Lec 43 Friday Dec 11

$U_q = U_q(sl_2)$ $\qquad H = SL_q(2)$

$A = k_q[x,y]$

Recall grading $A = \sum_{n \in \mathbb{N}} A_n$ ds

Recall A_n is H-comodule

$$\Delta_A \qquad A_n \longrightarrow H \otimes A_n$$

$$\underset{\substack{\parallel \\ u_i}}{y^i x^{n-i}} \longrightarrow \sum_j x^n_{ij} \otimes y^j x^{n-j}$$

So A_n^* becomes a H^*-module:

$$H^* \otimes A_n^* \xrightarrow[\tau_{H^*, A_n^*}]{} A_n^* \otimes H^* \cong (H \otimes A_n)^* \xrightarrow[\Delta_A^*]{} A_n^*$$

The duality $\langle , \rangle : U_q \times H \to k$ gives an alg morphism

$$\varphi : U_q \to H^*$$

Now A_n^* becomes a U_q-module:

$$U_q \otimes A_n^* \xrightarrow{\varphi \otimes id} H^* \otimes A_n^* \xrightarrow{\tau_{H^*, A_n^*}} A_n^* \otimes H^* \simeq (H \otimes A_n)^* \xrightarrow{\Delta_A^*} A_n^*$$

Describe the U_q-module A_n^*

Consider the basis $\{u^i\}_{i=0}^n$ for A_n^* dual to the basis $\{u_i\}_{i=0}^n$

Prop 9 With the above notation,

$$Ku^i = q^{n-2i} u^i \quad (0 \le i \le n)$$

$$eu^i = [n-i+1]\, u^{i-1} \quad (1 \le i \le n), \quad eu^0 = 0$$

$$fu^i = [i+1]\, u^{i+1} \quad (0 \le i \le n-1), \quad fu^n = 0$$

Moreover the U_q-module A_n^* is irred of type 1.

pf Recall the matrix

$$\left(\langle e, x^n_{ij} \rangle \right)_{0 \le i,j \le n} = \begin{pmatrix} 0 & [n] & & & 0 \\ & 0 & [n-1] & & \\ & & 0 & \ddots & \\ & 0 & & \ddots & [1] \\ & & & & 0 \end{pmatrix}$$

$$\left(\langle f, x^n_{ij} \rangle \right)_{0 \le i,j \le n} = \begin{pmatrix} 0 & & & & 0 \\ [1] & 0 & & & \\ & [2] & 0 & & \\ & 0 & \ddots & \ddots & \\ & & & [n] & 0 \end{pmatrix}$$

$$\left(\langle k, x^n_{ij} \rangle \right)_{0 \le i,j \le n} = \mathrm{diag}(q^n, q^{n-2}, \ldots, q^{-n})$$

For $\phi \in \{e, f, k\}$ show

$$\phi(u^j) = \sum_{i=0}^{n} \langle \phi, x^n_{ij} \rangle u^i \qquad (0 \le j \le n)$$

We have

$$\phi \otimes u^j \to \varphi(\phi) \otimes u^j \to u^j \otimes \varphi(\phi) \simeq F \to \Delta_A^* F \overset{?}{=} \sum_{i=0}^{n} \langle \phi, x^n_{ij} \rangle u^i$$

$$\Delta_A^* F \overset{?}{=} \sum_{i=0}^{n} \langle \phi, x^n_{ij} \rangle u^i$$

For $0 \le l \le n$,
Apply each side to u_l

$$\langle \Delta_A^* F, u_\ell \rangle \overset{?}{=} \left\langle \sum_{i=0}^{n} \langle \phi, x^n_{ij} \rangle u^i, u_\ell \right\rangle$$

$$\| \qquad\qquad\qquad\qquad \|$$

$$\langle F, \Delta_A u_\ell \rangle \qquad\qquad \langle \phi, x^n_{\ell j} \rangle$$

$$\|$$

$$\left\langle F, \sum_{r=0}^{n} x^n_{\ell r} \otimes u_r \right\rangle$$

$$\|$$

$$\underbrace{\sum_{r=0}^{n} \underbrace{\varphi(\phi)(x^n_{\ell r})}_{\| \atop \langle \phi, x^n_{\ell r} \rangle} \underbrace{u^j(u_r)}_{\| \atop \delta_{jr}}}$$

$$\|$$

$$\langle \phi, x^n_{\ell j} \rangle$$

OK.

