Lec 41 Monday Dec 7

$0 \neq q \in k$ q not a root of 1

Recall $\mathcal{U} = \mathcal{U}_q(sl_2)$

$$KK^{-1} = K^{-1}K = 1$$

$$KE = q^2 EK \qquad KF = q^{-2}FK$$

$$EF - FE = \frac{K - K^{-1}}{q - q^{-1}}$$

$$\Delta(K) = K \otimes K$$

$$\Delta(E) = E \otimes K + 1 \otimes E$$

$$\Delta(F) = F \otimes 1 + K^{-1} \otimes F$$

$$\varepsilon(K) = 1$$

$$\varepsilon(E) = 0$$

$$\varepsilon(F) = 0$$

$$S(K) = K^{-1} \qquad S(E) = -EK^{-1} \qquad S(F) = -KF$$

$$S^2(x) = KxK^{-1} \qquad \forall x \in \mathcal{U}$$

Recall algebra

$$A = k_q[x,y] \qquad yx = q\,xy$$

Next goal: turn A into a $\mathcal{U}$-module algebra.

Recall the requirements:

- A is $\mathcal{U}$-module
- $u(ab) = \sum_{(u)} u'(a)\, u''(b) \qquad u \in \mathcal{U},\ a,b \in A$
- $u(1_A) = \varepsilon(u)\, 1_A \qquad u \in \mathcal{U}$

<u>Motivation</u> Assume A is $\mathcal{U}$-module algebra

s.t.

$$e(x) = 0 \qquad e(y) = x$$

$$f(x) = y \qquad f(y) = 0$$

$$K(x) = qx \qquad K(y) = q^{-1}y$$

Find consequences.

Find action of e, f, k on

$$x^2, \quad xy, \quad y^2$$

$$e(x^2) = \underbrace{e(x)}_{=0} k(x) + x \underbrace{e(x)}_{=0}$$

$$= 0$$

$$e(xy) = \underbrace{e(x)}_{=0} k(y) + x \underbrace{e(y)}_{=x}$$

$$= x^2$$

$$e(y^2) = \underbrace{e(y)}_{=x} \underbrace{k(y)}_{=q^{-1}y} + \underbrace{y \underbrace{e(y)}_{=x}}_{qxy}$$

$$= \underbrace{(q+q^{-1})}_{=[2]} xy$$

$$f(x^2) = \underbrace{\underbrace{f(x)}_{=y} x}_{=qxy} + \underbrace{k^{-1}(x)}_{=q^{-1}x} \underbrace{f(x)}_{=y}$$

$$= (q+q^{-1}) xy$$

$$= [2] \, xy$$

$$f(xy) = \underset{\substack{\shortparallel \\ y}}{f(x)}\, y + k^{-1}(x) \underset{\substack{\shortparallel \\ 0}}{f(y)}$$

$$= y^2$$

$$f(y^2) = f(y)\, y + k^{-1}(y) f(y)$$

$$= 0$$

$K(x^2) = k(x)k(x) = q^2x^2$, $\quad k(xy) = xy$, $\quad k(y^2) = q^{-2}y^2$

Continuing along this line, we quickly guess

$$e(x^i y^j) = [j]\, x^{i+1} y^{j-1} \qquad i,j \in \mathbb{N}$$

$$f(x^i y^j) = [i]\, x^{i-1} y^{j+1}$$

$$K(x^i y^j) = q^{i-j} x^i y^j$$

where $[n] = \dfrac{q^n - q^{-n}}{q - q^{-1}}$

12/7/15

5

LEM 1 The algebra $A = k_q[x,y]$

is a U-module algebra such that

for $i,j \in \mathbb{N}$:

$$e(x^i y^j) = [j]\, x^{i+1} y^{j-1}$$

$$f(x^i y^j) = [i]\, x^{i-1} y^{j+1}$$

$$k(x^i y^j) = q^{i-j}\, x^i y^j$$

pf $\exists$ $e, f, k \in \mathrm{End}(A)$

that satisfy the above equations.

<u>Show A is U-module</u>

Recall grading

$$A = \sum_{n \in \mathbb{N}} A_n \qquad ds$$

A_n has basis

$$x^i y^j \qquad i+j = n$$

Fix $n \in \mathbb{N}$

Show A_n is U-module

For $0 \le i \le n$ define

$$v_i = \frac{x^{n-i}}{[n-i]!} \frac{y^i}{[i]!}$$

$\{v_i\}_{i=0}^n$ is basis for A_n.

