Lec 4 Friday Sept 11

Recall:

A is commutative algebra

$$GL_2(A) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(A) \,\middle|\, ad - bc \in A^\times \right\}$$

Describe the mult group str on $GL_2(A)$

Commuting indets

$x_{11}, x_{12}, x_{21}, x_{22}, \quad t$ $\quad *$

$GL(2)$ = com alg with gens $*$ and rels

$$(x_{11}x_{22} - x_{12}x_{21})t = 1$$

LEM 19 $\exists$ algebra homs

$$\Delta : GL(2) \to GL(2)^{\otimes 2}$$

$$x_{ij} \to \sum_{l=1}^{2} x'_{il} \, x''_{lj} \qquad 1 \le i,j \le 2$$

$$t \to t' t''$$

$$S : GL(2) \to GL(2)$$

$$x_{11} \to x_{22} t$$

$$x_{12} \to -x_{12} t$$

$$x_{21} \to -x_{21} t$$

$$x_{22} \to x_{11} t$$

$$t \to x_{11} x_{22} - x_{12} x_{21}$$

$$\varepsilon : GL(2) \to k$$

$$x_{11} \to 1$$

$$x_{12} \to 0$$

$$x_{21} \to 0$$

$$x_{22} \to 1$$

$$t \to 1$$

pf

Δ : one checks

$$\left(\Delta(x_{11})\, \Delta(x_{22}) - \Delta(x_{12})\, \Delta(x_{21}) \right) \Delta(t) = 1$$

✓

S : one checks

$$\left(S(x_{11})\, S(x_{22}) - S(x_{12})\, S(x_{21}) \right) \cdot S(t) = 1$$ ✓

ε : one checks

$$\left(\varepsilon(x_{11})\, \varepsilon(x_{22}) - \varepsilon(x_{12})\, \varepsilon(x_{21}) \right) \varepsilon(t) = 1$$

$\square$

Prop 20 the following diagrams commute:

(i)

$$F \mapsto F \circ \Delta$$

$$\begin{tikzcd}
\mathrm{Hom}_{alg}(GL(2)^{\otimes 2}, A) \arrow[r] \arrow[d, "\mathrm{Eval}_2"'] & \mathrm{Hom}_{alg}(GL(2), A) \arrow[d, "\mathrm{Eval}_1"] \\
GL_2(A)^2 \arrow[r, "\mathrm{mult}"'] & GL_2(A)
\end{tikzcd}$$

(ii)

$$F \mapsto F \circ S$$

$$\begin{tikzcd}
\mathrm{Hom}_{alg}(GL(2), A) \arrow[r] \arrow[d, "\mathrm{Eval}_1"'] & \mathrm{Hom}_{alg}(GL(2), A) \arrow[d, "\mathrm{Eval}_1"] \\
GL_2(A) \arrow[r, "\mathrm{Inv}"'] & GL_2(A)
\end{tikzcd}$$

(iii)

$$F \mapsto F \circ \varepsilon$$

$$\begin{tikzcd}
\mathrm{Hom}_{alg}(k, A) \arrow[r] \arrow[d, "\mathrm{Eval}_0"'] & \mathrm{Hom}_{alg}(GL(2), A) \arrow[d, "\mathrm{Eval}_1"] \\
GL_2(A)^0 \arrow[r, "\mathrm{ident}"'] & GL_2(A)
\end{tikzcd}$$

pf chase F around the diagrams

Details

(i)

$$
\begin{tikzcd}
F \arrow[r, mapsto] \arrow[d, mapsto] & F \circ \Delta \arrow[d, mapsto] \\
 & \left(F\left(\sum_{l=1}^{2} x'_{il} x''_{lj} \right) \right)_{1 \le i,j \le 2} \arrow[d, phantom, "\wr\wr"] \\
\left(F(x'_{ij}) \right), \ \left(F(x''_{ij}) \right) \arrow[r, mapsto] & \left(F(x'_{ij}) \right)_{1 \le i,j \le 2} \left(F(x''_{ij}) \right)_{1 \le i,j \le 2}
\end{tikzcd}
$$

(ii)

$$
\begin{tikzcd}
F \arrow[r, mapsto] \arrow[d, mapsto] & F \circ S \arrow[d, mapsto] \\
 & \begin{pmatrix} F(x_{22}) F(t) & F(-x_{12}) F(t) \\ F(-x_{21}) F(t) & F(x_{11}) F(t) \end{pmatrix} \arrow[d, phantom, "\wr\wr"] \\
\left(F(x_{ij}) \right)_{1 \le i,j \le 2} \arrow[r, mapsto] & \left(\left(F(x_{ij}) \right)_{1 \le i,j \le 2} \right)^{-1}
\end{tikzcd}
$$

