Lec 6 Wed, Sept 16

Ore extensions, cont.

Given algebra R, indet t

vector space $R[t]$ with new mult that satisfies

Ore 1 – Ore 3

Recall $\exists$ k-lin map $\delta : R \to R$ and

injective k-lin map $\alpha : R \to R$ s.t.

$$t\,a = \delta(a) + \alpha(a)\,t \qquad \forall a \in R$$

(iii) $\forall\, a, b \in R$

$\alpha(ab) = \alpha(a)\,\alpha(b)$ ie α is alg morphism

$\delta(ab) = \delta(a)b + \alpha(a)\,\delta(b)$ "δ is an α-derivation"

pf

$$t(ab) = (ta)b$$
$$\| \qquad\qquad \|$$
$$\delta(ab) + \alpha(ab)t \qquad (\delta(a) + \alpha(a)t)b$$
$$\|$$
$$\delta(a)b + \alpha(a)(\delta(b) + \alpha(b)t)$$
$$\|$$
$$\delta(a)b + \alpha(a)\delta(b) + \alpha(a)\alpha(b)t$$

We now reverse direction

Assume

- Algebra R (possibly with 0-div)
- $\alpha : R \to R$ is inj algebra morphism
- $\delta : R \to R$ is α-derivation

Show $R[t]$ has algebra structure that satisfies Ore 1, Ore 2 and

$$t a = \delta(a) + \alpha(a) t \qquad \forall a \in R$$

☆

Note $\delta(1) = 0$

pf
$$\delta(1) = \delta(1 \cdot 1) = \underbrace{\delta(1) 1}_{= \delta(1)} + \underbrace{\alpha(1)}_{= 1} \delta(1)$$

so
$$0 = \delta(1) \checkmark$$

Recall

$$\mathrm{End}(R) = \text{ k-alg of all k-lin trans } R \to R$$

So $\quad \alpha, \delta \in \mathrm{End}(R)$

Injective alg morphism

$$R \to \mathrm{End}(R)$$

$$a \to aI$$

Identify R with its image, so $R \subseteq \mathrm{End}(R)$

$\forall a \in R$

$$\delta a = \delta(a) + \alpha(a)\delta, \qquad *$$

$$\alpha a = \alpha(a)\alpha \qquad **$$

pf $*$: $\quad \forall b \in R$

$$\delta a(b) \overset{?}{=} \delta(a)(b) + (\alpha(a)\delta)(b)$$

$\delta a(b) = \delta(ab) = \delta(a)b + \alpha(a)\delta(b)$

$\delta(a)(b) = \delta(a)b$

$(\alpha(a)\delta)(b) = \alpha(a)\delta(b)$

✓

pf $**$:

$$\alpha_a(b) \overset{?}{=} (\alpha(a)\alpha)(b)$$

$$\begin{array}{ccc} \alpha_a(b) & \overset{?}{=} & (\alpha(a)\alpha)(b) \\ \| & & \| \\ \alpha(ab) & & \alpha(a)\alpha(b) \\ \| & & \\ \alpha(a)\alpha(b) & & \checkmark \end{array}$$

Define algebra $M = M_\infty(\mathrm{End}(R))$

consists of all matrices with

- entries in $\mathrm{End}(R)$
- rows/cols indexed by $\mathbb{N}$
- fin many nonzero entries in each row/col.

Define $\hat{t} \in M$ by

$$\hat{t} = \begin{pmatrix} \delta & & & & 0 \\ \alpha & \delta & & & \\ & \alpha & \delta & & \\ & & \ddots & \ddots & \\ & 0 & & & \end{pmatrix}$$

For $a \in R$ define $\hat{a} \in M$ by

$$\hat{a} = \operatorname{diag}(a, a, \ldots)$$

For $a, b \in R$ and $\gamma \in k$,

$$\widehat{a+b} = \hat{a} + \hat{b}$$

$$\widehat{ab} = \hat{a}\hat{b}$$

$$\widehat{\gamma a} = \gamma \hat{a}$$

By $*$, $**$

$$\hat{t}\hat{a} = \widehat{\delta(a)} + \widehat{\alpha(a)}\,\hat{t} \qquad \forall a \in R \qquad ***$$

For $i \in \mathbb{N}$

$$(\hat{t})^i \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \end{pmatrix} \leftarrow i \qquad \begin{bmatrix} \text{using} \\ \delta(1) = 0 \\ \alpha(1) = 1 \end{bmatrix}$$

so for $\displaystyle\sum_{i=0}^{n} a_i t^i \in R[t]$

$$\left(\sum_{i=0}^{n} \hat{a}_i (\hat{t})^i \right) \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix} = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \\ 0 \\ \vdots \end{pmatrix} \qquad ****$$

let $\widehat{R[t]}$ = subalg of M gen by

$\hat{t}$, $\{ \hat{a} \mid a \in R \}$

claim

$$\wedge : \begin{array}{ccc} R[t] & \longrightarrow & \widehat{R[t]} \\ \displaystyle\sum_{i=0}^{n} a_i t^i & \longrightarrow & \displaystyle\sum_{i=0}^{n} \hat{a}_i (\hat{t})^i \end{array}$$

is iso of k-vector spaces

pf of claim

Λ is k-linear ✓

Λ is surj by ***

Λ is inj by **** ✓

claim proved

Via Λ, pull back the algebra str from $\widehat{R[t]}$ to $R[t]$

Check algebra $R[t]$ satisfies ☆ and Ore 1, Ore 2

☆: by ***

Ore 1: since $\hat{a}\,\hat{b} = \widehat{ab}$ $\forall a, b \in R$

Ore 2: By constr.

