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VII.3 Action of $U_q(sl(2))$ on the Quantum Plane.

This section is the quantum version of V.6. We start with a few generalities on skew-derivations of an algebra A.

Let A = algebra. For $a \in A$, denote the left and right multiplication by the element a by $a_l \cdot$ and a_r, repectively.

If σ is an automorphism of the algebra A, then it's easy to ~~ver~~ chekk

$$\sigma a_l = \sigma(a)_l \sigma$$

check: $\forall x \in A$. $(\sigma a_l)\cdot x = \sigma(a_l \cdot x)$

$$= \sigma(ax)$$
$$= \sigma(a)\sigma(x)$$
$$= (\sigma(a)_l \sigma)\cdot x$$

Similarly, we have $\sigma a_r = \sigma(a)_r \sigma$.

Given two automorphisms σ and τ of the algebra A, a linear endomorphism δ of A is called a (σ,τ)-derivation if $\delta(aa') = \sigma(a)\delta(a') + \delta(a)\tau(a')$ for all $a, a' \in A$

Note that this relation is equivalent to $\delta a_l = \sigma(a)_l \delta + \delta(a)_l \tau$.

Check: $\delta(aa') = \delta a_l \cdot a' = \sigma(a)_l \delta \cdot a' + \delta(a)_l \tau \cdot a'$

$$= \sigma(a)\delta(a') + \delta(a)\tau(a')$$

Similarly, it's equivalent to $\delta a_r = ~~\sigma(a)~~\ \tau(a)_r \delta + \delta(a)_r \sigma$

Remark: It's well-known that, if δ is a derivation of a commutative algebra, then $a_\ell \delta$ is a derivation too.

Check: $(a_\ell\delta)(aa') = a\delta(aa')$
$$= a\sigma(a)\delta(a') + a\delta(a)\tau(a')$$
$$= \sigma(a)\, a\delta(a') + a\delta(a)\tau(a')$$
$$= \sigma(a)(a_\ell\delta)(a') + (a_\ell\delta)(a)\tau(a')$$

This is no longer true in a non-commutative case.

Lemma VII 3.1.
Let δ be a (σ,τ)-derivation of A and a be an element of A. If there exist algebra automorphisms σ' and τ' of A such that $a_r\sigma' = a_\ell\sigma$ and $a_\ell\tau' = a_r\tau$, then the linear endomorphism $a_\ell\delta$ is a (σ',τ)-derivation and $a_r\delta$ is a (σ,τ')-derivation.

Pf: For any $b, b' \in A$,
$$(a_\ell\delta)(bb') = a\delta(bb')$$
$$= a\sigma(b)\delta(b') + a\delta(b)\tau(b')$$
$$a_\ell\sigma = a_r\sigma' \Rightarrow \quad = \sigma'(b)a\delta(b') + a\delta(b)\tau(b')$$
$$= \sigma'(b)(a_\ell\delta)(b') + (a_\ell\delta)(b)\tau(b')$$

Thus, $a_\ell\delta$ is a (σ',τ)-derivation.

Similarly, $(a_r\delta)(bb') = \delta(bb')a$
$$= \sigma(b)\delta(b')a + \delta(b)\tau(b')a$$
$$\cancel{= \sigma(b)(a_r\delta)(b')}$$
$$a_r\tau = a_\ell\tau' \Rightarrow \quad = \sigma(b)\delta(b')a + \delta(b)a\tau'(b')$$
$$= \sigma(b)(a_r\delta)(b') + (a_r\delta)(b)\tau'(b') \quad \checkmark$$

~~We now return to the quantum~~
~~case~~

Recall that the quantum plane $A = k_q[x,y] = k\{x,y\}/I_q$, where I_q is the two-sided ideal of the free algebra $k\{x,y\}$ generated by the element $yx - qxy$.

k is the ground field

Let us consider its algebra automorphisms σ_x and σ_y defined by

$$\sigma_x(x) = qx, \quad \sigma_x(y) = y, \quad \sigma_y(x) = x, \quad \sigma_y(y) = qy.$$

Note that when $q=1$, we have $\sigma_x = \sigma_y = id$.