$\square$

$U_q = U_q(sl_2)$ $\qquad$ q not root of 1

LEM 10 $\exists$ anti aut $T: U_q \to U_q$ that sends

$$k \to k$$
$$k^{-1} \to k^{-1}$$
$$e \to kF$$
$$f \to ek^{-1}$$

Moreover $T^2 = id$

pf Check rels:

$$ke = q^2 ek$$

$$T(e)T(k) \overset{?}{=} q^2 T(k)T(e)$$

$$T(e)T(k) = kf\,k = q^2 k^2 f, \qquad q^2 T(k)T(e) = q^2 k\, kf \quad ✓$$

$$kf = q^{-2} fk$$

$$T(f)T(k) \overset{?}{=} q^{-2} T(k)T(f)$$

$$T(f)T(k) = ek^{-1}k = e, \qquad q^{-2}T(k)T(f) = q^{-2} k\, ek^{-1} = e \quad ✓$$

$$ef - fe = \frac{k - k^{-1}}{q - q^{-1}}$$

$$T(f)T(e) - T(e)T(f) \overset{?}{=} \frac{T(k) - T(k^{-1})}{q - q^{-1}}$$

$$T(f)T(e) = ek^{-1}kf = ef, \qquad T(e)T(f) = kf\,ek^{-1} = fe, \qquad \frac{T(k)-T(k^{-1})}{q-q^{-1}} = \frac{k-k^{-1}}{q-q^{-1}} \quad \text{ok.}$$

check $T^2 = id$:

$$e \xrightarrow{T} kf \xrightarrow{T} T(f)T(k) = ek^{-1}k = e$$

etc.

LEM 11 The antiaut T is an iso of coalgebras

pf

check

$$\begin{array}{ccc} U_q & \xrightarrow{T} & U_q \\ \Delta\downarrow & & \downarrow\Delta \\ U_q\otimes U_q & \xrightarrow[T\otimes T]{} & U_q\otimes U_q \end{array}$$

$$\begin{array}{ccc} e & \longrightarrow & kf \\ & & \downarrow \\ \downarrow & & (k\otimes k)(f\otimes 1 + k^{-1}\otimes f) \\ & & \shortparallel \checkmark \\ e\otimes k + 1\otimes e & \longrightarrow & kf\otimes k + 1\otimes kf \end{array}$$

$$\begin{array}{ccc} f & \longrightarrow & ek^{-1} \\ & & \downarrow \\ \downarrow & & (e\otimes k + 1\otimes e)(k^{-1}\otimes k^{-1}) \\ & & \shortparallel \checkmark \\ f\otimes 1 + k^{-1}\otimes f & \longrightarrow & ek^{-1}\otimes 1 + k^{-1}\otimes ek^{-1} \end{array}$$

$$\begin{array}{ccc} k & \longrightarrow & k \\ \downarrow & & \downarrow \\ k\otimes k & \longrightarrow & k\otimes k \end{array} \quad \checkmark$$

Recall the f.d. irred U_q-modules

$V_{n,\varepsilon}$ $\quad n \in \mathbb{N} \quad \varepsilon \in \{1,-1\}$

$V = V_{n,\varepsilon}$ has basis $\{v_i\}_{i=0}^{n}$ s.t.

$$K v_i = \varepsilon q^{n-2i} v_i \qquad (0 \le i \le n)$$

$$e v_i = \varepsilon [n-i+1] v_{i-1} \qquad (1 \le i \le n), \quad e v_0 = 0$$

$$f v_i = [i+1] v_{i+1} \qquad (0 \le i \le n-1), \quad f v_n = 0$$

Define a bilinear form

$$\langle , \rangle \quad V \times V \to K$$

s.t.

$$\langle v_i, v_j \rangle = \delta_{ij} \begin{bmatrix} n \\ i \end{bmatrix} q^{i(i-n+1)} \qquad (0 \le i,j \le n)$$

So $\langle , \rangle$ is symmetric and nondegenerate.

