Lec 44 Monday Dec 14 Hans Chaumont 12/14/15

Quantum Groups in Statistical Mechanics

Outline: - What is statistical mechanics? What kinds of problems are studied?
- Define the 6-vertex model + its partition function.
- Transfer matrices and R matrices.
- How quantum groups fit in.

What is statistical mechanics?

- studies systems of large number of particles (atoms or molecules), using statistical + probabilistic methods.
- Want to relate microscopic behavior to macroscopic properties.

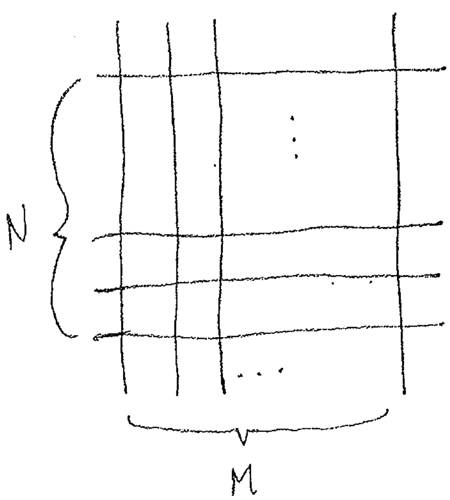
  microscopic mechanical laws $\overset{?}{\longleftrightarrow}$ temperature, pressure, density, free energy, etc.

- Examples of problems:
  - What happens w/ phase changes: eg water boiling or freezing.
  - Magnetization of an iron bar.

6-vertex model

- Ice-type model. First introduced by Pauling in 1935.
  Used to model ice, KDP crystals, ferroelectric or antiferroelectric crystals.

- 2 dimensional lattice.



Periodic boundary conditions.

For each edge we associate variable which can have values + or −. (spin).

A configuration of the system is an assignment of spins to each edge. 2^{NM} different configurations.

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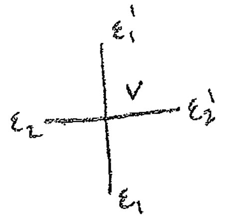
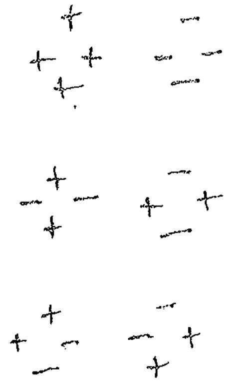
Define a probability measure on the set of all configurations.



$R_{\varepsilon_1,\varepsilon_2}^{\varepsilon_1'\varepsilon_2'}$ - Boltzmann weights.

At each vertex v we'll allow 6 possible configurations, each weighted a, b, or c.



note that interchanging spins preserves weights.

Weight of configuration C is product of Boltzmann weights over all vertices.

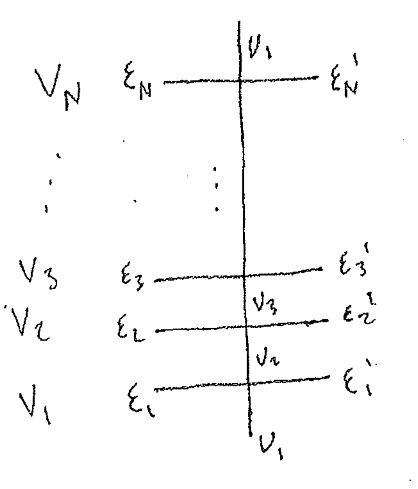
$$W(C)=\prod_v R_{\varepsilon_1\varepsilon_2}^{\varepsilon_1'\varepsilon_2'}(v).$$

If we let $Z_{M,N}=\sum_C W(C)$, (partition function), then define our probability measure by giving probability $\frac{W(C)}{Z_{M,N}}$ to configuration C.

Turns out that the partition function $Z_{M,N}$ is the main object of study.
(Somehow captures the important thermodynamic properties of the system).

Transfer Matrix + R matrices.

Suppose we look at just one column



V_0 - auxillary

For each choice of $\varepsilon_1,\ldots,\varepsilon_N$, $\varepsilon_1',\ldots,\varepsilon_N'$

Define $T_{\varepsilon_1,\ldots,\varepsilon_N}^{\varepsilon_1',\ldots,\varepsilon_N'} = \sum_{\nu_1,\ldots,\nu_N} R_{\nu_1\varepsilon_1}^{\nu_2\varepsilon_1'} R_{\nu_2\varepsilon_2}^{\nu_3\varepsilon_2'} \cdots R_{\nu_N\varepsilon_N}^{\nu_1\varepsilon_N'}$.

