Combinatorics Seminar

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the equitable presentation for the quantum algebra $U_q(\mathfrak{sl}_2)$

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Fix $0 \neq q \in \mathbb{C}$ $\quad q^2 \neq 1$

__DEF__ $U_q = U_q(sl_2)$ is the $\mathbb{C}$-alg with gens

$$e, f, k, k^{-1}$$

and relations

$$kk^{-1} = 1, \qquad k^{-1}k = 1,$$

$$ke = q^2 ek, \qquad kf = q^{-2} fk,$$

$$ef - fe = \frac{k - k^{-1}}{q - q^{-1}}$$

U_q is Hopf algebra with

coproduct

$$\Delta : \begin{array}{ccc} U_q & \to & U_q \otimes U_q \\ k & \to & k \otimes k \\ k^{-1} & \to & k^{-1} \otimes k^{-1} \\ e & \to & e \otimes k + 1 \otimes e \\ f & \to & f \otimes 1 + k^{-1} \otimes f \end{array}$$

counit

$$\varepsilon : \begin{array}{ccc} U_q & \to & \mathbb{C} \\ k & \to & 1 \\ k^{-1} & \to & 1 \\ e & \to & 0 \\ f & \to & 0 \end{array}$$

antipode

$$S: \begin{aligned} U_q &\to U_q \\ k &\to k^{-1} \\ k^{-1} &\to k \\ e &\to -ek^{-1} \\ f &\to -kf \end{aligned}$$

vs U_q is ∞-dimensional, with basis

$$e^r k^s f^t \qquad r,t \in \mathbb{N}, \; s \in \mathbb{Z}$$

LEM Assume q not a root of 1.

Then $\exists$ family of f.d. irred U_q-modules

$*$

$$\mathbb{V}_{d,\varepsilon} \qquad d \in \mathbb{N} \qquad \varepsilon \in \{1,-1\}$$

with this property: $\mathbb{V}_{d,\varepsilon}$ has a basis $\{v_i\}_{i=0}^{d}$ s.t.

$$k v_i = \varepsilon q^{d-2i} v_i \qquad (0 \le i \le d)$$

$$e v_i = \varepsilon [d-i+1] v_{i-1} \qquad (1 \le i \le d), \qquad e v_0 = 0$$

$$f v_i = [i+1] v_{i+1} \qquad (0 \le i \le d-1), \qquad f v_d = 0$$

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}$$

Each f.d. irred U_q-module is iso exactly one of $(*)$

Write $\mathbb{V}_d = \mathbb{V}_{d,1}$

LEM Each f.d. U_q-module is direct sum of irred U_q-modules

View of $V = \mathbb{V}_d$

For $0 \le i \le d$ define

$$V_i = \mathbb{C} v_i$$

so

$$V = \sum_{i=0}^{d} V_i \qquad \text{ds}$$

"decomposition of V"

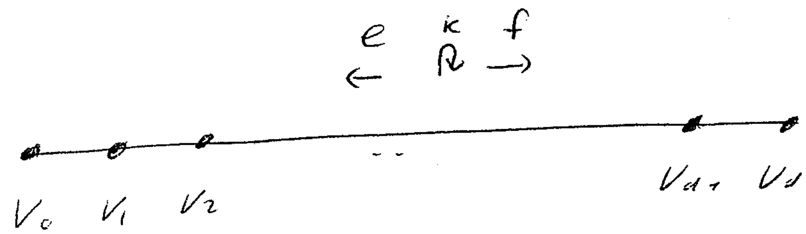
$$K V_i = V_i \qquad 0 \le i \le d$$

$$e V_i = V_{i-1} \qquad (1 \le i \le d), \qquad e V_0 = 0$$

"lowering map"

$$f V_i = V_{i+1} \qquad (0 \le i \le d-1) \qquad f V_d = 0$$

"raising ''"



the equitable presentation of U_q (Ito, Weng, Ter 2006)

