Noetherian by LEM 32

and induction on n

The algebra $R[x_1, \dots, x_n] / I$ is left Noeth by LEM 31 \square

Ch II tensor Products

Def Given vector spaces U, V

$\text{Hom}(U, V)$ is the vector space of all linear maps $U \rightarrow V$

So $\text{End}(U) = \text{Hom}(U, U)$

For a vector space W

$\text{Hom}^2(U, V; W)$ is the vector space of all bilinear maps $U \times V \rightarrow W$

Given vector spaces U, V we now consider
the tensor product

$$U \otimes V = U \otimes_K V$$

Motivation (informal view of \otimes)

Assume U has a basis X and
 V has a basis Y

Define $U \otimes V$ to be the vector space with
basis $x \otimes y$ $x \in X, y \in Y$ *

For $u = \sum_{x \in X} a_x x \in U,$

$v = \sum_{y \in Y} b_y y \in V$

define $u \otimes v = \sum_{\substack{x \in X \\ y \in Y}} a_x b_y x \otimes y$

then $U \times V \rightarrow U \otimes V$

maps $u, v \rightarrow u \otimes v$

is bilinear and image spans $U \otimes V$.

The map $\varphi_0: U \times V \rightarrow U \otimes V$ has the following

property:

Given a vector space W and a bilinear map

$$\theta: U \times V \rightarrow W,$$

\exists unique lin trans

$$\bar{\theta}: U \otimes V \rightarrow W$$

st

$$U \times V \xrightarrow{\varphi_0} U \otimes V$$

commutes

$$\theta \searrow \quad \downarrow \bar{\theta}$$

W

Just define $\bar{\theta}$ on the basis x by

$$\bar{\theta}(x \otimes y) = \theta(x, y) \quad \forall x \in X, y \in Y.$$

Note that θ is recovered from $\bar{\theta}$ by

$$\theta = \bar{\theta} \circ \varphi_0$$

The linear map

$$\begin{aligned} \text{Hom}(U \otimes V, W) &\longrightarrow \text{Hom}^2(U, V; W) \\ f &\longrightarrow f \circ \varphi_0 \end{aligned}$$

**

Sends $\bar{\theta} \longrightarrow \theta$

One checks $\bar{\theta}$ is an iso of vs.

Next goal:

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Replace above view of $U \otimes V$ by

"basis free" approach

Until further notice fix u, v

LEM 1 Given a vector space $U \otimes V$ and a bilinear map

$$\begin{aligned} \varphi_0 : U \times V &\rightarrow U \otimes V \\ u, v &\rightarrow u \otimes v. \end{aligned}$$

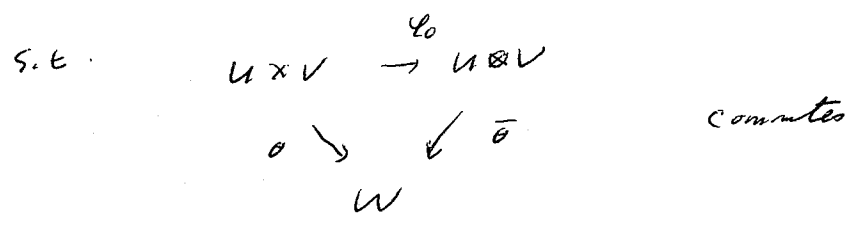
Then \forall vector spaces W TFAE:

(i) \forall bilinear maps

$$\theta : U \times V \rightarrow W.$$

\exists unique linear map

$$\bar{\theta} : U \otimes V \rightarrow W$$



(ii) the map

$$\begin{aligned} \text{Hom}(U \otimes V, W) &\rightarrow \text{Hom}^2(U, V; W) \\ f &\rightarrow f \circ \varphi_0 \end{aligned} \quad \star$$

is a vector space isom.

