NOTES: HOPF ALGEBRA STRUCTURE OF U_q AND SEMISIMPLICITY OF U_q-MODULES

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1. Hopf algebra structure of U_q

Throughout these notes, assume $k = \mathbb{C}$ and that q is not a root of unity.

Recall U_q is the algebra generated by symbols E, F, K, K^{-1} subject to the constraints

$$\begin{array}{ll} KE = q^2 EK & KK^{-1} = K^{-1}K = 1 \\ KF = q^{-2} FK & [E,F] = \frac{K-K^{-1}}{q-q^{-1}}, \end{array}$$

where $[E,F] = EF - FE$ denotes the commutator.

Theorem 1. *U_q has a Hopf algebra structure with*

$$\begin{array}{llllll} \Delta: & E \mapsto 1\otimes E + E \otimes K & \varepsilon: & E, F \mapsto 0 & S: & E \mapsto -EK^{-1} \\ & F \mapsto K^{-1}\otimes F + F\otimes 1 & & K, K^{-1} \mapsto 1 & & F \mapsto -KF \\ & K \mapsto K\otimes K & & & & K \mapsto K^{-1} \\ & K^{-1} \mapsto K^{-1}\otimes K^{-1} & & & & K^{-1} \mapsto K. \end{array}$$

Proof. Claim 1: Δ and ε are algebra morphisms.

Proof of claim 1: We check that Δ and ε preserve the defining relations for U_q. For instance,

$$\begin{array}{rcl} \Delta(K)\Delta(E) & = & (K\otimes K)(1\otimes E + E\otimes K) \\ & = & K\otimes KE + KE\otimes K^2 \\ & = & K\otimes q^2EK + q^2EK\otimes K^2 \\ & = & q^2(K\otimes EK + EK\otimes K^2) \\ & = & q^2(1\otimes E + E\otimes K)(K\otimes K) \\ & = & q^2\Delta(E)\Delta(K). \end{array}$$

The other relations are similarly preserved by Δ, and similarly for ε.

Claim 2: Δ is coassociative and ε satisfies the counit axiom. Thus U_q is a bialgebra.

Proof of claim 2: now that we know Δ and ε are algebra morphisms, it suffices to show that these properties hold for the generators of U_q. For example, to show that Δ is coassociative when applied to E, we have

$$\begin{array}{rcl} (\Delta\otimes \mathrm{id})\circ\Delta(E) & = & (\Delta\otimes\mathrm{id})(1\otimes E + E\otimes K) \\ & = & \Delta(1)\otimes E + \Delta(E)\otimes K \\ & = & 1\otimes 1\otimes E + (1\otimes E + E\otimes K)\otimes K \\ & = & 1\otimes 1\otimes E + 1\otimes E\otimes K + E\otimes K\otimes K \\ (\mathrm{id}\otimes\Delta)\circ\Delta(E) & = & (\mathrm{id}\otimes\Delta)(1\otimes E + E\otimes K) \\ & = & 1\otimes\Delta(E) + E\otimes\Delta(K) \\ & = & 1\otimes(1\otimes E + E\otimes K) + E\otimes K\otimes K \\ & = & 1\otimes 1\otimes E + 1\otimes E\otimes K + E\otimes K\otimes K. \end{array}$$

A similar computation suffices for the rest of the generators, proving coassociativity. Similarly we can prove the counit axiom.

Claim 3: $S : U_q \to U_q^{\mathrm{op}}$ is an algebra morphism.

Proof of claim 3: We show that S preserves the defining relations for U_q. For example,

$$\begin{array}{rcl} S(KE) & = & S(E)S(K) \\ & = & (-EK^{-1})(K^{-1}) \\ & = & -K^{-1}KEK^{-2} \\ & = & -q^2K^{-1}EKK^{-2} \\ & = & -q^2K^{-1}EK^{-1} \\ & = & q^2S(K)S(E) \\ & = & S(q^2EK). \end{array}$$

A similar computation suffices for the remaining relations.

Claim 4: S is an antipode.

