Lec 5 Monday Sept 14

We continue to discuss gradings and filtrations

Given algebra A

Given filtration $\{A_i\}_{i \in \mathbb{N}}$ of A

Define a vector space

$$gr(A) = \bigoplus_{i \in \mathbb{N}} A_i / A_{i-1} \qquad (A_{-1} = 0) \qquad *$$

$*$ becomes a graded algebra with product

$$A_i/A_{i-1} \times A_j/A_{j-1} \longrightarrow A_{i+j}/A_{i+j-1}$$

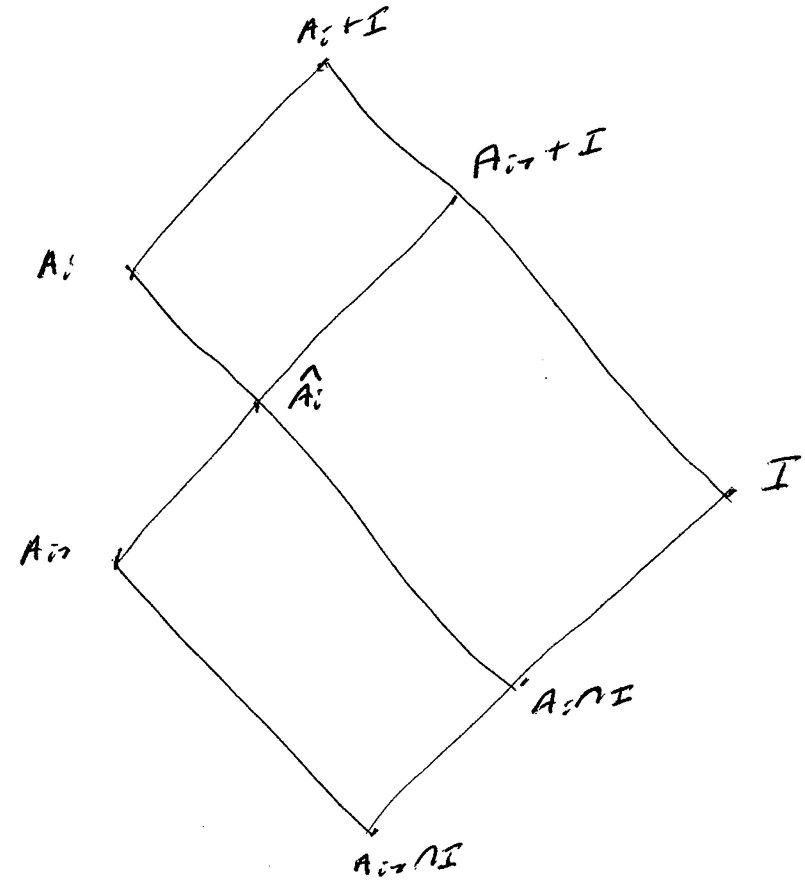
$$a + A_{i-1} \qquad b + A_{j-1} \longrightarrow ab + A_{i+j-1}$$

Given: alg A

filtration $\{A_i\}_{i \in \mathbb{N}}$ of A

2-sided ideal $I \trianglelefteq A$ with $I \neq A$

For $i \in \mathbb{N}$ we have inclusions



where
$$\hat{A}_i = A_i \cap (A_{i-1} + I) = A_{i-1} + A_i \cap I$$

Filtratn $\{A_i\}_{i\in\mathbb{N}}$ gives graded algebra

$$gr(A) = \bigoplus_{i\in\mathbb{N}} A_i/A_{i-1}$$

Ideal I gives quot algebra

$$\bar{A} = A/I$$

We have canon map

$$\mathrm{can}: \begin{array}{ccc} A & \to & \bar{A} \\ a & \mapsto & \bar{a} \end{array}$$

For $i\in\mathbb{N}$ let

$$\begin{aligned} \bar{A}_i &= \text{image of } A_i \text{ under can} \\ &= \frac{A_i + I}{I} \end{aligned}$$

one checks that

$$\{\bar{A}_i\}_{i\in\mathbb{N}} \text{ is filtratn of } \bar{A}$$

this filtr gives graded algebra

$$gr(\bar{A}) = \bigoplus_{i\in\mathbb{N}} \bar{A}_i/\bar{A}_{i-1}$$

For $i \in \mathbb{N}$ the canon maps

$$A_i \to \overline{A_i}, \qquad A_{i+1} \to \overline{A_{i+1}}$$

induce surj linear trans

$$A_i / A_{i+1} \longrightarrow \overline{A_i} / \overline{A_{i+1}} \qquad *$$

$$a + A_{i+1} \longrightarrow \bar{a} + \overline{A_{i+1}}$$

$*$ has kernel $\hat{A}_i / A_{i+1}$.