Get

$$e v_i = v_{i-1} [n-i+1] \qquad (1 \le i \le n), \quad e v_0 = 0$$

$$f v_i = v_{i+1} [i+1] \qquad (0 \le i \le n-1), \quad f v_n = 0$$

$$k v_i = q^{n-2i} v_i \qquad (0 \le i \le n)$$

We recognize A_n is irred U-module of type 1.

We have shown A is U-module.

<u>Show</u> $u(ab) = \sum_{(u)} u'(a)\, u''(b)$ $\quad \forall u \in U$ $\quad \forall a,b \in A$

WLOG $u \in \{e, f, k\}$ and a, b are basis elements for A

$a = x^i y^j$ $\quad b = x^r y^s$ $\quad i,j,r,s \in \mathbb{N}$

$ab = x^{i+r} y^{j+s} q^{jr}$

$$e(ab) \stackrel{?}{=} e(a)\,k(b) + a\,e(b)$$

$\parallel \quad [j+s]\, x^{i+r+1} y^{j+s-1} q^{jr}$

$\parallel \quad [j]\, x^{i+1} y^{j-1} q^{r-s} x^r y^s + x^i y^j [s]\, x^{r+1} y^{s-1}$

$$x^{i+r+1} y^{j+s-1} \left([j]\, q^{r-s} q^{(j-1)r} + [s]\, q^{j(r+1)} \right)$$

$$[j+s] \stackrel{?}{=} [j]\, q^{-s} + [s]\, q^j$$

$$q^{j+s} - q^{-j-s} = (q^j - q^{-j})\, q^{-s} + (q^s - q^{-s})\, q^j$$

✓

$$f(ab) \overset{?}{=} f(a)b + k^{-1}(a)\, f(b)$$

$$\|$$

$$[i+r]\, x^{i+r-1} y^{j+s+1} q^{jr}$$

$$\|$$

$$[i]\, x^{i-1} y^{j+1} x^r y^s + q^{j-i} x^i y^j [r]\, x^{r-1} y^{s+1}$$

$$\|$$

$$x^{i+r-1} y^{j+s+1} \left([i]\, q^{(j+1)r} + [r]\, q^{j-i} q^{j(r-1)} \right)$$

$$[i+r] \overset{?}{=} [i]\, q^r + [r]\, q^{-i}$$

$$q^{i+r} - q^{-i-r} \overset{?}{=} (q^i - q^{-i})\, q^r + (q^r - q^{-r})\, q^{-i}$$

✓

$$k(ab) \overset{?}{=} k(a)\,k(b)$$

$$\|$$

$$q^{i+r-j-s}\, x^{i+r} y^{j+s} q^{jr} \qquad q^{i-j} x^i y^j \; q^{r-s} x^r y^s$$

$$\| $$

$$x^{i+r} y^{j+s}\, q^{i-j} q^{r-s} q^{jr}$$

ok

<u>Show</u> $\quad u(1_A) = \varepsilon(u)\, 1_A \qquad u \in U$

WLOG $\quad u \in \{e, f, k\}$

$$e(1) \overset{?}{=} \varepsilon(e)\, 1 \qquad \varepsilon(e) = 0$$

$$e(1) = e(x^0 y^0) = 0$$

$$f(1) = \varepsilon(f)\, 1 \qquad \varepsilon(f) = 0$$

$$f(1) = f(x^0 y^0) = 0$$

$$k(1) = \varepsilon(k)\, 1 \qquad \varepsilon(k) = 1$$

$$k(1) = k(x^0 y^0) = q^{0-0}(x^0 y^0) = 1 \quad \checkmark$$

$\square$

(Aside) Interpreting the $U_q(sl_2)$-action on $K_q[x,y]$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| A$

in terms of derivations.

