Lec 42 Wed Dec 9 12/9/15

We can now easily show the standard duality exists.

thm 6 For the bialgebras

$U_q = U_q(sl_2)$ $H = M_q(2)$ q not root of 1

$\exists$ unique duality $\langle , \rangle : U_q \; H \to k$

s.t

$\langle , \rangle$	a	b	c	d
e	0	1	0	0
f	0	0	1	0
k	q	0	0	q^{-1}
k^{-1}	q^{-1}	0	0	q

☆

pf Recall a basis for $H = M_q(2)$:

$x_{ij}^{n} \, det_q^{t}$ $n, t \in \mathbb{N}$ $0 \le i, j \le n$.

Define $e^\vee, f^\vee, k^\vee, (k^{-1})^\vee \in H^*$

as follows:

$$\left(e^{\vee}\left(x_{ij}^{n} \det{}_q^{t} \right) \right)_{0 \le i,j \le n} = \begin{pmatrix} 0 & [n] & & & \\ & 0 & [n-1] & & \\ & & 0 & \ddots & \\ & & & \ddots & [1] \\ & & & & 0 \end{pmatrix}$$

$$\left(f^{\vee}\left(x_{ij}^{n} \det{}_q^{t} \right) \right)_{0 \le i,j \le n} = \begin{pmatrix} 0 & & & & \\ [1] & 0 & & & \\ & [2] & 0 & & \\ & & \ddots & \ddots & \\ & & & [n] & 0 \end{pmatrix}$$

$$\left(k^{\vee}\left(x_{ij}^{n} \det{}_q^{t} \right) \right)_{0 \le i,j \le n} = \mathrm{diag}\left(q^{n}, q^{n-2}, q^{n-4}, \ldots, q^{-n} \right)$$

$$\left((k^{-1})^{\vee}\left(x_{ij}^{n} \det{}_q^{t} \right) \right)_{0 \le i,j \le n} = \mathrm{diag}\left(q^{-n}, q^{2-n}, \ldots, q^{n} \right)$$

(Recall $[n] = \dfrac{q^{n}-q^{-n}}{q-q^{-1}}$)

In the algebra H^*

$e^\vee, f^\vee, k^\vee, (k^{-1})^\vee$ satisfy the defining relations for U_q

So $\exists$ alg morphism $\varphi: U_q \to H^*$

that sends

$e \to e^\vee, \quad f \to f^\vee, \quad k \to k^\vee, \quad (k^{-1}) \to (k^{-1})^\vee$

Define

$$\langle , \rangle : \quad \begin{array}{l} U_q \times H \to k \\ u, \quad x \to \varphi(u)(x) \end{array}$$

So $\langle , \rangle$ is bilinear.

By constr $\langle , \rangle$ satisfies ☆

Since φ is an alg morphism,

$$\langle uv, x \rangle = \sum_{(x)} \langle u, x' \rangle \langle v, x'' \rangle \qquad u, v \in U_q,\ x \in H$$

and

$$\langle 1, x \rangle = \varepsilon(x) \qquad x \in H$$

One checks

$$\langle u, 1 \rangle = \varepsilon(u) \qquad \forall u \in U_q$$

For $u \in U_q$ show

$$\langle u, xy \rangle = \sum_{(u)} \langle u', x \rangle \langle u'', y \rangle \qquad \forall x, y \in H$$

"Condition $C(u)$"

By Constr

$C(e)$, $C(f)$, $C(k)$, $C(k^{-1})$ hold.

For $u, v \in U_q$ s.t. $C(u)$, $C(v)$ hold, show $C(uv)$ holds, pf is identical to case $q=1$, and omitted.

We have shown $\langle , \rangle$ is a standard duality.

One checks $\langle , \rangle$ is unique.

$\square$

let $I =$ 2-sided ideal of $M_q(2)$ gen by

$$\underset{H}{\overset{\shortparallel}{}}$$

$$ad - q^{-1}bc - 1$$

So $\quad SL_q(2) = M_q(2)/I$

LEM X $\quad$ For the duality $\langle , \rangle$ in thm 6,

$$\langle U_q, I \rangle = 0$$

pf $\quad$ <u>claim 1</u>

$$\langle 1, I \rangle = 0$$

<u>pf cl1</u> $\quad \forall\, x, y \in H$

$$\langle 1, x(\det_q - 1)y \rangle = \varepsilon\left(x(\det_q - 1)y \right)$$

$$= \varepsilon(x)\left(\underbrace{\varepsilon(\det_q)}_{1} - 1\right)\varepsilon(y)$$