(i) \rightarrow (ii)

\star is linear \checkmark

\star is inj: by uniqueness of $\bar{\theta}$ in (i)

\star is surj: by existence of $\bar{\theta}$ in (i)

(ii) \rightarrow (i)

$\bar{\theta}$ exists: since \star is surj

$\bar{\theta}$ is unique: since \star is inj



LEM 2 Referring to LEM 1, assume (i), (ii) hold
V vs W. Then

the image $\varphi_0(U \times V)$ spans $U \otimes V$

pf Let $S = \text{Span}(\varphi_0(U \times V))$

show $S = U \otimes V$

Suppose not

In LEM 1 (i), take $W = K$ and

$$\theta : \begin{matrix} U \times V \rightarrow K \\ u, v \rightarrow 0 \end{matrix}$$

\exists lin trans

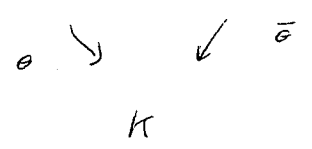
$$\bar{\theta} : U \otimes V \rightarrow K$$

that sends

$$S \rightarrow 0$$

$\bar{\theta}$ not unique

Obs
$$U \times V \xrightarrow{\varphi_0} U \otimes V$$



commutes

cont.

$$S_0 = U \otimes V$$

□

LEM 3 Given

- (i) vs $U \otimes V$ and bil map $\varphi_0: U \times V \rightarrow U \otimes V$
 that satisfies LEM 1 (i). (ii) V vs W
 (iii) vs $U \hat{\otimes} V$ and bil map $\hat{\varphi}_0: U \times V \rightarrow U \hat{\otimes} V$

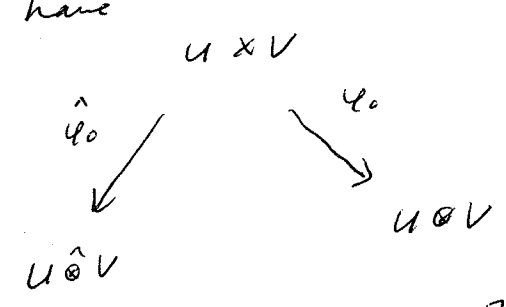
Then \exists iso of vectn spaces

$$U \otimes V \rightarrow U \hat{\otimes} V$$

that sends

$$u \otimes v \rightarrow u \hat{\otimes} v \quad \forall u \in U \quad \forall v \in V$$

pf We have



$\varphi_0, \hat{\varphi}_0$ bilin.

By Lem 1 with $W = U \hat{\otimes} V \quad \exists$ lin map

$U \otimes V \rightarrow U \hat{\otimes} V$ that sends $u \otimes v \rightarrow u \hat{\otimes} v \quad \forall u \in U \quad \forall v \in V$.

Swapping roles $U \hat{\otimes} V \leftrightarrow U \otimes V \quad \exists$ lin map

$U \hat{\otimes} V \rightarrow U \otimes V$ that sends $u \hat{\otimes} v \rightarrow u \otimes v \quad \forall u \in U \quad \forall v \in V$.

Above 2 maps are inverses by LEM 2; hence bijections. \square

DEF 4. By the tensor product of U, V

we mean a vector space $U \otimes V$ together

with a linear map

$$\begin{aligned} \varphi_0 : U \times V &\rightarrow U \otimes V \\ u, v &\rightarrow u \otimes v \end{aligned}$$

that satisfies

LEM 1 (i), (ii)

\forall vector spaces W .

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LEM 5 Given vector spaces U, V

The tensor product of U, V exists.

Pf Start with Cartesian product $U \times V$

Let $k U \times V$ denote the vector space with basis $U \times V$

Let J denote the subspace of $k U \times V$ spanned by

all vectors of form

$$(\alpha u, v) - \alpha(u, v) \quad \alpha \in k$$

$$u, u' \in U$$

$$(u + u', v) - (u, v) - (u', v) \quad u, v' \in V$$

$$(u, \alpha v) - \alpha(u, v)$$

$$(u, v + v') - (u, v) - (u, v')$$

Define the quotient vector space

$$U \otimes V = \frac{k U \times V}{J}$$

Define $\varphi_0: U \times V \rightarrow U \otimes V$ to be the composition

$$\varphi_0: U \times V \xrightarrow{\text{incl}} k U \times V \xrightarrow{a} U \otimes V$$

$$u \rightarrow u + J$$

φ_0 is bilinear by def of J_0

Obs: image $\varphi_0(U \times V)$ spans $U \otimes V$.