Define algebra automorphisms σ_x, σ_y of A

such that

$$\sigma_x(x) = qx \qquad \sigma_x(y) = y$$

$$\sigma_y(x) = x \qquad \sigma_y(y) = qy$$

Define

$$D_x, D_y \in \mathrm{End}(A)$$

s.t. for $i, j \in \mathbb{N}$,

$$D_x(x^i y^j) = \frac{1-q^{-2i}}{1-q^{-2}} x^{i-1} y^j$$

$$D_y(x^i y^j) = q^{-i} \frac{1-q^{-2j}}{1-q^{-2}} x^i y^{j-1}$$

Note that

$$D_y(y^j x^i) = \frac{1-q^{-2j}}{1-q^{-2}} y^{j-1} x^i$$

LEM 2 With the above notation,

(i) D_x is a $(\sigma_x^{-2}\sigma_y)$ - derivation of A

(ii) D_y is a $(\sigma_x^{-1}\sigma_y^{-2})$ - derivation of A

pf (i) Show

$$D_x(ab) = \alpha(a) D_x(b) + D_x(a) b \qquad a,b \in A$$

$$\alpha = \sigma_x^{-2}\sigma_y$$

WLOG take

$$a = x^i y^j \qquad b = x^r y^s$$

$$ab = x^{i+r} y^{j+s} q^{jr}$$

$$D_x(ab) = q^{jr} \frac{1-q^{-2i-2r}}{1-q^{-2}} x^{i+r-1} y^{j+s}$$

$$\alpha(a) = q^{-2i+j} x^i y^j$$

$$D_x(b) = \frac{1-q^{-2r}}{1-q^{-2}} x^{r-1} y^s$$

$$\alpha(a) D_x(b) = q^{-2i+j} \frac{1-q^{-2r}}{1-q^{-2}} q^{jr-j} x^{i+r-1} y^{j+s}$$

$$D_x(a) = \frac{1-q^{-2i}}{1-q^{-2}} x^{i-1} y^j$$

$$b = x^r y^s$$

$$D_x(a) b = \frac{1-q^{-2i}}{1-q^{-2}} x^{i+r-1} y^{j+s} \quad q^{jr}$$

Require

$$q^{jr} \frac{1-q^{-2i-2r}}{1-q^{-2}} \overset{?}{=} q^{-2i+j} \frac{1-q^{-2r}}{1-q^{-2}} q^{jr} q^{-j}$$

$$+ \frac{1-q^{-2i}}{1-q^{-2}} q^{jr}$$

$$1-q^{-2i-2r} \overset{?}{=} q^{-2i}\left(1-q^{-2r}\right) + 1-q^{-2i}$$

ok

(ii) Show

$$D_y(ab) = \alpha(a)\, D_y(b) + D_y(a)\, b$$

$$\alpha = \sigma_x^{-1} \sigma_y^{-2}$$

Take $a = x^i y^j$ $\quad b = x^r y^s$

$$ab = x^{i+r} y^{j+s} q^{jr}$$

$$D_y(ab) = q^{jr} q^{-i-r} \frac{1-q^{-2j-2s}}{1-q^{-2}} x^{i+r} y^{j+s-1}$$

$$\alpha(a) = q^{-i-2j} x^i y^j$$

$$D_y(b) = q^{-r} \frac{1-q^{-2s}}{1-q^{-2}} x^r y^{s-1}$$

$$\alpha(a) D_y(b) = q^{-i-2j} q^{-r} \frac{1-q^{-2s}}{1-q^{-2}} x^{i+r} y^{j+s-1} q^{jr}$$

$$D_y(a) = q^{-i} \frac{1-q^{-2j}}{1-q^{-2}} x^i y^{j-1}$$

$$b = x^r y^s$$

$$D_y(a)\, b = q^{-i} \frac{1-q^{-2j}}{1-q^{-2}} x^{i+r} y^{j+s-1} q^{jr-r}$$

Require

$$q^{jr}\, q^{-i-r}\, \frac{1-q^{-2j-2s}}{1-q^{-2}} \stackrel{?}{=} q^{-i-2j}\, q^{-r}\, q^{jr}\, \frac{1-q^{-2s}}{1-q^{-2}}$$

$$+\; q^{-i}\, q^{jr}\, q^{-r}\, \frac{1-q^{-2j}}{1-q^{-2}}$$

$$1-q^{-2j-2s} \stackrel{?}{=} q^{-2j}\left(1-q^{-2s}\right) + 1-q^{-2j}$$

ok

$\square$

LEM 3 With the above notation,

$$D_x y = q y D_x; \qquad D_x x - q^{-2} x D_x = 1,$$

$$D_y x = q^{-1} x D_y, \qquad D_y y - q^{-2} y D_y = \sigma_x^{-1}.$$

pf Apply each side to $x^i y^j$ for $i, j \in \mathbb{N}$.