We define q-analogues $\frac{\partial_q}{\partial x}$ and $\frac{\partial_q}{\partial y}$ of the classical partial derivatives by

$$\frac{\partial_q}{\partial x}(x^m y^n) = [m]\, x^{m-1} y^n \quad \text{and} \quad \frac{\partial_q}{\partial y}(x^m y^n) = [n]\, x^m y^{n-1}$$

where $[n] = \dfrac{q^n - q^{-n}}{q - q^{-1}}$.

Proposition VII 3.2.

(a) $x_\ell, x_r, y_\ell, y_r, \sigma_x, \sigma_y, \frac{\partial_q}{\partial x}, \frac{\partial_q}{\partial y}$ have the following relations

$$y_\ell x_\ell = q x_\ell y_\ell, \qquad x_r y_r = q y_r x_r$$

$$\sigma_x x_{\ell,r} = q x_{\ell,r} \sigma_x, \qquad \sigma_y y_{\ell,r} = q y_{\ell,r} \sigma_y$$

$$\frac{\partial_q}{\partial x}\sigma_x = q\sigma_x \frac{\partial_q}{\partial x}, \qquad \frac{\partial_q}{\partial y}\sigma_y = q\sigma_y \frac{\partial_q}{\partial y}$$

$$\frac{\partial_q}{\partial x} y_\ell = q y_\ell \frac{\partial_q}{\partial x}, \qquad \frac{\partial_q}{\partial y} x_r = q x_r \frac{\partial_q}{\partial y}$$

$$\frac{\partial_q}{\partial x} x_\ell = q^{-1} x_\ell \frac{\partial_q}{\partial x} + \sigma_x = q x_\ell \frac{\partial_q}{\partial x} + \sigma_x^{-1}$$

$$\frac{\partial_q}{\partial x} y_r = q^{-1} y_r \frac{\partial_q}{\partial y} + \sigma_y = q y_r \frac{\partial_q}{\partial y} + \sigma_y^{-1},$$

We

$$x_\ell \frac{\partial_q}{\partial x} = \frac{\sigma_x - \sigma_x^{-1}}{q - q^{-1}}, \qquad y_r \frac{\partial_q}{\partial y} = \frac{\sigma_y - \sigma_y^{-1}}{q - q^{-1}}.$$

(b) The endomorphism $\frac{\partial_q}{\partial x}$ is a $(\sigma_x^{-1}\sigma_y, \sigma_x)$-derivation and, similarly, $\frac{\partial_q}{\partial y}$ is a $(\sigma_y, \sigma_x\sigma_y^{-1})$-derivation.

Pf of (a):

By Prop. IV.1.1(b), we know that $y^j x^i = q^{ij} x^i y^j$, so we only need to verify the relations for $x^i y^j$.

① $$\begin{aligned}(y_\ell x_\ell)(x^i y^j) &= y x^{i+1} y^j \\ &= q^{i+1} x^{i+1} y^{j+1} \\ &= q \cdot q^i x^{i+1} y^{j+1} \\ &= q x (q^i x^i y) y^j \\ &= q x (y x^i) y^j \\ &= (q x y) x^i y^j = (q x_\ell y_\ell)(x^i y^j)\end{aligned}$$

② $$\begin{aligned}(\sigma_x x_\ell)(x^i y^j) &= \sigma_x (x^{i+1} y^j) \\ &= q^{i+1} x^{i+1} y^j \\ &= q x (q^i x^i y^j) \\ &= (q x_\ell \sigma_x)(x^i y^j)\end{aligned}$$

$\begin{cases} \sigma_x(x) = qx \\ \sigma_x(y) = y \end{cases}$

③ $$\begin{aligned}\left(\frac{\partial_q}{\partial x}\sigma_x\right)(x^i y^j) &= \frac{\partial_q}{\partial x}\left(\sigma_x (x^i y^j)\right) \\ &= \frac{\partial_q}{\partial x}(q^i x^i y^j) \\ &= q^i [i] x^{i-1} y^j \\ &= q \sigma_x ([i] x^{i-1} y^j) \\ &= q \sigma_x \left(\frac{\partial_q}{\partial x}(x^i y^j)\right)\end{aligned}$$