Def 23 Above algebra structure on $R[t]$
is called the <u>Ore extension</u> with data t, α, δ
Written $R[t, \alpha, \delta]$

Example of α-derivation

Given algebra R

Given $a, b \in R$ with a invertible

$$\alpha : \begin{array}{l} R \to R \\ x \to a x a^{-1} \end{array}$$ is an algebra iso

$$\delta : \begin{array}{l} R \to R \\ x \to a[b, x] \end{array}$$ is α-derivation

$[b,x] = bx - xb$

check

$$\delta(xy) \overset{?}{=} \delta(x) y + \alpha(x) \delta(y)$$

$$\delta(xy) = a(bxy - xyb)$$

$$\delta(x) y + \alpha(x) \delta(y) = a(bx - xb) y + a x a^{-1} a(by - yb)$$

$$= abxy - \cancel{axby} + \cancel{axby} - axyb$$

✓

LEM 24 Assume

- algebra R has no 0-divisors
- $\alpha : R \to R$ is inj alg morphism
- $\delta : R \to R$ is α-derivation.

then $R[t, \alpha, \delta]$ satisfies Ore 3

pf Given $p, q \in R[t, \alpha, \delta]$ show

$$\deg(pq) = \deg(p) + \deg(q)$$

Assume $p \neq 0$, $q \neq 0$ else triv.

Write
$$p = \sum_{i=0}^{n} a_i t^i \qquad a_i \in R \qquad a_n \neq 0$$
$$q = \sum_{j=0}^{m} b_j t^j \qquad b_j \in R \qquad b_m \neq 0$$

$$pq = a_n t^n b_m t^m + LT$$
$$= \underbrace{a_n \underbrace{\alpha^n(b_m)}_{\neq 0 \text{ since } \alpha \text{ inj}}}_{\neq 0 \text{ since } R \text{ has no 0-div}} t^{n+m} + LT$$

So $\deg(pq) = n+m$ $\square$

Given Ore ext $R[t, \alpha, \delta]$

LEM 25 For $n \geq 0$ and $a \in R$

$$t^n a = \sum_{\ell=0}^{n} S_{n,\ell}(a)\, t^{n-\ell}$$

where for $0 \leq \ell \leq n$,

$$S_{n,\ell}(a) = \sum x_1 x_2 \cdots x_n (a)$$

sum over all sequences $x_1, x_2, \ldots, x_n$

with $x_i \in \{\alpha, \delta\}$ for $1 \leq i \leq n$

and $\ell = \left|\{ i \mid 1 \leq i \leq n,\ x_i = \delta \}\right|$

pf Routine ind on n. $\square$

LEM 26 Assume the algebra R has no 0-divisors, then any Ore ext $R[t;\alpha,\delta]$ has no 0-divisors.

pf Given $0 \neq p, q \in R[t;\alpha,\delta]$ show

$$pq \neq 0.$$

as by LEM 24
$$\deg(pq) = \underbrace{\deg p}_{\geq 0} + \underbrace{\deg q}_{\geq 0}$$

$$\geq 0$$

So $pq \neq 0$. $\square$

In the Ore ext $R[t, \alpha, \delta]$ we saw

$$tR \subseteq R + Rt$$

So for $n \geq 0$,

$$t^n R \subseteq R + Rt + Rt^2 + \cdots + Rt^n$$

More generally

$$R + tR + t^2R + \cdots + t^nR \subseteq R + Rt + Rt^2 + \cdots + Rt^n \qquad *$$

LEM 27 With above notation, assume α is surjective (and hence algebra iso), then equality in $*$ for $n \geq 0$.

pf By ind on n.

$n=0$ ✓

$n \geq 1$: Given $a \in R$ suf to show

$$at^n \in R + tR + t^2R + \cdots + t^nR$$

$$\exists\, b \in R \quad \text{s.t} \quad \alpha^n(b) = a$$

obs

$$t^n b = \underset{\overset{\shortparallel}{a}}{\alpha^n(b)}\, t^n + LT$$

ie

$$t^n b - a t^n \in \underbrace{R + Rt + Rt^2 + \dots + Rt^{n-1}}_{\parallel \text{ by ind}}$$

$$\cap$$

$$t^n R \qquad\qquad R + tR + t^2R + \dots + t^{n-1}R$$

So

$$a t^n \in R + tR + t^2R + \dots + t^n R$$

$\square$