<u>Thm</u> U_q has a presentation by gens

$$x, y, y^{-1}, z$$

and rels

$$yy^{-1} = 1, \qquad y^{-1}y = 1,$$

$$\frac{qxy - q^{-1}yx}{q - q^{-1}} = 1, \qquad \frac{qyz - q^{-1}zy}{q - q^{-1}} = 1, \qquad \frac{qzx - q^{-1}xz}{q - q^{-1}} = 1$$

"q-weyl rels"

An iso with the orig pres sends

$$x \longrightarrow k^{-1} - k^{-1} e \, q(q - q^{-1})$$

$$y \longrightarrow k$$

$$z \longrightarrow k^{-1} + f(q - q^{-1})$$

Inverse iso sends

$$e \longrightarrow (1 - yx) q^{-1} (q - q^{-1})^{-1}$$

$$f \longrightarrow (z - y^{-1})(q - q^{-1})^{-1}$$

$$k \longrightarrow y$$

Equitable view of Hopf structure

We have

$$\Delta(x) = x \otimes 1 + y^{-1} \otimes (x-1)$$

$$\Delta(y) = y \otimes y$$

$$\Delta(z) = z \otimes 1 + y^{-1} \otimes (z-1)$$

$$\varepsilon(x) = 1, \qquad \varepsilon(y) = 1, \qquad \varepsilon(z) = 1,$$

$$S(x) = 1 + y - yx$$

$$S(y) = y^{-1}$$

$$S(z) = 1 + z - yz$$

the vs U_q has basis

$$x^r y^s z^t \qquad r, t \in \mathbb{N}, \quad s \in \mathbb{Z}$$

Equitable view of irred U_q-modules

V_d has a basis with respect to which

$$X: \quad \begin{pmatrix} q^{-d} & q^{d}-q^{-d} & & & 0 \\ & q^{2-d} & q^{d}-q^{2-d} & & \\ & & q^{4-d} & \ddots & \\ & 0 & & \ddots & \ddots \\ & & & & q^{d} \end{pmatrix}$$

constant row sum q^{d}

$$Y: \quad \mathrm{diag}\left(q^{d}, q^{d-2}, q^{d-4}, \ldots, q^{-d}\right)$$

$$Z: \quad \begin{pmatrix} q^{-d} & & & & 0 \\ q^{-d}-q^{2-d} & q^{2-d} & & & \\ & q^{-d}-q^{4-d} & q^{4-d} & & \\ & 0 & \ddots & \ddots & \\ & & & & q^{d} \end{pmatrix}$$