4 $\left(\frac{\partial_q}{\partial x} y_\ell\right)(x^i y^j) = \frac{\partial_q}{\partial x}(y x^i y^j)$

$= \frac{\partial_q}{\partial x}(q^i x^i y^{j+1})$

$= q^i [i] x^{i-1} y^{j+1}$

$= q[i](q^{i-1} x^{i-1} y) y^j$ [Prop IV 1.1(b)]

$= q[i](y x^{i-1}) y^j$

$= q y ([i] x^{i-1} y^j)$

$= \left(q y_\ell \frac{\partial_q}{\partial x}\right)(x^i y^j)$

5 $\left(\frac{\partial_q}{\partial x} x_\ell\right)(x^i y^j) = \frac{\partial_q}{\partial x}(x^{i+1} y^j)$

$= [i+1] x^i y^j$

$= (q^{-1}[i] + q^i) x^i y^j$

$= q^{-1}[i] x^i y^j + q^i x^i y^j$

$= q^{-1} x [i] x^{i-1} y^j + q^i x^i y^j$

$= \left(q^{-1} x_\ell \frac{\partial_q}{\partial x}\right)(x^i y^j) + \sigma_x(x^i y^j)$

6 $\left(x_\ell \frac{\partial_q}{\partial x}\right)(x^i y^j) = x [i] x^{i-1} y^j$

$= [i] x^i y^j$

$= \frac{q^i x^i y^j - q^{-i} x^i y^j}{q - q^{-1}}$

$= \frac{\sigma_x(x^i y^j) - \sigma_x^{-1}(x^i y^j)}{q - q^{-1}}$

~~(b)~~ Pf of (b). Recall that δ is a (σ, τ)-derivation if

$$\delta(aa') = \sigma(a)\delta(a') + \delta(a)\tau(a'),$$ and it's equivalent to $$\delta a_\ell = \sigma(a)_\ell \delta + \delta(a)_\ell \tau. \quad (*)$$

In addition, we note that if relation $(*)$ holds for a, a', then it holds for aa'.

Check: $\delta(aa')_\ell = \delta a_\ell a'_\ell$

$= \sigma(a)_\ell \delta a'_\ell + \delta(a)_\ell \tau a'_\ell$

$\delta a'_\ell = \sigma(a')_\ell \delta + \delta(a')_\ell \tau \Rightarrow \quad = \sigma(a)_\ell \sigma(a')_\ell \delta + \sigma(a)_\ell \delta(a')_\ell \tau + \delta(a)_\ell \tau a'_\ell$

$\tau a'_\ell = \tau(a')_\ell \tau \Rightarrow \quad = \sigma(a)_\ell \sigma(a')_\ell \delta + \sigma(a)_\ell \delta(a')_\ell \tau + \delta(a)_\ell \tau(a')_\ell \tau$

$\delta(aa') = \sigma(a)\delta(a') + \delta(a)\tau(a') \Rightarrow \quad = \sigma(aa')_\ell \delta + \delta(aa')_\ell \tau$

We only need to check relation $(*)$ for $a = x$ and $a = y$ when $\delta = \frac{\partial_q}{\partial x}$, $\sigma = \sigma_x^{-1}\sigma_y$, $\tau = \sigma_x$.

Check:

$$\sigma_x^{-1}\sigma_y(x)_\ell \frac{\partial_q}{\partial x} + \frac{\partial_q}{\partial x}(x)_\ell \sigma_x = q^{-1} x_\ell \frac{\partial_q}{\partial x} + \sigma_x$$

$$= \frac{\partial_q}{\partial x} x_\ell \quad \text{by } (5)$$

$$\sigma_x^{-1}\sigma_y(y)_\ell \frac{\partial_q}{\partial x} + \frac{\partial_q}{\partial x}(y)_\ell \sigma_x = q\, y_\ell \frac{\partial_q}{\partial x} = \frac{\partial_q}{\partial x} y_\ell \quad \text{by } (2).$$

Therefore relation $(*)$ holds for $a = x$ & $a = y$ when $\delta = \frac{\partial_q}{\partial x}$, $\sigma = \sigma_x^{-1}\sigma_y$, $\tau = \sigma_x$. Hence, $\frac{\partial_q}{\partial x}$ is a $(\sigma_x^{-1}\sigma_y, \sigma_x)$-derivation.

