Lec 7 Friday Sept 18

More on Ore extensions

Cor 28 For an Ore extension $R[t;\alpha,\delta]$ assume that α is surjective (and hence alg iso)
Then each element of $R[t;\alpha,\delta]$ is uniquely written as

$$\sum_{i \in \mathbb{N}} t^i a_i \qquad a_i \in R \qquad \text{fin many } a_i \text{ nonzero} \qquad ☆$$

" $R[t;\alpha,\delta]$ is a free right R-module "

pf By LEM 27

$$R + tR + \cdots + t^n R = R + Rt + \cdots + Rt^n \qquad n \in \mathbb{N} \qquad *$$

Given $f \in R[t;\alpha,\delta]$. Using $*$, put f in form ☆

Show form ☆ is unique. Suppose not. Then $\exists$ $n \in \mathbb{N}$ and $a_i \in R$ $(0 \le i \le n)$ s.t. $a_n \neq 0$ and

$$0 = \sum_{i=0}^{n} t^i a_i$$

$$= t^n a_n + \sum_{i=0}^{n-1} t^i a_i$$

$$\| $$

$$\underbrace{\alpha^n(a_n)}_{\neq 0} t^n + LT$$

$$\neq 0 \quad \text{contr.}$$

□

Noetherian Rings

LEM 29 Fa a ring R TFAE

(i) $\forall$ chains of left ideals "ascending chain condition"

$I_0 \subseteq I_1 \subseteq I_2 \subseteq \cdots$

$\exists\, n \geq 0$ s.t

$I_n = I_{n+1} = I_{n+2} = \cdots$

(ii) Each left ideal of R is fin. generated.

pf (i) $\to$ (ii)

Pick $a_0 \in I$

$I_0 = Ra_0$ is left ideal of R

$I_0 \subseteq I$

Assume $I_0 \neq I$, else done

Pick $a_1 \in I \setminus I_0$

$I_1 = Ra_0 + Ra_1$ is left ideal of R

$I_0 \subsetneq I_1 \subseteq I$

Assume $I_1 \neq I$, else done

Pick $a_2 \in I \setminus I_1$

$\exists\, n \geq 0$ s.t. $I_n = I$; else

$$I_0 \subsetneq I_1 \subsetneq I_2 \subsetneq \cdots$$

violates asc chain cond.

Now

$$I = I_n = Ra_0 + \cdots + Ra_n \quad \text{is fin. gen.}$$

(ii) $\to$ (i) Given left ideals of R

$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \cdots \qquad \ast$$

Define $$I = \bigcup_{i=0}^{\infty} I_i$$

I is left ideal of R

$\exists\, n \geq 0$ $\exists\, a_0, a_1, \ldots, a_n \in R$ s.t.

$$I = Ra_0 + Ra_1 + \cdots + Ra_n.$$

For $0 \leq i \leq n$ a_i contained in some term of $\ast$

Since n finite, some term I_N contains all of $a_0, a_1, \ldots, a_n$. So $I = I_N = I_{N+1} = \cdots$ $\square$

Def 30 A ring R is called <u>left-Noetherian</u> whenever (i), (ii) hold in LEM. 29

R is <u>right Noetherian</u> whenever R^{op} is left Noetherian

R is <u>Noetherian</u> whenever R is both left and right Noetherian

LEM 31 Given rings R, S and surjective ring hom $\theta : R \to S$.

Assume

R is left (resp. right) Noetherian. Then

S is left (resp. right) Noetherian.

pf Let I = left ideal of S

Show I is finitely generated.

Let J = preimage of I under θ

J is left ideal of R

So J is fin generated

$\exists$ $a_1, a_2 \ldots a_n \in R$ such that

$$J = \sum_{i=1}^{n} R a_i$$

Now
$$\begin{aligned} I &= \theta(J) \\ &= \sum_{i=1}^{n} S\,\theta(a_i) \end{aligned}$$

is fin generated ✓

$\square$

Given algebra R

LEM 32 Given Ore extension $R[t, \alpha, \delta]$

Assume R is left Noetherian, and α is surjective.

then $R[t, \alpha, \delta]$ is left Noetherian.

$\overset{\parallel}{A}$

pf By LEM 27

for $n \geq 0$,

$$R + Rt + Rt^2 + \cdots + Rt^n = R + tR + t^2R + \cdots + t^nR \quad (= A_n)$$

So $A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots$ $\quad A = \bigcup_{n \in \mathbb{N}} A_n$

$RA_n \subseteq A_n$, $\quad A_nR \subseteq A_n$

$tA_n \subseteq A_{n+1}$, $\quad A_nt \subseteq A_{n+1}$

For $f \in A_n$ write

$$f = \sum_{i=0}^{n} t^i c_i(f) \qquad c_i(f) \in R$$

Obs

$$c_{i+1}(tf) = c_i(f) \qquad 0 \leq i \leq n$$

Claim 1 For $n \geq 0$ and $f \in A_n$ and $a \in R$,

$$c_n(af) = \alpha^{-n}(a)\, c_n(f)$$

pf cl 1

$$f - t^n c_n(f) \in A_{n-1}$$

$$af - t^n c_n(af) \in A_{n-1}$$

Also

$$af - \underbrace{a t^n}_{= t^n \alpha^{-n}(a) + LT} c_n(f) \in a A_{n-1} \subseteq A_{n-1}$$

so

$$af - t^n \underbrace{\alpha^{-n}(a)\, c_n(f)}_{= c_n(af)} \in A_{n-1}$$

✓

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Given a left ideal I of A,

show that I is fin. generated.