$$= 0$$

claim 2

$$\langle e, I\rangle = 0 \qquad \langle f, I\rangle = 0$$

$$\langle k, I\rangle = 0 \qquad \langle k^{-1}, I\rangle = 0$$

pf cl 2 $\quad \forall\, x, y \in H$

$$\langle e,\ x(\det_q - 1)y\rangle =$$

$$\langle e, x\rangle \underbrace{\langle k, \det_q - 1\rangle}_{1 - 1 = 0} \langle k, y\rangle +$$

$$\varepsilon(x) \underbrace{\langle e, \det_q - 1\rangle}_{0 - 0 = 0} \langle k, y\rangle +$$

$$\varepsilon(x)\, \underbrace{\varepsilon(\det_q - 1)}_{1 - 1 = 0} \langle e, y\rangle$$

$$= 0$$

other cases sim.

claim 3 $\quad \Delta(I) \subseteq I \otimes H + H \otimes I$

pf cl 3 $\quad \forall \quad x, y \in H$

$$\Delta(x(\det_q - 1)y) = \Delta(x) \underbrace{\Delta(\det_q - 1)}_{\det_q \otimes \det_q - 1 \otimes 1 \;=\; (\det_q - 1)\otimes \det_q + 1 \otimes (\det_q - 1)} \Delta(y)$$

$$\subseteq I \otimes H + H \otimes I \checkmark$$

claim 4 Given $u, v \in U_q$ s.t

$$\langle u, I \rangle = 0 = \langle v, I \rangle$$

then $\langle uv, I \rangle = 0$

pf cl4 $\forall\ x \in I$,

$$\langle uv, x \rangle = \sum_{(x)} \underbrace{\langle u, x' \rangle \langle v, x'' \rangle}_{\substack{\| \\ 0}}$$

by cl3 $x' \in I$ or $x'' \in I$

$= 0$

Result follows. $\square$

Recall $I =$ 2-sided ideal of $M_q(2)$ gen by
$ad - q^{-1}bc - 1$

$SL_q(2) = M_q(2)/I$

By LEM X and the construction, the duality $\langle , \rangle$ in Thm 6 induces a duality

$$\langle , \rangle : U_q \times SL_q(2) \to k$$

Recall $SL_q(2)$ is Hopf alg. Its antipode S satisfies

$S(a) = d$ $\qquad$ $S(b) = -qb$

$S(c) = -q^{-1}c$ $\qquad$ $S(d) = a$

Recall U_q is also a Hopf alg. Its antipode S satisfies

$S(e) = -ek^{-1}$ $\qquad$ $S(f) = -kf$

$S(k) = k^{-1}$ $\qquad$ $S(k^{-1}) = k$

LEM 8 For $u \in U_q$ and $x \in SL_q(2)$

$$\langle S(u), x\rangle = \langle u, S(x)\rangle \qquad *$$

pf One checks $*$ holds for

$u \in \{e, f, k, k^{-1}\}$, $x \in \{a, b, c, d\}$

ex $u=e$, $x=a$

$$\langle S(e), a\rangle \overset{?}{=} \langle e, S(a)\rangle$$

$$\langle S(e), a\rangle = \underbrace{-\langle ek^{-1}, a\rangle}_{} = \sum_{(a)} \langle e, a'\rangle\langle k^{-1}, a''\rangle = \underbrace{\underbrace{\langle e, a\rangle}_{=0}\langle k^{-1}, a\rangle + \underbrace{\langle e, b\rangle}_{=1}\underbrace{\langle k^{-1}, c\rangle}_{=0}}_{=0} = 0$$

$$\langle e, S(a)\rangle = \langle e, d\rangle = 0$$

ok

Given $u \in \{e, f, k, k^{-1}\}$ and $x, y \in SL_q(2)$

s.t.

$$\langle S(u), x\rangle = \langle u, S(x)\rangle$$

$$\langle S(u), y\rangle = \langle u, S(y)\rangle$$

Show

$$\langle S(u), xy\rangle \overset{?}{=} \langle u, S(xy)\rangle$$

Case $u = e$

$$\langle S(e), xy \rangle \overset{?}{=} \langle e, S(xy) \rangle$$

$$\langle e, S(xy) \rangle = \langle e, S(y) S(x) \rangle$$

$$= \sum_{(e)} \langle e', S(y) \rangle \langle e'', S(x) \rangle$$

$\Delta(e) = e \otimes k + 1 \otimes e$

$$= \underbrace{\langle e, S(y) \rangle}_{= \langle S(e), y \rangle = \langle -ek^{-1}, y \rangle} \underbrace{\langle k, S(x) \rangle}_{= \langle S(k), x \rangle = \langle k^{-1}, x \rangle} + \underbrace{\langle 1, S(y) \rangle}_{= \varepsilon(S(y)) = \varepsilon(y)} \underbrace{\langle e, S(x) \rangle}_{= \langle S(e), x \rangle = \langle -ek^{-1}, x \rangle}$$