LEM 12 With the above notation,

$$\langle xu, v\rangle = \langle u, T(x)v\rangle \qquad \forall x \in U_q \quad \forall u, v \in V$$

pf WLOG

$u = v_i, \quad v = v_j.$

WLOG $x \in \{e, f, k, k^{-1}\}$

Case $x = k$ $\quad \langle k v_i, v_j\rangle \overset{?}{=} \langle v_i, \bar{k} v_j\rangle$

$$\varepsilon q^{n-2i}\langle v_i, v_j\rangle \qquad \varepsilon q^{n-2j}\langle v_i, v_j\rangle$$

$\langle v_i, v_j\rangle = 0$ if $i \neq j$.

ok

Case $x = e$ $\quad \langle e v_i, v_j\rangle \overset{?}{=} \langle v_i, k f v_j\rangle$

$$\varepsilon[n-i+1]\langle v_{i-1}, v_j\rangle \qquad \varepsilon[j+1]\, q^{n-2j-2}\langle v_i, v_{j+1}\rangle$$

[Assume $i = j+1$, else both sides 0]

$$\varepsilon[n-i+1]\begin{bmatrix} n \\ i-1\end{bmatrix} q^{(i-1)(i-n)} \qquad \varepsilon[i]\, q^{n-2i}\begin{bmatrix} n \\ i\end{bmatrix} q^{i(i-n+1)}$$

$$\cancel{[n-i+1]}\,\frac{\cancel{[n]!}}{\cancel{[i-1]!}\,\cancel{[n-i+1]!}}\;q^{(i-1)(i-n)} \stackrel{?}{=} \cancel{[i]}\,\frac{\cancel{[n]!}}{\cancel{[i]!}\,\cancel{[n-i]!}}\;q^{n-2i}\,q^{i(i-n+1)}$$

$$i^2 - i - in + n \stackrel{?}{=} n - 2i + i^2 - in + i$$

Case $x = f$ is sim. $\square$

COR 13 Let V denote a f.d. U_q-module.

Then $\exists$ symmetric, nondeg bil form

$$\langle \, , \, \rangle : V \times V \to K$$

s.t

$$\langle xu, v \rangle = \langle u, T(x) v \rangle \qquad \forall x \in U_q \quad \forall u, v \in V$$

pf. Express V as a direct sum of irred U_q-modules. Each summand supports a bilinear form of desired type. Result follows. $\square$

The adjoint action

Given Lie algebra $\mathcal{L}$

univ env alg $\mathcal{U} = \mathcal{U}(\mathcal{L})$

LEM 14 $\mathcal{U}$ is a $\mathcal{U}$-module algebra such that each $u \in \mathcal{L}$ acts on $\mathcal{U}$ as

$$\begin{aligned} \mathcal{U} &\longrightarrow \mathcal{U} \\ a &\longrightarrow ua - au \end{aligned}$$

"adjoint action"

pf Recall requirements:

- $\mathcal{U}$ is $\mathcal{U}$-module
- $u(ab) = \sum_{(u)} u'(a)\, u''(b)$ $\quad \forall u \in \mathcal{U}$, $\forall a, b \in \mathcal{U}$
- $u(1_{\mathcal{U}}) = \varepsilon(u)\, 1_{\mathcal{U}}$ $\quad \forall u \in \mathcal{U}$

• Show $\mathcal{U}$ is $\mathcal{U}$-module:

Recall defining relations for $\mathcal{U}$

$$uv - vu = [u,v] \qquad \forall u,v \in \mathcal{L}$$

Actions on $\mathcal{U}$

$$\begin{array}{lccccl}
 & \mathcal{U} & \longrightarrow & \mathcal{U} & \longrightarrow & \mathcal{U} \\
uv: & a & \longrightarrow & va - av & \longrightarrow & u(va-av) - (va-av)u \\
 & & & & & = uva - uav - vau + avu \\
vu: & a & \longrightarrow & ua - au & \longrightarrow & vua - vau - uav + auv \\
[u,v]: & a & & \longrightarrow & & [u,v]a - a[u,v] \\
 & & & & & = (uv-vu)a - a(uv-vu)
\end{array}$$