Given vector space W and bilinear map

$$\theta: U \times V \rightarrow W$$

define lin map

$$\hat{\theta}: K U \times V \rightarrow W$$

that sends basis vectors $(u, v) \rightarrow \theta(u, v) \quad u \in U, v \in V$

claim $\hat{\theta}(J) = 0$

pf of $\hat{\theta}$ sends

$$(\alpha u, v) - \alpha(u, v) \rightarrow \begin{matrix} \theta(\alpha u, v) - \alpha \theta(u, v) = 0 \\ \parallel \\ \alpha \theta(u, v) \end{matrix}$$

$$(u + u', v) - (u, v) - (u', v) \rightarrow \begin{matrix} \theta(u + u', v) - \theta(u, v) - \theta(u', v) = 0 \\ \parallel \\ \theta(u, v) + \theta(u', v) \end{matrix}$$

$$(u, \alpha v) - \alpha(u, v) \rightarrow \begin{matrix} \theta(u, \alpha v) - \alpha \theta(u, v) = 0 \\ \parallel \\ \alpha \theta(u, v) \end{matrix}$$

$$(u, v + v') - (u, v) - (u, v') \rightarrow \begin{matrix} \theta(u, v + v') - \theta(u, v) - \theta(u, v') = 0 \\ \parallel \\ \theta(u, v) + \theta(u, v') \end{matrix}$$

claim proved ✓

By the claim, $\hat{\sigma}$ induces a linear map

$$\bar{\sigma} : \begin{array}{ccc} U \otimes V & \longrightarrow & W \\ a + J & \longrightarrow & \hat{\sigma}(a) \end{array}$$

Check this diag commutes:

$$\begin{array}{ccc} U \times V & \xrightarrow{\varphi_0} & U \otimes V \\ & \searrow \sigma & \swarrow \bar{\sigma} \\ & & W \end{array}$$

*

$$\begin{array}{ccc} (u, v) & \longrightarrow & (u, v) + J \\ & \searrow & \downarrow \bar{\sigma} \\ & & \hat{\sigma}(u, v) \\ & & \sigma(u, v) = v \end{array}$$

OK

Above map $\bar{\sigma}$ is unique lin map $U \otimes V \rightarrow W$ that makes * commute, since $\varphi_0(U \times V)$ spans $U \otimes V$.

We have shown that the vs $U \otimes V$ together with the bilin map $\varphi_0: U \times V \rightarrow U \otimes V$ is a tensor product of U, V .

□

LEM 6 Given vectn spaces $U, V, W.$

Then \exists vectn space iso

$$\text{Hom}(U \otimes V, W) \longrightarrow \text{Hom}(U, \text{Hom}(V, W))$$

$$f \longrightarrow F$$

s.t

$$F(u)(v) = f(u \otimes v) \quad \forall u \in U \quad \forall v \in V$$

pf We obtain the desired iso as a

composition

$$\begin{array}{ccc} \text{Hom}(U \otimes V, W) & \xrightarrow{\text{iso}} & \text{Hom}^2(U, V; W) & \rightarrow & \text{Hom}(U, \text{Hom}(V, W)) \\ f & \rightarrow & f \circ \varphi_0 & \rightarrow & \bar{g} \\ & & & & \text{(Fund)} \end{array}$$

For $g \in \text{Hom}^2(U, V; W)$ find \bar{g}

Require lin map

$$\bar{g} : U \rightarrow \text{Hom}(V, W)$$

$\forall u \in U$ require lin map

$$\bar{g}(u) : V \rightarrow W$$

For $v \in V$ require

$$\bar{g}(u)(v) \in W$$

Declare

$$\bar{g}(u)(v) = g(u, v)$$