(iii)

$$
\begin{tikzcd}
\eta_A \arrow[r, mapsto] \arrow[d, mapsto] & \eta_A \circ \varepsilon \arrow[d, mapsto] \\
 & \begin{pmatrix} \eta_A(1) & \eta_A(0) \\ \eta_A(0) & \eta_A(1) \end{pmatrix} \arrow[d, phantom, "\wr\wr"] \\
\phi \arrow[r, mapsto] & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\end{tikzcd}
$$

Given commutative algebra A

Define
$$SL_2(A) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(A) \,\middle/\, ad - bc = 1 \right\}$$

Next goal: describe mult. group str on $SL_2(A)$

Mult, Inv, Ident

Commuting indets *

$x_{11},\ x_{12},\ x_{21},\ x_{22}$

Define *

$SL(2)$ = comm algebra with gens

and rels

$$x_{11} x_{22} - x_{12} x_{21} = 1$$

So

$$SL(2) = M(2)/I,$$

ideal I gen by

$$x_{11} x_{22} - x_{12} x_{21} - 1$$

Comm indets

$$x'_{ij}, \quad x''_{ij}$$ **

Def $SL(2)^{\otimes 2}$ = comm alg with gens **

and rels $$x'_{11}x'_{22} - x'_{12}x'_{21} = 1, \qquad x''_{11}x''_{22} - x''_{12}x''_{21} = 1$$

Bijections

$$\mathrm{Hom}_{alg}(k, A) \longleftrightarrow SL_2(A)^0$$

Eval0 $\quad \overset{\shortparallel}{\{3_A\}} \qquad \qquad \overset{\shortparallel}{\emptyset}$

$$\mathrm{Hom}_{alg}(SL(2), A) \longleftrightarrow SL_2(A)$$

Eval1 $\quad F \longleftrightarrow \left(F(x_{ij}) \right)_{1 \le i,j \le 2}$

$$\mathrm{Hom}_{alg}(SL(2)^{\otimes 2}, A) \longleftrightarrow SL_2(A)^2$$

Eval2 $\quad F \longleftrightarrow \left(F(x_{ij}') \right)_{1 \le i,j \le 2}, \left(F(x_{ij}'') \right)_{1 \le i,j \le 2}$

LEM 21 $\exists$ alg homs

$$\Delta : \quad SL(2) \longrightarrow SL(2)^{\otimes 2}$$

$$x_{ij} \longrightarrow \sum_{l} x'_{il}\, x''_{lj} \qquad 1 \le i,j \le 2$$

$$S : \quad SL(2) \longrightarrow SL(2)$$

$$\begin{array}{ccc} x_{11} & & x_{22} \\ x_{12} & & -x_{12} \\ x_{21} & & -x_{21} \\ x_{22} & & x_{11} \end{array}$$

$$\varepsilon : \quad SL(2) \longrightarrow k$$

$$\begin{array}{ccc} x_{11} & & 1 \\ x_{12} & & 0 \\ x_{21} & & 0 \\ x_{22} & & 1 \end{array}$$

pf one checks

$$\Delta(x_{11})\Delta(x_{22}) - \Delta(x_{12})\Delta(x_{21}) = 1$$

$$S(x_{11})S(x_{22}) - S(x_{12})S(x_{21}) = 1 \quad \checkmark$$

$$\varepsilon(x_{11})\varepsilon(x_{22}) - \varepsilon(x_{12})\varepsilon(x_{21}) = 1$$

$\square$

Prop 22 The following diagrams commute

(i)

$$\begin{tikzcd}
\mathrm{Hom}_{alg}(SL(2)^{\otimes 2}, A) \arrow[r, "F \,\mapsto\, F\circ\Delta"] \arrow[d, "\mathrm{Eval}_2"'] & \mathrm{Hom}_{alg}(SL(2), A) \arrow[d, "\mathrm{Eval}_1"] \\
SL_2(A)^2 \arrow[r, "\mathrm{mult}"'] & SL_2(A)
\end{tikzcd}$$

(ii)

$$\begin{tikzcd}
\mathrm{Hom}_{alg}(SL(2), A) \arrow[r, "F \,\mapsto\, F\circ S"] \arrow[d, "\mathrm{Eval}_1"'] & \mathrm{Hom}_{alg}(SL(2), A) \arrow[d, "\mathrm{Eval}_1"] \\
SL_2(A) \arrow[r, "\mathrm{inv}"'] & SL_2(A)
\end{tikzcd}$$

(iii)

$$\begin{tikzcd}
\mathrm{Hom}_{alg}(k, A) \arrow[r, "F \,\mapsto\, F\circ\varepsilon"] \arrow[d, "\mathrm{Eval}_0"'] & \mathrm{Hom}_{alg}(SL(2), A) \arrow[d, "\mathrm{Eval}_1"] \\
SL_2(A)^0 \arrow[r, "\mathrm{ident}"'] & SL_2(A)
\end{tikzcd}$$

pf Chase F around each diag $\square$

Gradings and filtrations

Given algebra A.