Actions of $uv - vu$ on $\mathcal{U}$ and $[u,v]$ on $\mathcal{U}$ agree ✓

- Show

$$u(ab) = \sum_{(u)} u'(a)\, u''(b) \qquad \forall u \in U \quad \forall a, b \in U$$

WLOG $u \in \mathcal{L}$

$$\Delta(u) = u \otimes 1 + 1 \otimes u$$

Require

$$u(ab) = u(a)\, b + a\, u(b)$$

$u(ab) = uab - abu$, $u(a) = ua - au$ so $u(a)\,b = (ua - au)b$, $u(b) = ub - bu$ so $a\,u(b) = a(ub - bu)$

ok

- Show

$$u(1_U) = \varepsilon(u)\, 1_U \qquad \forall u \in U$$

WLOG $u \in \mathcal{L}$

$$u(1_U) \overset{?}{=} \varepsilon(u)\, 1_U$$

$\varepsilon(u) = 0$

$u(1_U) = u 1_U - 1_U u = 0$

ok.

$\square$

We now generalize the adjoint action to any Hopf algebras

Prop 15 Given Hopf algebra H

Then H becomes an H-module algebra with action

$$H \times H \longrightarrow H$$
$$u \quad a \longrightarrow \sum_{(u)} u' a\, S(u'')$$

pf • Show H is H-module

$\forall\, u, v \in H \quad \forall\, a \in H$ show

$$(uv)(a) \overset{?}{=} u(v(a))$$

$$u(v(a)) = u\left(\sum_{(v)} v' a\, S(v'')\right)$$
$$= \sum_{(u)} \sum_{(v)} u'v' a\, S(v'')\, S(u'')$$

$$(uv)(a) = \sum_{(uv)} (uv)' a\, S((uv)'')$$

$$\left[\Delta(uv) = \Delta(u)\Delta(v)\right]$$

$$= \sum_{(u)} \sum_{(v)} u'v' a\, S(u''v'')$$

$$S(u''v'') = S(v'')\, S(u'') \quad \text{ok.}$$

$\forall a \in H$ show

$$1_H(a) \stackrel{?}{=} a$$

$$\begin{aligned}
&\| \\
&\sum_{(1)} 1' a\, S(1'') \\
&\| \quad \left[\Delta(1) = 1 \otimes 1, \quad S(1) = 1\right] \\
&1 \cdot a \cdot 1 \\
&\| \\
&a
\end{aligned}$$

OK

H is H-module ✓

• Show

$$u(ab) \stackrel{?}{=} \sum_{(u)} u'(a)\, u''(b) \qquad \forall u \in H \quad \forall a, b \in H$$

$$u(ab) = \sum_{(u)} u' ab\, S(u'')$$

$$\sum_{(u)} u'(a)\, u''(b) = \sum_{(u)} \left(\sum_{(u')} (u')'\, a\, S((u')'') \right) \left(\sum_{(u'')} (u'')'\, b\, S((u'')'') \right)$$

$$= \sum_{(u)} u'\, a\, \underbrace{S(u'')\, u'''}\, b\, S(u'''')$$

$$\left[\quad \varepsilon(h)\, 1_H = \sum_{(h)} S(h')\, h'' \qquad h \in H \quad \right]$$

$$= \sum_{(u)} u'\, a\, \varepsilon(u'')\, b\, S(u''')$$

$$= \sum_{(u)} u'\, ab\, \underbrace{\varepsilon(u'')\, S(u''')}$$

$$\left[\begin{array}{ll} h = \sum_{(h)} \varepsilon(h')\, h'' & \forall h \in H \\ \text{so } S(h) = \sum_{(h)} \varepsilon(h')\, S(h'') & \end{array} \right]$$

$$= \sum_{(u)} u'\, ab\, S(u'')$$

ok.