~~Similar~~.

Similarly, we can show that $\frac{\partial_q}{\partial y}$ is a derivation of $(\sigma_y, \sigma_x\sigma_y^{-1})$-derivation. We omit the details

The following theorem shows that $\frac{\partial_q}{\partial x}$ and $\frac{\partial_q}{\partial y}$ endow the quantum plane with the structure of a module-algebra over the Hopf algebra U_q.

Theorem VII 3.3. For any $P \in k_q[x,y]$, set

$$\left.\begin{array}{ll} EP = x\frac{\partial_q}{\partial y}P, & FP = \frac{\partial_q}{\partial x}P \cdot y \\ KP = (\sigma_x\sigma_y^{-1})(P), & K^{-1}P = (\sigma_y\sigma_x^{-1})(P) \end{array}\right\} (*)$$

(a) Formulas $(*)$ define the structure of a U_q-module-algebra on $k_q[x,y]$.

(b) The subspace $k_q[x,y]_n$ of homogeneous elements of degree n is a U_q-submodule of the quantum plane. It is generated by the highest weight vector x^n and is isomorphic to the simple module $V_{1,n}$.

~~Remark: Theorem VII 3.3 is the quantum version of Theorem V 6.4. It shows that the quantum plane contains all finite-dimensional simple U_q-modules.~~

~~Pf of (a): We first show that the formulas $(*)$ equip~~

Pf of (a):

We first show that the formulas (*) equip $k_q[x,y]$ with a U_q-module structure. So we have to check Relations VI 1.10 – 1.12.

For VI 1.10, $KK^{-1} = \sigma_x \sigma_y^{-1} \sigma_y \sigma_x^{-1} = 1.$

$K^{-1}K = \sigma_y \sigma_x^{-1} \sigma_x \sigma_y^{-1} = 1$

For VI 1.11, $KEK^{-1} = \sigma_x \sigma_y^{-1} x \frac{\partial_q}{\partial y} \sigma_y \sigma_x^{-1}$

$$= \sigma_x \sigma_y^{-1} x_\ell \frac{\partial_q}{\partial y} \sigma_y \sigma_x^{-1}$$

$$= \sigma_x \sigma_y^{-1} (x)_\ell \sigma_y^{-1} \frac{\partial_q}{\partial y} \sigma_y \sigma_x^{-1}$$

by def. of σ_y

$$= \sigma_x x_\ell \sigma_y^{-1} \frac{\partial_q}{\partial y} \sigma_y \sigma_x^{-1}$$

by ①

$$= q x_\ell \sigma_x \sigma_y^{-1} \frac{\partial_q}{\partial y} \sigma_y \sigma_x^{-1}$$

by ③

$$= q^2 x_\ell \sigma_x \frac{\partial_q}{\partial y} \sigma_x^{-1}$$

$$= q^2 x_\ell \frac{\partial_q}{\partial y} = q^2 E.$$

$$\cancel{KFK^{-1} = \sigma_x \sigma_y^{-1} \frac{\partial_q}{\partial x} y \sigma_y \sigma_x^{-1}} \quad \cancel{\text{by } ④, ②}$$

$$\cancel{= q^{-1} \sigma_x \sigma_y^{-1} \frac{\partial_q}{\partial x} \sigma_y y^{-1} \sigma_x^{-1}}$$

$$\cancel{= q^{-1} \sigma_x \frac{\partial_q}{\partial x}}$$

$$KFK^{-1} = \sigma_x \sigma_y^{-1} \frac{\partial_q}{\partial x} y \sigma_y \sigma_x^{-1}$$

$$= q^{-1} \sigma_x \sigma_y^{-1} \frac{\partial_q}{\partial x} \sigma_y y_\ell \sigma_x^{-1}$$

$$= q^{-1} \sigma_x \frac{\partial_q}{\partial x} y_\ell \sigma_x^{-1}$$

$$= q^{-1} \sigma_x \frac{\partial_q}{\partial x} \sigma_x^{-1} y_\ell$$

$$= q^{-2} \frac{\partial_q}{\partial x} y = q^{-2} F$$

~~For [E,F]~~

For 1.12.