Prop 4 On the $U_q(sl_2)$-module $K_q[x,y]$

(i) $K = \sigma_x \sigma_y^{-1}$

(ii) $E = \sigma_x \sigma_y D_y x$

(iii) $F = y D_x$

pf (i) ✓

(ii), (iii) Apply each side to $x^i y^j$ for $i,j \in \mathbb{N}$

Note 5 Referring to the $U_q(sl_2)$-module $K_q[x,y]$,

Each element of $U_q(sl_2)$ commutes with $\sigma_x \sigma_y$.

Also

$$q x \sigma_x = \sigma_x x \qquad x \sigma_y = \sigma_y x$$

$$q y \sigma_y = \sigma_y y \qquad y \sigma_x = \sigma_x y$$

$$D_x \sigma_x = q \sigma_x D_x \qquad D_x \sigma_y = \sigma_y D_x$$

$$D_y \sigma_y = q \sigma_y D_y \qquad D_y \sigma_x = \sigma_x D_y$$

Next goal: Display a duality

$$\langle\, , \,\rangle : U_q \times M_q(2) \to k$$

$M_q(2) = H$

$U_q = U_q(sl_2)$ $\quad$ q not a root of 1

Recall

	U_q	$M_q(2)$
gens	e, f, k, k^{-1}	a, b, c, d
Δ	$\Delta(e) = e\otimes k + 1\otimes e$	$\Delta(a) = a\otimes a + b\otimes c$
	$\Delta(f) = f\otimes 1 + k^{-1}\otimes f$	$\Delta(b) = a\otimes b + b\otimes d$
	$\Delta(k) = k\otimes k$	$\Delta(c) = c\otimes a + d\otimes c$
	$\Delta(k^{-1}) = k^{-1}\otimes k^{-1}$	$\Delta(d) = c\otimes b + d\otimes d$
	$\Delta(1) = 1\otimes 1$	$\Delta(1) = 1\otimes 1$
ε	$\varepsilon(e) = 0$	$\varepsilon(a) = 1$ $\quad$ $\varepsilon(b) = 0$
	$\varepsilon(f) = 0$	$\varepsilon(c) = 0$ $\quad$ $\varepsilon(d) = 1$
	$\varepsilon(k) = 1$	
	$\varepsilon(k^{-1}) = 1$	
	$\varepsilon(1) = 1$	$\varepsilon(1) = 1$

For the time being assume a duality $\langle\ \rangle$ exists and consider the implications.

For $x, y \in H$

$$\langle e, xy\rangle = \langle e, x\rangle\langle k, y\rangle + \varepsilon(x)\langle e, y\rangle$$

$$\langle f, xy\rangle = \langle f, x\rangle\,\varepsilon(y) + \langle k^{-1}, x\rangle\langle f, y\rangle$$

$$\langle k, xy\rangle = \langle k, x\rangle\langle k, y\rangle$$
$$\langle k^{-1}, xy\rangle = \langle k^{-1}, x\rangle\langle k^{-1}, y\rangle$$

Iterating, we find that for $x_1, x_2, \ldots, x_n \in H$

$$\langle e, x_1 x_2 \cdots x_n\rangle = \sum_{i=1}^{n} \langle e, x_i\rangle\ \varepsilon(x_1)\cdots\varepsilon(x_{i-1})\langle k, x_{i+1}\rangle\cdots\langle k, x_n\rangle$$

$$\langle f, x_1 x_2 \cdots x_n\rangle = \sum_{i=1}^{n} \langle f, x_i\rangle\langle k^{-1}, x_1\rangle\cdots\langle k^{-1}, x_{i-1}\rangle\,\varepsilon(x_{i+1})\cdots\varepsilon(x_n)$$

$$\langle k, x_1 \cdots x_n\rangle = \langle k, x_1\rangle\cdots\langle k, x_n\rangle$$

$$\langle k^{-1}, x_1 \cdots x_n\rangle = \langle k^{-1}, x_1\rangle\cdots\langle k^{-1}, x_n\rangle$$

For $u, v \in U_q$,

$$\langle uv, a\rangle = \langle u, a\rangle\langle v, a\rangle + \langle u, b\rangle\langle v, c\rangle$$

$$\langle uv, b\rangle = \langle u, a\rangle\langle v, b\rangle + \langle u, b\rangle\langle v, d\rangle$$

$$\langle uv, c\rangle = \langle u, c\rangle\langle v, a\rangle + \langle u, d\rangle\langle v, c\rangle$$

$$\langle uv, d\rangle = \langle u, c\rangle\langle v, b\rangle + \langle u, d\rangle\langle v, d\rangle$$