constant row sum q^{-d}

COR Each of X, Y, Z is semisimple on V_d with eigenvalues

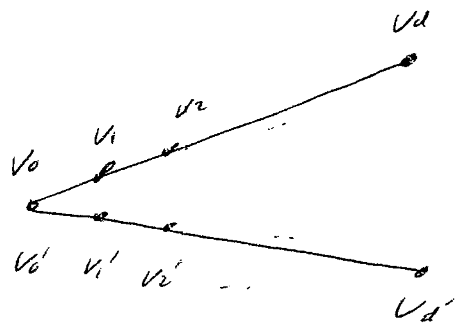
$$q^{d-2i} \qquad 0 \le i \le d.$$

A diagram

Given two decomps of $V = V_d$

$$\{V_i\}_{i=0}^{d}, \qquad \{V_i'\}_{i=0}^{d}$$

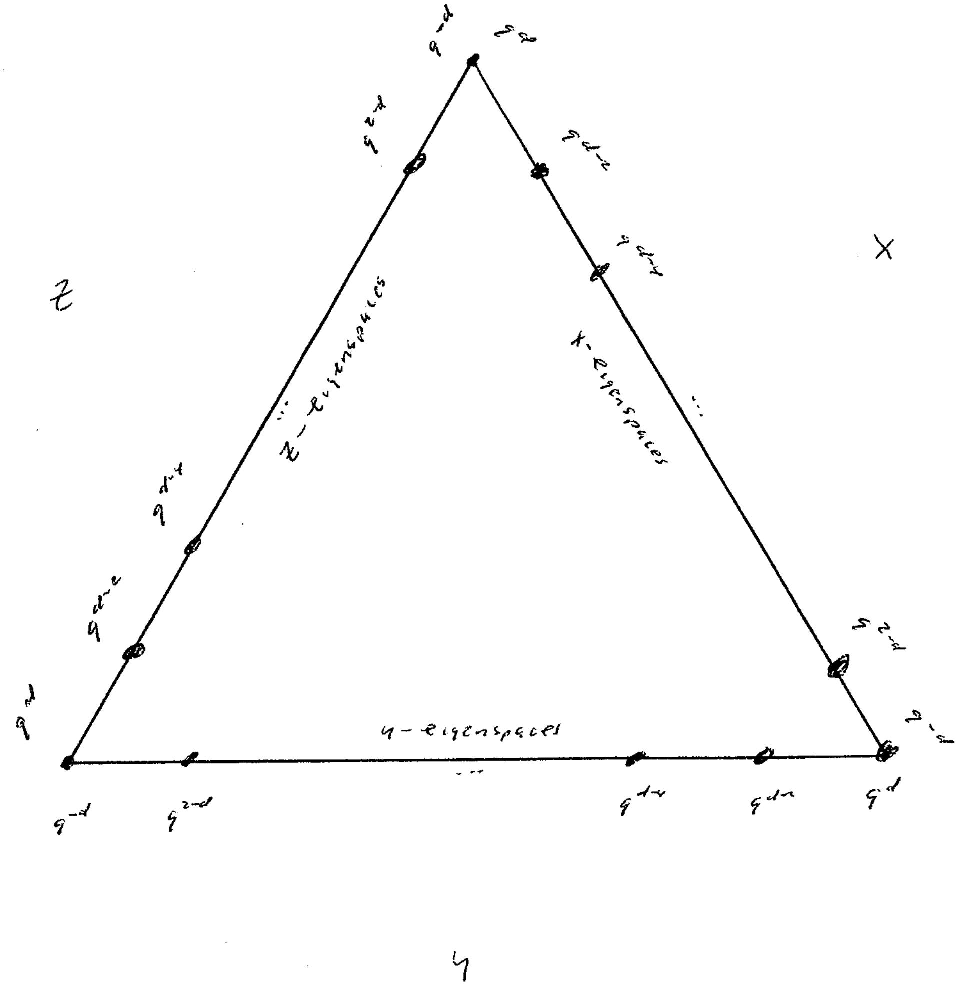
the diagram



means

$$V_0 + V_1 + \cdots + V_i = V_0' + V_1' + \cdots + V_i' \qquad 0 \le i \le d.$$

For V_d.



q^{-d}
q^{d}
q^{2-d}
q^{d-2}
q^{d-4}
q^{d-2}
q^{d}
q^{2-d}
q^{-d}
q^{-d}
q^{2-d}
q^{d-4}
q^{d-2}
q^{d}
z-eigenspaces
x-eigenspaces
y-eigenspaces
x
z
y

Define 3 decompositions of V_d :

decomp name	ith component of decomp $(0 \le i \le d)$
$[x]$	eigenspace for x with eigval q^{d-2i}
$[y]$	-- y --
$[z]$	-- z --