- Show

$$u(1_H) \overset{?}{=} \varepsilon(u) 1_H \qquad \forall u \in H$$

$$u(1_H) = \sum_{(u)} u' 1_H S(u'')$$

$$= \sum_{(u)} u' S(u'')$$

$$= \varepsilon(u) 1_H \qquad \text{by def of } S$$

$\square$

Example $\qquad H = U_q(sl_2)$

For u among e, f, k, k^{-1} describe

$$u(a) = \sum_{(u)} u' a S(u'') \qquad \forall a \in$$

Recall
$$\Delta(e) = e \otimes k + 1 \otimes e$$
$$\Delta(f) = f \otimes 1 + k^{-1} \otimes f$$
$$\Delta(k) = k \otimes k$$

$$S(e) = -ek^{-1}, \qquad S(f) = -kf, \qquad S(k) = k^{-1}$$

$\forall a \in U_q$,
$$e(a) = (ea - ae)k^{-1},$$

$$f(a) = fa - k^{-1}akf,$$

$$k(a) = kak^{-1},$$

$$k^{-1}(a) = k^{-1}ak.$$

check rels directly

$$ke = q^2 ek.$$

$$ke(a) \stackrel{?}{=} q^2 ek(a)$$

$$\| \qquad\qquad\qquad \|$$

$$k\left((ea - ae)k^{-1}\right) \qquad q^2 e\left(kak^{-1}\right)$$

$$\| \qquad\qquad\qquad \|$$

$$k(ea-ae)k^{-1}k^{-1} \qquad q^2\left(e(kak^{-1}) - (kak^{-1})e\right)k^{-1}$$

$$\| \qquad\qquad\qquad \|$$

$$keak^{-2} - kaek^{-2} \qquad q^2 ekak^{-2} - q^2 ka\underbrace{k^{-1}ek^{-1}}_{= q^{-2}ek^{-1}}$$

$$\| \qquad\qquad\qquad \underbrace{\qquad\qquad}_{kaek^{-2}}$$

$$q^2 ekak^{-2}$$

ok

$kf = q^{-2} fk$

$$kf(a) \stackrel{?}{=} q^{-2} fk(a)$$

$$kf(a) = k(fa - k^{-1}akf) = k(fa - k^{-1}akf)k^{-1}$$

$$q^{-2} fk(a) = q^{-2} f(kak^{-1}) = q^{-2}(f\,kak^{-1} - k^{-1}kak^{-1}kf)$$

$$q^{-2} fkak^{-1} - q^{-2} af$$

$$kfak^{-1} - a\underbrace{kf}_{q^{-2}fk}k^{-1}$$

$$= q^{-2}fkak^{-1} \qquad = q^{-2}af$$

ok.

$$ef - fe = \frac{k - k^{-1}}{q - q^{-1}}$$

$$ef(a) - fe(a) \overset{?}{=} \frac{k - k^{-1}}{q - q^{-1}}(a)$$

$$ef(a) = e\left(fa - k^{-1}akf\right)$$

$$= \left(e\left(fa - k^{-1}akf\right) - \left(fa - k^{-1}akf\right)e\right)k^{-1}$$

$$= efak^{-1} - ek^{-1}akfk^{-1} - faek^{-1} + k^{-1}akfek^{-1}$$

$$fe(a) = f\left((ea - ae)k^{-1}\right)$$

$$= f(ea - ae)k^{-1} - k^{-1}(ea - ae)k^{-1}kf$$

$$= feak^{-1} - faek^{-1} - k^{-1}eaf + k^{-1}aef$$

$$\frac{k - k^{-1}}{q - q^{-1}}(a) = \frac{kak^{-1} - k^{-1}ak}{q - q^{-1}}$$

$\frac{k-\hat{k}^{-1}}{q-q^{-1}}\, ak^{-1}$

$k^{-1}a\, fe$

$=$

$\hat{e}fak^{-1}\ - ek^{-1}a\underbrace{kfk^{-1}}_{q^{-2}f} - fa\, ek^{-1} + k^{-1}ak\, fek^{-1}$

$-\ \hat{f}eak^{-1} + fa\, ek^{-1} + \underbrace{k^{-1}e}_{q^{-2}ek^{-1}}af - k^{-1}a\check{e}\hat{f}$

$-k^{-1}a\ \frac{k-k^{-1}}{q-q^{-1}}$

$$\stackrel{?}{=}\ \frac{kak^{-1}-k^{-1}a\check{k}}{q-q^{-1}}$$

ok