A grading of A is a sequence of subspaces

$\{A_i\}_{i\in\mathbb{N}}$ of A s.t

(i) $1 \in A_0$

(ii) $A = \sum_{i\in\mathbb{N}} A_i$ ds

(iii) $A_i A_j \subseteq A_{i+j}$ for $i,j \in \mathbb{N}$

A graded algebra is an algebra together with a grading.

Ex let X = set

Free alg $T = k\{X\}$

For $i \in \mathbb{N}$ define

T_i = subspace of T spanned by words over X that have length i

So T_0 has basis 1
T_1 has basis X

The sequence $\{T_i\}_{i \in \mathbb{N}}$ is a grading of T

Given an algebra A with grading $\{A_i\}_{i\in\mathbb{N}}$.

For $i\in\mathbb{N}$, the elements of A_i are called <u>i-homogeneous</u>

An element $a\in A$ is called <u>homogeneous</u> whenever $\exists\, i\in\mathbb{N}$

s.t. a is i-homogeneous.

Let I = 2-sided ideal of A

I called <u>homogeneous</u> whenever I is generated by

its homogeneous elements.

(ex)

I is homog iff $\quad I = \sum_{i\in\mathbb{N}} I\cap A_i$

Assume I is homog.

Given $\{a_i\}_{i\in\mathbb{N}}$ in A, fin many nonzero.

TFAE:

(ex)

(i) $a_i\in A_i$ for $i\in\mathbb{N}$ and $\sum_{i\in\mathbb{N}} a_i\in I$

(ii) $a_i\in A_i\cap I$ for $i\in\mathbb{N}$

Until further notice assume

I is homogeneous and $I\neq A$

Consider quotient algebra

$$\bar{A} = A/I$$

We now give $\bar{A}$ a grading.

The canonical map

$$\mathrm{can}: \begin{array}{ccc} A & \to & \bar{A} \\ a & \mapsto & a+I \ (=\bar{a}) \end{array}$$

is alg hom.

For $i \in \mathbb{N}$ let

$$\bar{A}_i = \text{image of } A_i \text{ under can} = \frac{A_i + I}{I}$$

then $\{\bar{A}_i\}_{i \in \mathbb{N}}$ is a grading of $\bar{A}$ (ex)

We now describe the grading $\{\bar{A}_i\}_{i\in\mathbb{N}}$ from another point of view.

For $i \in \mathbb{N}$ consider the restriction

$$\mathrm{can}\big/_{A_i} : A_i \longrightarrow \overline{A_i} \qquad *$$

$*$ is surjective.

Kernel of $*$ is $A_i \cap I$

$*$ induces vector space iso

$$\text{natural: } \quad \underbrace{A_i \big/ A_i \cap I}_{A_i^\vee} \longrightarrow \overline{A_i} \qquad **$$

Define vector space

$$A^\vee = \bigoplus_{i\in\mathbb{N}} A_i^\vee$$

$**$ induces vector space iso

$$\text{natural: } \quad A^\vee \longrightarrow \overline{A}$$

Via natural, pull back the algebra structure on $\bar{A}$ to turn $A^\vee$ into an algebra with grading $\{A_i^\vee\}_{i\in\mathbb{N}}$.

The mult on $A^\vee$ is

$$A_i^\vee \times A_j^\vee \longrightarrow A_{i+j}^\vee$$

$$a + A_i \cap I \qquad b + A_j \cap I \longrightarrow ab + A_{i+j} \cap I$$

For the algebra $A^\vee$,

nat : $A^\vee \to \bar{A}$

is an iso of graded algebras.

Given algebra A

A <u>filtration</u> of A is a sequence of subspaces

$\{A_i\}_{i\in\mathbb{N}}$ of A s.t.

$1 \in A_0$

$A_{i-1} \subseteq A_i$ for $i \in \mathbb{N}$ $\qquad (A_{-1} = 0)$

$A_i A_j \subseteq A_{i+j}$ $\forall\, i,j \in \mathbb{N}$

$\bigcup_{i\in\mathbb{N}} A_i = A$

Ex Suppose $\{A_i\}_{i\in\mathbb{N}}$ is a grading of A.

then $\{A_0 + A_1 + \cdots + A_i\}_{i\in\mathbb{N}}$ is a filtration of A.