We have $AI \subseteq I$

so

$$RI \subseteq I, \qquad tI \subseteq I$$

For $n \geq 0$ define

$$\begin{aligned} I_n &= I \cap A_n \\ &= \{ f \in I \mid c_i(f) = 0 \text{ for } i > n \} \end{aligned}$$

Obs

$$I_n \subseteq I_{n+1} \qquad \forall n \in \mathbb{N} \tag{1}$$

$$I = \bigcup_{n \in \mathbb{N}} I_n \tag{2}$$

Also $RI_n \subseteq I_n, \qquad tI_n \subseteq I_{n+1} \qquad \forall n \in \mathbb{N}$

So for $n \in \mathbb{N}$,

$$I_n + tI_n \subseteq I_{n+1} \qquad *$$

Going to show equality holds in $*$ for n suf large.

Fix $n \in \mathbb{N}$ define

$$L_n = \{ c_n(f) \mid f \in I_n \}$$

<u>claim 2</u> Fix $n \in \mathbb{N}$

L_n is a left ideal of R

<u>pf cl 2</u> L_n is subspace of vector space R ✓

show $R L_n \subseteq L_n$

Given $a \in R$ and $c \in L_n$ show $ac \in L_n$.

$\exists f \in I_n$ s.t.

$$c = c_n(f)$$

Obs

$$\alpha^n(a) f \in R I_n \subseteq I_n$$

$$c_n(\alpha^n(a) f) = a\, c_n(f) \quad \text{by cl 1}$$
$$= ac$$

So $ac \in L_n$ ✓

Claim 3 $\quad L_n \subseteq L_{n+1}$ for $n \in \mathbb{N}$

pf cl 3 $\quad$ Given $c \in L_n$ $\quad$ show $c \in L_{n+1}$

$\exists \; f \in I_n$ s.t.

$$c = c_n(f)$$

Obs

$$tf \in I_{n+1}$$

$$c_{n+1}(tf) = c_n(f)$$
$$= c$$

So

$$c \in L_{n+1}$$

Since R is left Noetherian,

$\exists$ $N \geq 0$ s.t. $L_n = L_{n+1}$ for $n \geq N$.

<u>Claim 4</u> For $n \geq N$,

$$I_n + t I_n = I_{n+1}$$

<u>pf cl4</u> $\subseteq$ ✓

$\supseteq$: For $f \in I_{n+1}$

$$f - t^{n+1} \underbrace{c_{n+1}(f)}_{\substack{\cap \\ L_{n+1} = L_n}} \in A_n \qquad (3)$$

$\exists$ $g \in I_n$ s.t.

$$c_n(g) = c_{n+1}(f) \qquad (4)$$

By constr

$$g - t^n c_n(g) \in A_{n-1}$$

So

$$tg - t^{n+1} c_n(g) \in t A_{n-1} \subseteq A_n \qquad (5)$$

By (3)-(5)

$$\underset{\overset{\uparrow}{I}}{f} - t\underset{\overset{\uparrow}{I}}{g} \in A_n$$

so

$$\underbrace{f - tg}_{\overset{\shortparallel}{h}} \in I \cap A_n = I_n$$

We have

$$f = \underset{\overset{\uparrow}{I_n}}{h} + \underset{\overset{\uparrow}{tI_n}}{tg}$$

✓

<u>Claim 5</u> the ideal I of A is generated by I_N

<u>pf cl 5</u> By Claim 4 and (1), (2). ✓

For $0 \le n \le N$ the ideal L_n of R is fin gen

so $\exists$ finite subset G_n of I_n s.t.

$\{c_n(g) \mid g \in G_n\}$ is a gen set for ideal L_n

Claim 6 For $0 \le n \le N$,

$$I_n = I_{n-1} + \sum_{g \in G_n} Rg \qquad (I_{-1} = 0)$$

pf cl 6 $\supseteq$ ✓

$\subseteq$ Given $f \in I_n$

$$L_n = \sum_{g \in G_n} R c_n(g)$$

$$f - t^n \underbrace{c_n(f)}_{\in L_n} \in A_{n-1}$$

$$c_n(f) = \sum_{g \in G_n} r_g c_n(g) \qquad r_g \in R$$

Define

$$F = f - \sum_{g \in G_n} \alpha^n(r_g)\, g$$

show $F \in I_{n-1}$

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$$F \in I_n \subseteq I$$

Show

$$F \in A_{n+1}$$

obs

$$c_n(F) = c_n(f) - \sum_{g \in G_n} c_n\left(\alpha^n(r_g)\, g\right)$$

$$c_n\left(\alpha^n(r_g)\, g\right) = \alpha^{-n+n}(r_g)\, c_n(g) \quad \text{by claim 1}$$

$$= c_n(f) - \sum_{g \in G_n} r_g\, c_n(g)$$

$$= 0$$

So $F \in A_{n+1}$

So $F \in I \cap A_{n+1} = I_{n+1}$

So

$$f = \underset{\substack{\uparrow \\ I_{n+1}}}{F} + \sum_{g \in G_n} \underbrace{\alpha^n(r_g)\, g}_{\substack{\uparrow \\ R_g}}$$

✓

Claim 7 $\quad \bigcup_{n=0}^{N} G_n$ is gen set for ideal I

$\quad\quad\quad\quad \parallel$

$\quad\quad\quad\quad G$

pf cl 7 $\quad$ By claim 6

$$I_N = \sum_{n=0}^{N} \sum_{g \in G_n} R g$$

$$= \sum_{g \in G} R g$$

By claim 5

$$I = A I_N$$
$$= \sum_{g \in G} A g$$

— o —

$|G| < \infty$ so I is fin gen $\square$