In other words

$$\begin{pmatrix} \langle uv, a\rangle & \langle uv, b\rangle \\ \langle uv, c\rangle & \langle uv, d\rangle \end{pmatrix} = \begin{pmatrix} \langle u, a\rangle & \langle u, b\rangle \\ \langle u, c\rangle & \langle u, d\rangle \end{pmatrix}\begin{pmatrix} \langle v, a\rangle & \langle v, b\rangle \\ \langle v, c\rangle & \langle v, d\rangle \end{pmatrix}$$

So the map

$$U_q \longrightarrow Mat_2(K)$$

$$u \longmapsto \begin{pmatrix} \langle u, a\rangle & \langle u, b\rangle \\ \langle u, c\rangle & \langle u, d\rangle \end{pmatrix}$$

is an alg morphism.

Assume

$$\begin{pmatrix} \langle e, a \rangle & \langle e, b \rangle \\ \langle e, c \rangle & \langle e, d \rangle \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \langle f, a \rangle & \langle f, b \rangle \\ \langle f, c \rangle & \langle f, d \rangle \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \langle k, a \rangle & \langle k, b \rangle \\ \langle k, c \rangle & \langle k, d \rangle \end{pmatrix} = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}$$

$$\begin{pmatrix} \langle k^{-1}, a \rangle & \langle k^{-1}, b \rangle \\ \langle k^{-1}, c \rangle & \langle k^{-1}, d \rangle \end{pmatrix} = \begin{pmatrix} q^{-1} & 0 \\ 0 & q \end{pmatrix}$$

"Standard duality"

Then for $i,j,k,l \in \mathbb{N}$ and $\phi \in \{e, f, k, k^{-1}\}$,

$$\langle \phi, a^i b^j c^k d^l \rangle$$

is given in table below

j k	$\langle e, a^i b^j c^k d^l \rangle$	$\langle f, a^i b^j c^k d^l \rangle$	$\langle k, a^i b^j c^k d^l \rangle$	$\langle k^{-1}, a^i b^j c^k d^l \rangle$
0 0	0	0	q^{i-l}	q^{l-i}
1 0	q^{-l}	0	0	0
0 1	0	q^{-i}	0	0
$j+k \geq 2$	0	0	0	0

Recall the $M_q(2)$ - comodule algebra $k_q[x,y]$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \overset{\shortparallel}{A}$

$$\Delta_A : \quad A \longrightarrow M_q(2)\otimes A$$
$$x \longrightarrow a\otimes x + b\otimes y$$
$$y \longrightarrow c\otimes x + d\otimes y$$

For $n\in\mathbb{N}$ and $0\le i,j\le n$ define

$$X_{ij}^n \in M_q(2)$$

by

$$\Delta_A(y^i x^{n-i}) = \sum_{j=0}^{n} X_{ij}^n \otimes y^j x^{n-j} \qquad 0\le i\le n$$

Using $\quad yx = qxy$ and

$$\Delta_A(x^{n-i}y^i) = (\Delta_A(x))^{n-i}(\Delta_A(y))^i$$
$$= (a\otimes x + b\otimes y)^{n-i}(c\otimes x + d\otimes y)^i$$

we get

$$X^n_{ij} = \sum_z \begin{bmatrix} i \\ z \end{bmatrix} \begin{bmatrix} n-i \\ j-z \end{bmatrix} q^{i(n-i) - z(n-i-j+z)} a^{n-i-j+z} b^{j-z} c^{i-z} d^z$$

sum is over all $z \in \mathbb{N}$ s.t.

$$i+j-n \le z \le i, j$$

As a poly in a, b, c, d

X^n_{ij} is homog with total degree n.