The nilpotent relatives of x, y, z

In U_q

$$\frac{q y z - q^{-1} z y}{q - q^{-1}} = 1$$

Rewrite:

$$q(1 - yz) = q^{-1}(1 - zy)$$

Define

$$n_x = \frac{q(1-yz)}{q-q^{-1}} = \frac{q^{-1}(1-zy)}{q-q^{-1}}$$

n_y, n_z similarly defined

Get

$$x n_y = q^2 n_y x, \qquad x n_z = q^{-2} n_z x$$

$$y n_z = q^2 n_z y, \qquad y n_x = q^{-2} n_x y$$

$$z n_x = q^2 n_x z, \qquad z n_y = q^{-2} n_y z$$

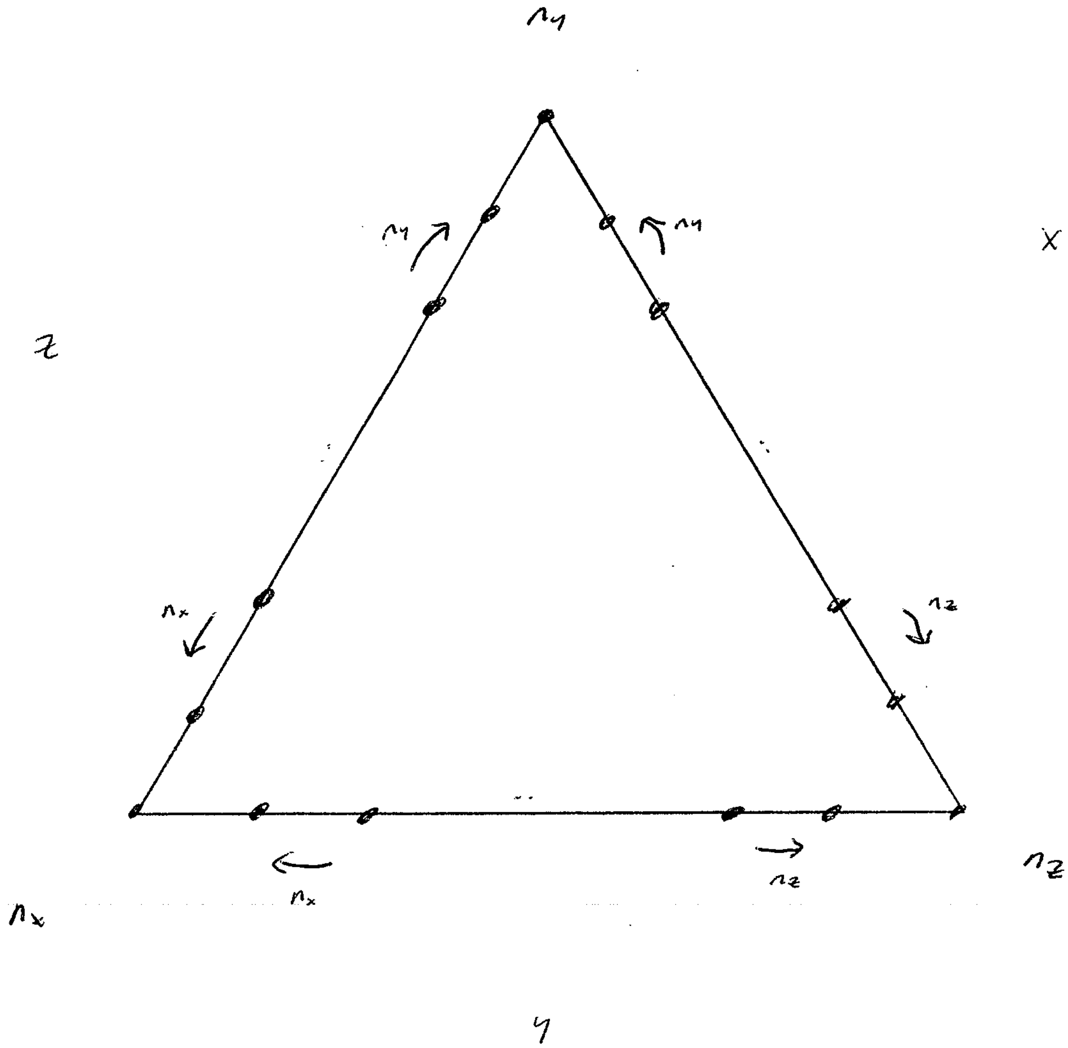
For $\mathbb{V}_d$,

A_x, A_y, A_z act on the decomps $[x], [y], [z]$

as follows:

$\mathcal{U}$	action of $\mathcal{U}$ on $[x]$	$[y]$	$[z]$
A_x	tridiag	raising	lowering
A_y	lowering	tridiag	raising
A_z	raising	lowering	tridiag

Action of n_x, n_y, n_z on $\mathbb{V}_d$:



"Each of n_x, n_y, n_z pulls towards its corner"

$\mathbb{V}_d$ and LR triples

Let V = any vector space dim $d+1$

Given $A, B \in \mathrm{End}(V)$

Call A, B an LR pair whenever $\exists$ decomp $\{V_i\}_{i=0}^{d}$ of V

that is lowered by A and raised by B:

$$A V_i = V_{i-1} \quad (1 \le i \le d), \qquad A V_0 = 0$$
$$B V_i = V_{i+1} \quad (0 \le i \le d-1), \qquad B V_d = 0$$

Given $A, B, C \in \mathrm{End}(V)$, they form an LR-triple

whenever any two of A, B, C form an LR pair.

COR On the U_q module $\mathbb{V}_d$ the elements

n_x, n_y, n_z act as an LR triple.

Note The LR triples are classified up to iso (Ter 2015)

Get the above family and 8 more families.

$\mathbb{V}_d$ and Billiard arrays

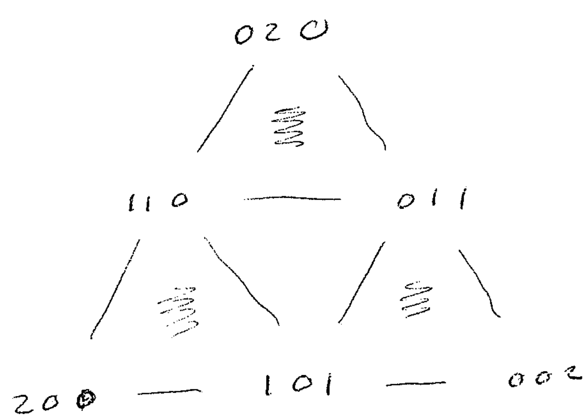
<u>Notation</u> Define

$$\Delta_d = \left\{ (r,s,t) \;\middle|\; r,s,t \in \mathbb{N}, \quad r+s+t=d \right\}$$