(can think of this as prob of going from $\varepsilon_1,\ldots,\varepsilon_N$ to $\varepsilon_1',\ldots,\varepsilon_N'$.)

If we associate to each row vector space $V_i=\mathbb{C}v_+\oplus\mathbb{C}v_-$, then this defines an operator on $V^{\otimes N}$, T.

This is the column transfer matrix.

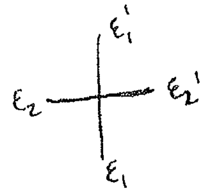
There are M columns, let $\mathcal{E}_i = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N)$ be the configuration of edges between columns i and $i+1$. Then

$$Z_{M,N} = \sum_{\mathcal{E}_1}{}' \sum_{\mathcal{E}_2}{}' \cdots \sum_{\mathcal{E}_M}{}' T_{\mathcal{E}_1}^{\mathcal{E}_2} T_{\mathcal{E}_2}^{\mathcal{E}_3} \cdots T_{\mathcal{E}_N}^{\mathcal{E}_1}$$

$$= \mathrm{tr}(T^M).$$

We want to understand spectrum of T.

Operator T counts contribution from vertices on a column. If we want to look at contribution from a single vertex, we define another operator $R \in \mathrm{End}(V \otimes V)$.



$$R(v_{\varepsilon_1'} \otimes v_{\varepsilon_2'}) = \sum_{\varepsilon_1, \varepsilon_2}{}' v_{\varepsilon_1} \otimes v_{\varepsilon_2} R_{\varepsilon_1 \varepsilon_2}^{\varepsilon_1' \varepsilon_2'}$$

In basis $\{v_+ \otimes v_+, v_+ \otimes v_-, v_- \otimes v_+, v_- \otimes v_-\}$

$$R = \begin{bmatrix} a & & & \\ & b & c & \\ & c & b & \\ & & & a \end{bmatrix}.$$

Can then define $\overset{i \neq j}{\vee}$ R_{ij} on $V_1 \otimes \cdots \otimes V_n$ which acts as R on V_i and V_j and the identity for all V_k $k \neq i, j$.

- Relation between R and T: Monodromy matrix. ⟶ to column.

If we identify another v.s. $V_0 = \mathbb{C} v_+ \oplus \mathbb{C} v_-$, then we can define the monodromy matrix

$$\mathcal{T} = R_{01} R_{02} \cdots R_{0N} \quad \text{acts on } V_0 \otimes V_1 \otimes \cdots \otimes V_N.$$

transfer matrix $T = \mathrm{tr}_{V_0}(\mathcal{T})$

<u>Relation to Quantum Groups</u> Understand T by Bethe Ansatz. (a way to diagonalize T).

Recall that the model depends on Boltzmann weights a, b, c.

When do transfer matrices of different parameters $T(a,b,c)$, $T(a',b',c')$ commute?

$\Delta = \frac{a^2+b^2-c^2}{ab}$ the anisotropy parameter. When $\Delta(a,b,c) = \Delta(a',b',c')$.

Furthermore, scaling a,b,c by ρ scales partition function. $Z_{M,N}(\rho a, \rho b, \rho c) = \rho^{MN} Z(a,b,c)$.

Fixing scale and fixing Δ, we can parametrize weights:

one way to do so is the following:

$a = \rho \sin(u+\eta)$
$b = \rho \sin(u)$
$c = \rho \sin(\eta)$.

ρ fixes scale, η fixes Δ.
u parameter.

Commutativity of $T(u)$ and $T(u')$ follows from the fact that $R(u)$ satisfies

Yang-Baxter equation: (see Zhaoxden Wang's notes)
(w/ parameter)
Given any nonzero complex parameters
ξ_1, ξ_2, ξ_3

$$R_{12}(\xi_1/\xi_2) R_{23}(\xi_3/\xi_1) R_{12}(\xi_2/\xi_1)$$
$$= R_{23}(\xi_2/\xi_1) R_{12}(\xi_3/\xi_1) R_{23}(\xi_3/\xi_2).$$

R matrices can be seen as intertwining maps for modules over $U_q(\hat{sl_2})$.
affine quantum enveloping algebra of sl_2.

Given a solution to Yang Baxter equation, we can get a solvable model, can build or solve other models this way. So studying representations of $U_q(\hat{sl_2})$ becomes important for this.
(e.g. eight vertex model).