Also the elements

$$X^n_{ij} \qquad 0 \le i, j \le n$$

are lin indep

n	$\left(X^n_{ij} \right)_{0 \le i,j \le n}$			
0	1			
1	a	b		
	c	d		
2	a^2	$[2]\,ab$	b^2	
	qac	$ad + qbc$	$q\,bd$	
	c^2	$[2]\,cd$	d^2	
3	a^3	$[3]\,a^2b$	$[3]\,ab^2$	b^3
	q^2a^2c	$a^2d + q^3[2]abc$	$q[2]abd + q^2b^2c$	q^2b^2d
	q^2ac^2	$q[2]acd + q^2bc^2$	$ad^2 + q^2[2]bcd$	q^2bd^2
	c^3	$[3]\,c^2d$	$[3]\,cd^2$	d^3

For the standard duality $\langle\, , \rangle$,

$$\left(\langle e, x^n_{ij}\rangle\right)_{0\le i,j\le n} = \begin{pmatrix} 0 & [n] & & & \\ & 0 & [n-1] & & 0 \\ & & \ddots & \ddots & \\ & 0 & & \ddots & [1] \\ & & & & 0 \end{pmatrix}$$

$$\left(\langle f, x^n_{ij}\rangle\right)_{0\le i,j\le n} = \begin{pmatrix} 0 & & & & 0 \\ [1] & 0 & & & \\ & [2] & \ddots & & \\ & & \ddots & \ddots & \\ 0 & & & & \\ & & & [n] & 0 \end{pmatrix}$$

$$\left(\langle k, x^n_{ij}\rangle\right)_{0\le i,j\le n} = \mathrm{diag}\left(q^n, q^{n-2}, \ldots \qquad q^{-n}\right)$$

$$\left(\langle k^{-1}, x^n_{ij}\rangle\right)_{0\le i,j\le n} = \mathrm{diag}\left(q^{-n}, q^{2-n}, \ldots \qquad q^{n}\right)$$

Recall

$$\det_q = ad - q^{-1}bc$$

$$\Delta(\det_q) = \det_q \otimes \det_q$$

$$\varepsilon(\det_q) = 1$$

The following results resemble the case $q=1$.

The proofs are similar, and omitted.

- The following is a basis for vector space $H = M_q(2)$:

$$x^n_{ij} \det_q{}^t \qquad n, t \in \mathbb{N} \quad 0 \le i, j \le n$$

- For $n \in \mathbb{N}$ and $0 \le i, j \le n$,

$$\Delta(x^n_{ij}) = \sum_{l=0}^{n} x^n_{il} \otimes x^n_{lj}$$

- For $n, t \in \mathbb{N}$ and $0 \le i, j \le n$.

$$\Delta(x^n_{ij} \det_q{}^t) = \sum_{l=0}^{n} (x^n_{il} \det_q^t) \otimes (x^n_{lj} \det_q^t) \qquad ☆$$

- For $n, t \in \mathbb{N}$ and $0 \le i, j \le n$,

$$\varepsilon\left(x_{ij}^{n} \det_q^{t} \right) = \delta_{ij}$$

- For the standard duality $\langle , \rangle$

$$\langle e, \det_q \rangle = 0 \qquad \langle k, \det_q \rangle = 1$$

$$\langle f, \det_q \rangle = 0 \qquad \langle k^{-1}, \det_q \rangle = 1$$

For $n, t \in \mathbb{N}$ and $0 \le i, j \le n$

$$\langle e, x_{ij}^{n} \det_q^{t} \rangle = \langle e, x_{ij}^{n} \rangle$$

$$\langle f, x_{ij}^{n} \det_q^{t} \rangle = \langle f, x_{ij}^{n} \rangle$$

$$\langle k, x_{ij}^{n} \det_q^{t} \rangle = \langle k, x_{ij}^{n} \rangle$$

$$\langle k^{-1}, x_{ij}^{n} \det_q^{t} \rangle = \langle k^{-1}, x_{ij}^{n} \rangle$$

- Given any duality $\langle , \rangle$,

  For $n, t \in \mathbb{N}$ and $0 \le i, j \le n$

  by ✩, for $u, v \in U_q$ we have

$$\langle uv, x^n_{ij} \det_q^t \rangle = \sum_{l=0}^{n} \langle u, x^n_{il} \det_q^t \rangle \langle v, x^n_{lj} \det_q^t \rangle$$

So the matrix

$$\Big(\langle uv, x^n_{ij} \det_q^t \rangle \Big)_{0 \le i,j \le n} = \Big(\langle u, x^n_{ij} \det_q^t \rangle \Big)_{0 \le i,j \le n} \Big(\langle v, x^n_{ij} \det_q^t \rangle \Big)_{0 \le i,j \le n}$$

So the map

$$U_q \longrightarrow \mathrm{Mat}_{n+1}(k)$$

$$u \longmapsto \Big(\langle u, x^n_{ij} \det_q^t \rangle \Big)_{0 \le i,j \le n}$$

is an alg morphism