So $*$ induces vector space iso

$$\theta_i \quad \frac{A_i / A_{i+1}}{\hat{A}_i / A_{i+1}} \longrightarrow \frac{\overline{A_i}}{\overline{A_{i+1}}} \qquad **$$

$**$ induces vector space iso

$$\theta : \bigoplus_{i \in \mathbb{N}} \frac{A_i / A_{i+1}}{\hat{A}_i / A_{i+1}} \longrightarrow \bigoplus_{i \in \mathbb{N}} \frac{\overline{A_i}}{\overline{A_{i+1}}} = gr(\overline{A})$$

Also, $*$ induces surj lin trans

$$\underbrace{\bigoplus_{i \in \mathbb{N}} A_i / A_{i+1}}_{= gr(A)} \longrightarrow \underbrace{\bigoplus_{i \in \mathbb{N}} \overline{A_i} / \overline{A_{i+1}}}_{= gr(\overline{A})} \qquad ☆$$

One checks that $\star$ is an algebra hom.

the kernel is

$$J = \bigoplus_{i \in \mathbb{N}} \frac{\hat{A}_i}{A_{i-1}}$$

So J is a homog 2-sided ideal of $gr(A)$

Moreover $\star$ induces iso of graded algebras

$$gr(A)/J \longrightarrow gr(\bar{A})$$

This iso is the composition

$$gr(A)/J \xrightarrow[nat^{-1}]{} \bigoplus \frac{A_i/A_{i-1}}{\hat{A}_i/A_{i-1}} \xrightarrow[\theta]{} gr(\bar{A})$$

<u>Summary</u>

Alg A

filtr $\{A_i\}$ of A

ideal I

graded alg

$gr(A) = \bigoplus_i A_i/A_{i-1}$

alg $\bar{A} = A/I$

filtr $\{\bar{A}_i\}$ of $\bar{A}$

hom ideal

$J = \bigoplus_i \hat{A}_i / A_{i-1}$

graded alg

$gr(\bar{A}) = \bigoplus_i \bar{A}_i/\bar{A}_{i-1}$

quot alg

$gr(A)/J$

is graded

$\longleftrightarrow$

iso of graded algebras

(Aside) the following diagram commutes.

Each map is iso of vector spaces

$$\bar{A}_i \Big/ \bar{A}_{i+1} = \frac{(A_i+I)/I}{(A_{i+1}+I)/I}$$

$$\begin{tikzcd}
\dfrac{A_i/A_{i+1}}{\hat{A}_i/A_{i+1}} \arrow[r, "\theta_i"] \arrow[d, "\text{canon bij}"'] & \dfrac{\bar{A}_i}{\bar{A}_{i+1}} \arrow[d, "\text{canon bij}"] \\
\dfrac{A_i}{\hat{A}_i} \arrow[r] & \dfrac{A_i+I}{A_{i+1}+I}
\end{tikzcd}$$

$$a+\hat{A}_i \to a+A_{i+1}+I$$

(Aside) we have iso of vs

$$\frac{A_i \cap I}{A_{i-1} \cap I} \xleftrightarrow[\text{canon bij}]{} \frac{\hat{A}_i}{A_{i-1}}$$