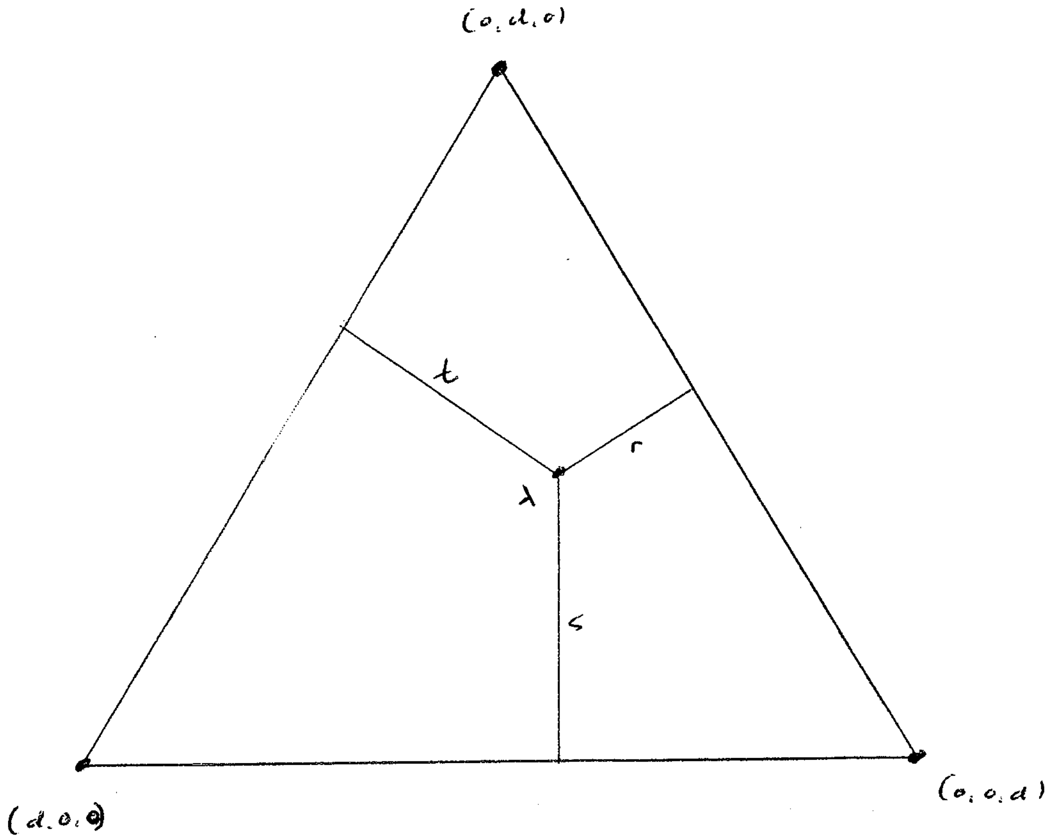
Element in Δ_d called a <u>location</u>

Ex $d=2$

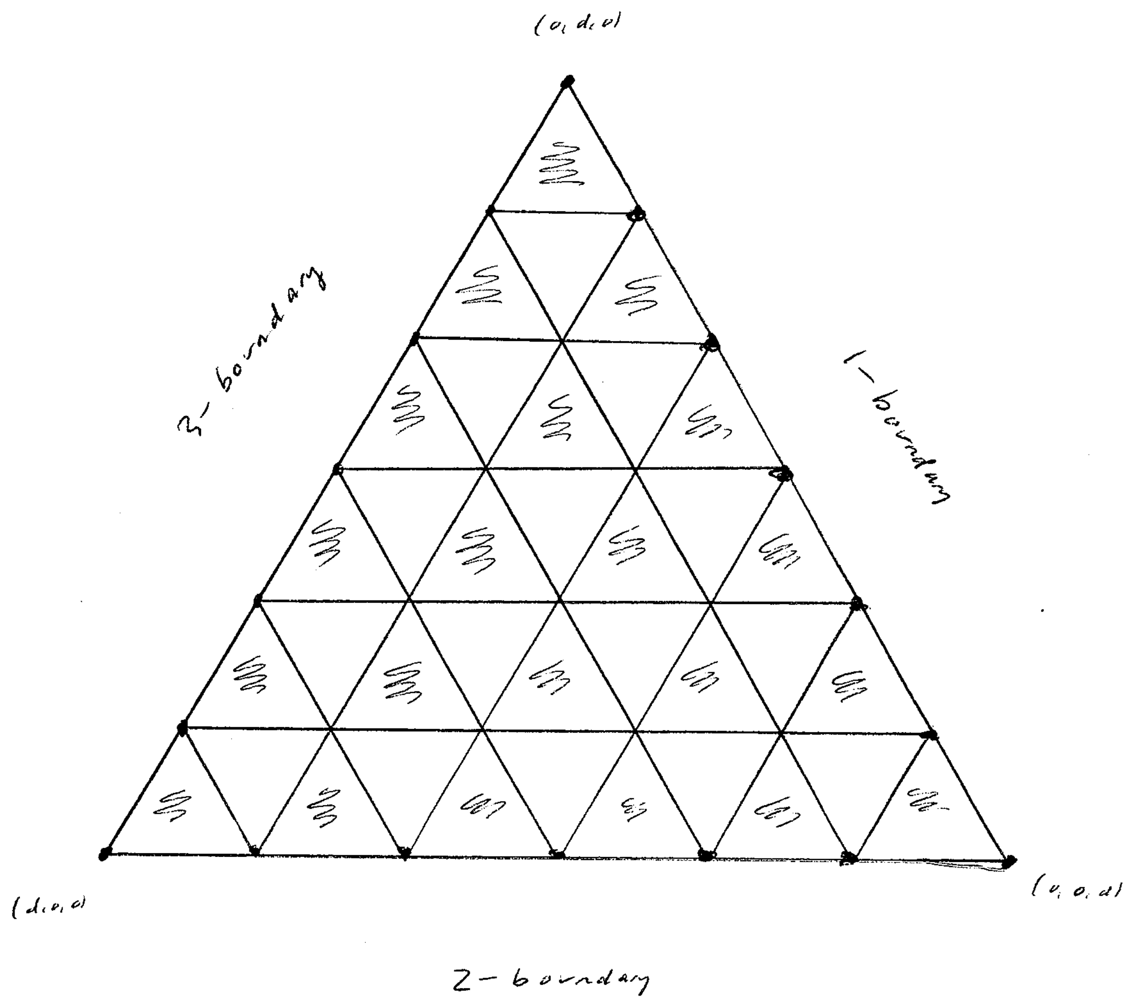
Δ_2 :