$$[E,F] = x_\ell \frac{\partial_q}{\partial y} y_r \frac{\partial_q}{\partial x} - \underbrace{y_r \frac{\partial_q}{\partial y} x_\ell \frac{\partial_q}{\partial y}}_{\text{same}}$$

by ①

$$= q^{-1} x_\ell y_r \frac{\partial_q}{\partial y}\frac{\partial_q}{\partial x} + x_\ell \sigma_y \frac{\partial_q}{\partial x} - q^{-1} y_r x_\ell \frac{\partial_q}{\partial x}\frac{\partial_q}{\partial y} - y_r \sigma_x \frac{\partial_q}{\partial y}$$

$$= x_\ell \sigma_y \frac{\partial_q}{\partial x} - y_r \sigma_x \frac{\partial_q}{\partial y}$$

$$= \sigma_y x_\ell \frac{\partial_q}{\partial x} - \sigma_x y_r \frac{\partial_q}{\partial y}$$

$$= \frac{\sigma_y(\sigma_x - \sigma_x^{-1})}{q-q^{-1}} - \frac{\sigma_x(\sigma_y - \sigma_y^{-1})}{q-q^{-1}}$$

$$= \frac{\sigma_x\sigma_y^{-1} - \sigma_y\sigma_x^{-1}}{q-q^{-1}} = \frac{K-K^{-1}}{q-q^{-1}}$$

We now prove that the quantum plane is a U_q-algebra. By Lemma V 6.2, it's enough to check that for any $u \in U_q$, we have

① $u1 = \varepsilon(u)1$,

② $K(PQ) = K(P)K(Q)$

③ $E(PQ) = PE(Q) + E(P)K(Q)$

④ $F(PQ) = K^{-1}(P)F(Q) + F(P)Q$

for any pair (P, Q) of elements of $k_q[x,y]$.

$(\varepsilon(E)=\varepsilon(F)=0,\ \varepsilon(K)=\varepsilon(K^{-1})=1)$

① follows from VII 1 (1.3) and the definitions of σ_x, σ_y, $\frac{\partial_q}{\partial x}$, $\frac{\partial_q}{\partial y}$, E, F, ~~and~~ K, and K^{-1}

② follows from the definition of K and that σ_x, σ_y are algebra automorphisms

③ ~~follows from~~ By Lemma 3.1. and Prop 3.2 (b), we note that $x_\ell \frac{\partial_q}{\partial y}$ is a $(id, \sigma_x\sigma_y^{-1})$-derivation and $y_r\frac{\partial_q}{\partial x}$ is a $(\sigma_x^{-1}\sigma_y, id)$-derivation. These together with the definitions of E, F, K & K^{-1} & $x_\ell\sigma_y = x_\ell\, id$, $y_r\sigma_x = y_r\, id$ imply ③ & ④. This completes the pf of (a).

~~Pf of (b)~~

Pf of (b)

By the definitions of E, ~~F~~, K, it's easy to show that

$$Ex^n = x\frac{\partial_q}{\partial y}(x^n) = 0$$

$$Kx^n = \sigma_x\sigma_y^{-1}(x^n) = \sigma_x(x^n) = q^n x^n.$$

Thus, x^n is a highest weight vector of weight q^n

By the definition of F, we have

$$\frac{1}{[p]!}F^p(x^n) = \frac{1}{[p]!}\frac{\partial_q^p}{\partial x^p}(x^n) = \frac{[n]!}{[p]![n-p]!}x^{n-p}y^p.$$

A basis for $k_q[x,y]_n$ is generated by x^n under action of F.

Therefore, x^n generates $k_q[x,y]_n$.