Special Case

Start with free alg $T = k\{X\}$ set X

I = any 2-sided ideal of T s.t. $I \neq T$

Filtration of T:

$$\{ \underbrace{T_0 + T_1 + \cdots + T_i}_{T^{(i)}} \}_{i \in \mathbb{N}}$$

T_i spanned by words over X length i

Alg $A = T/I$

A has filtr $\left\{ \dfrac{T^{(i)} + I}{I} \right\}_{i \in \mathbb{N}}$

Desc corresp graded alg $gr(A)$

For $i \in \mathbb{N}$ consider v.s. iso

$$T_i \longrightarrow \frac{T^{(i)}}{T^{(i-1)}} \qquad *$$

$$a \longrightarrow a + T^{(i-1)}$$

$*$ induces vs iso

$$T = \sum_{i \in \mathbb{N}} T_i \longrightarrow \bigoplus_{i \in \mathbb{N}} \frac{T^{(i)}}{T^{(i-1)}} \quad (**)$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad " \; gr(T)$$

One checks
$**$ is alg iso

the alg $gr(A)$ is iso $gr(T)/J$

where J is homog 2-sided ideal of $gr(T)$:

$$J = \bigoplus_{i \in \mathbb{N}} \frac{T^{(i-1)} + T^{(i)} \cap I}{T^{(i-1)}}$$

let $\mathcal{J}$ = preimage of J under $**$

= hom 2-sided ideal of T

So for $i \in \mathbb{N}$

$\mathcal{J} \cap T_i$ = preimage of

$$\frac{T^{(i-1)} + T^{(i)} \cap I}{T^{(i-1)}}$$

under $*$
the alg $gr(A)$ is iso $T/\mathcal{J}$ $\square$

Ex Algebra $SL(2)$

Gen set

$$X = \{ x_{ij} \mid 1 \le i, j \le 2 \}$$

Free alg $T = k\{X\}$ with filtr $\{T^{(i)}\}_{i \in \mathbb{N}}$

I = 2-sided ideal of T gen by

$xy - yx \qquad x, y \in X$

$x_{11} x_{22} - x_{12} x_{21} - 1$

$$SL(2) = T/I$$

Find $gr(SL(2))$

$\mathcal{J}$ = 2-sided ideal of T gen by

$xy - yx \qquad x, y \in X$

$x_{11} x_{22} - x_{12} x_{21}$

Then $gr(SL(2))$ iso $T/\mathcal{J}$ (ex)

Ore extensions

Given algebra R

t an indet not assumed to commute with elements in R

$R[t]$ consists of formal linear combinations

degree $\rightarrow$ $\sum_{i=0}^{n} a_i t^i$ $\quad a_i \in R, \quad n \geq 0, \quad a_n \neq 0$

Obs each nonzero element of R has degree 0

Declare $\deg(0) = -\infty$,

Replace usual algebra structure on $R[t]$ by new one.

Retain usual k-vector space structure, but use new mult.

Requirements of new mult:

Given $\sum_{i=0}^{n} a_i t^i \in R[t]$, Fa $a \in R$

Ore 1: $a\left(\sum_{i=0}^{n} a_i t^i\right) = \sum_{i=0}^{n} (a a_i) t^i$

Ore 2: $\left(\sum_{i=0}^{n} a_i t^i\right) t = \sum_{i=0}^{n} a_i t^{i+1}$

Fa $P, Q \in R[t]$

Ore 3: $\deg(PQ) = \deg P + \deg Q$

Given algebra str on $R[t]$ that satisfies Ore 1 - Ore 3

Find consequences:

(i) Fa nonzero $a, b \in R$

$ab \neq 0$ "R has no zero divisor"

pf $\deg(ab) = \underset{\geq 0}{\deg(a)} + \underset{\geq 0}{\deg(b)}$

$= 0$

so $ab \neq 0$

(ii) $\exists$ k-linear map $\delta : R \to R$ and injective k-lin map $\alpha : R \to R$ s.t

$$ta = \delta(a) + \alpha(a)\, t \qquad \forall a \in R$$

pf Given $0 \neq a \in R$

Let $n = \deg(ta)$

Write $ta = \sum_{i=0}^{n} a_i t^i \qquad a_i \in R \qquad a_n \neq 0$

$$n = \deg(ta) = \underbrace{\deg t}_{1} + \underbrace{\deg a}_{0}$$

$$= 1$$

So $ta = a_0 + a_1 t \qquad a_1 \neq 0$

Define $\delta(a) = a_0, \qquad \alpha(a) = a_1$

By constr δ, α are k-linear and α is inj.