For Loc $\lambda = (r, s, t) \in \Delta_d$



View of Δ_d



We now put a polynomial at each location as follows:

Given commuting indets x, y, z (lower case)

Given scalars $\bar{x}, \bar{y}, \bar{z}$ s.t. $\bar{x}\bar{y}\bar{z} = q^{2-2d}$

For $\lambda = (r,s,t) \in \Delta_d$ define a polynomial

$$B_\lambda = (x\bar{x}, y; q^{-2})_r \, (y\bar{y}, z; q^{-2})_s \, (z\bar{z}, x; q^{-2})_t$$

where

$$(a, b; q)_n = (a-b)(a-qb) \cdots (a-q^{n-1}b)$$

For $d=2$ the B_λ are:

$$\begin{tikzcd}[column sep=small]
 & & (y\bar y - z)(y\bar y - q^{-2}z) \arrow[dl, dash] \arrow[dr, dash] & & \\
 & (x\bar x - y)(y\bar y - z) \arrow[rr, dash] \arrow[dl, dash] \arrow[dr, dash] & & (y\bar y - z)(z\bar z - x) \arrow[dl, dash] \arrow[dr, dash] & \\
(x\bar x - y)(x\bar x - q^{-2}y) \arrow[rr, dash] & & (z\bar z - x)(x\bar x - y) \arrow[rr, dash] & & (z\bar z - x)(z\bar z - q^{-2}x)
\end{tikzcd}$$

Features

- For each black 3-clique Δ in Δ_d, the corner polynomials are lin dependent

- For each line parallel to a boundary in Δ_d, the corresp polys are lin indep.

these features make the vectors $\{B_\lambda \mid \lambda \in \Delta_d\}$ a (concrete) Billiard array (Ter 2014)

For the above Billiard Array define

$$V = \mathrm{Span}(B_\lambda \mid \lambda \in \Delta_d)$$

Obs

$$\dim V = d+1$$

Pick $\eta \in \{1,2,3\}$. For the η-boundary of Δ_d the corresp B_λ form a basis for V "boundary basis"

We now turn V into a U_q-module such that the 3 boundary bases are eigenbases for the equit gens X, Y, Z.

Thm (Ter 2014) The above vector space V becomes a U_q-module such that the boundary locations λ satisfy:

Loc λ	B_λ description
$(0, d-i, i)$	eigvector for X with eigval q^{d-2i}
$(i, 0, d-i)$	eigvector for Y with eigval q^{d-2i}
$(d-i, i, 0)$	eigvector for Z with eigval q^{d-2i}

—— $\circ$ ——