$$\langle S(e), xy \rangle = -\langle ek^{-1}, xy \rangle$$

$$\Delta(ek^{-1}) = \Delta(e) \Delta(k)^{-1}$$
$$= (e \otimes k + 1 \otimes e)(k^{-1} \otimes k^{-1})$$
$$= ek^{-1} \otimes 1 + k^{-1} \otimes ek^{-1}$$

$$= -\langle ek^{-1}, x \rangle \underbrace{\langle 1, y \rangle}_{= \varepsilon(y)} - \langle k^{-1}, x \rangle \langle ek^{-1}, y \rangle$$

ok

Case $u = f$

$$\langle S(f), xy \rangle \overset{?}{=} \langle f, S(xy) \rangle$$

$$\langle f, S(xy) \rangle = \langle f, S(y) S(x) \rangle$$

$$= \sum_{(f)} \langle f', S(y) \rangle \langle f'', S(x) \rangle$$

$$\Delta(f) = k^{-1} \otimes f + f \otimes 1$$

$$= \langle k^{-1}, S(y) \rangle \langle f, S(x) \rangle + \langle f, S(y) \rangle \langle 1, S(x) \rangle$$

$$\langle k^{-1}, S(y) \rangle = \langle S(k^{-1}), y \rangle = \langle k, y \rangle$$

$$\langle f, S(x) \rangle = \langle S(f), x \rangle = -\langle kf, x \rangle$$

$$\langle f, S(y) \rangle = \langle S(f), y \rangle = -\langle kf, y \rangle$$

$$\langle 1, S(x) \rangle = \varepsilon(S(x)) = \varepsilon(x)$$

$$\langle S(f), xy \rangle = \langle -kf, xy \rangle$$

$$\Delta(kf) = \Delta(k)\Delta(f)$$
$$= (k \otimes k)(k^{-1} \otimes f + f \otimes 1)$$
$$= 1 \otimes kf + kf \otimes k$$

$$= -\langle 1, x \rangle \langle kf, y \rangle - \langle kf, x \rangle \langle k, y \rangle$$

$$\langle 1, x \rangle = \varepsilon(x)$$

ok

Case $u=k$

$$\langle S(k), xy\rangle \overset{?}{=} \langle k, S(xy)\rangle$$

$$= \langle k^{-1}, xy\rangle \qquad \Delta(k^{-1}) = k^{-1}\otimes k^{-1}$$

$$= \langle k^{-1}, x\rangle\langle k^{-1}, y\rangle$$

$$\langle k, S(xy)\rangle = \langle k, S(y)S(x)\rangle = \langle k, S(y)\rangle\langle k, S(x)\rangle$$

$$\langle k, S(y)\rangle = \langle S(k), y\rangle = \langle k^{-1}, y\rangle$$

$$\langle k, S(x)\rangle = \langle S(k), x\rangle = \langle k^{-1}, x\rangle$$

ok

So far, $*$ hols for $u\in\{e, f, k, k^{-1}\}$ and $\forall x\in SL_q(2)$

Given $u, v \in U_q$ s.t.

$$\langle S(u), x\rangle = \langle u, S(x)\rangle \quad \forall x\in SL_q(2)$$

$$\langle S(v), x\rangle = \langle v, S(x)\rangle \quad ''$$

Show $\langle S(uv), x\rangle = \langle uv, S(x)\rangle \quad \forall x\in SL_q(2)$

$$\langle S(uv), x \rangle = \langle S(v) S(u), x \rangle$$

$$= \sum_{(x)} \langle S(v), x' \rangle \langle S(u), x'' \rangle$$

$$= \sum_{(x)} \langle v, S(x') \rangle \langle u, S(x'') \rangle$$

Also

$$\langle uv, S(x) \rangle = \sum_{(S(x))} \langle u, S(x)' \rangle \cdot \langle v, S(x)'' \rangle$$

$$\left[\Delta^{op} \circ S \otimes S = S \circ \Delta \right]$$

$$= \sum_{(x)} \langle u, S(x'') \rangle \langle v, S(x') \rangle$$

ok

Result follows $\square$