Proof of claim 4: by claim 3, it suffices to show

$$\sum_{(x)} x'S(x'') = \varepsilon(x)1 = \sum_{(x)} S(x')x''$$

for all x in a generating set for U_q, namely for $x = E, F, K, K^{-1}$. Here is the computation for $x = E$:

$$\begin{aligned}\sum_{(E)} E'S(E'') &= 1S(E) + ES(K) \\ &= -EK^{-1} + EK^{-1} \\ &= 0 = \varepsilon(E)1. \\ \sum_{(E)} S(E')E'' &= S(1)E + S(E)K \\ &= E + (-EK^{-1}K) \\ &= 0 = \varepsilon(E)1.\end{aligned}$$

□

Notice that U_q is neither commutative nor cocommutative. Additionally, $S^2 \neq \text{id}$. For instance, $S^2(E) = S(-EK^{-1}) = -S(K^{-1})S(E) = -K(-EK^{-1}) = KEK^{-1} = q^2E$. However, we do have that $S^2(u) = KuK^{-1}$ for all $u \in U_q$, and so in that sense S^2 is an inner automorphism of U_q.

Our next goal is to show that all finite dimensional U_q modules are semisimple, i.e. a direct sum of simple modules. We recall that every simple U_q-module is isomorphic to exactly one module $V_{\varepsilon,n}$, with $\varepsilon \in \{1, -1\}$ and $n \in \mathbb{N}$, which has the following properties:

- $V_{\varepsilon,n}$ has dimension $n + 1$.
- $V_{\varepsilon,n}$ is generated as a U_q-module by some $v \in V_{\varepsilon,n}$ with $Kv = \varepsilon q^n v$. We call εq^n the *weight* of $V_{\varepsilon,n}$.
- $\exists\ C_q$ in the center of U_q that acts on $V_{\varepsilon,n}$ as multiplication by

$$\varepsilon\frac{q^{n+1} + q^{-(n+1)}}{(q - q^{-1})^2}.$$

Lemma 1. *Fix $\varepsilon \in \{\pm 1\}$. There exists C_ε in the center of U_q which acts on $V_{\varepsilon,0}$ as 0 and on $V_{\varepsilon',n}$ as multiplication by a nonzero scalar for any $\varepsilon' \in \{\pm 1\}$ and $n > 0$.*

Proof. Let

$$C_\varepsilon = C_q - \varepsilon\frac{q + q^{-1}}{(q - q^{-1})^2}.$$

Then C_ε is in the center of U_q and acts on $V_{\varepsilon,0}$ as 0. For $\varepsilon' \in \{\pm 1\}$ and $n > 0$, C_ε acts on on $V_{\varepsilon',n}$ as multiplication by

$$\varepsilon'\frac{q^{n+1} + q^{-(n+1)}}{(q - q^{-1})^2} - \varepsilon\frac{q + q^{-1}}{(q - q^{-1})^2},$$

so it suffices to show that $\varepsilon'(q^{n+1} + q^{-(n+1)}) - \varepsilon(q + q^{-1})$ is nonzero. Multiplying this expression by $\varepsilon' q^{n+1}$, we get $q^{2n+2} + 1 - \varepsilon\varepsilon'(q^{n+2} - q^n) = (q^{n+2} - \varepsilon\varepsilon')(q^n - \varepsilon\varepsilon')$, which is nonzero because q is not a root of unity. □

We now prove the main theorem.

Theorem 2. *Any finite-dimensional U_q-module is a direct sum of simple modules.*

Proof. Let V be a finite dimensional module and $V' \subseteq V$ a submodule. Then it suffices to show that there is a submodule $V'' \subseteq V$ such that $V = V' \oplus V''$. We proceed in two steps.

Step 1: Suppose that V' has codimension 1 in V. We will use induction on $\dim V'$. If $\dim V' = 0$, we can just take $V'' = V$.

To complete the base case, suppose $\dim V' = 1$. Then V' and V/V' are both modules of dimension 1, hence simple. So we have $V' \simeq V_{\varepsilon_1,0}$ and $V/V' \simeq V_{\varepsilon_2,0}$ for some $\varepsilon_1, \varepsilon_2 \in \{\pm 1\}$. We have two cases:

- Case 1: $\varepsilon_1 \neq \varepsilon_2$. Then the action of K on V has two distinct eigenvalues ε_1 and ε_2, and so is diagonalizable. So V has a basis $\{v_1, v_2\}$ with $Kv_i = \varepsilon_i v_i$, and $V' = kv_1$. Note that $K(Ev_i) = q^2EKv_i = q^2\varepsilon_i(Ev_i)$, so Ev_i is an eigenvector for the action of K with eigenvalue $q^2\varepsilon v_i \notin \{\varepsilon_1, \varepsilon_2\}$. Thus $Ev_i = 0$. Similarly $Fv_i = 0$. Thus kv_2 is a submodule of V, and so, letting $V'' = kv_2$, we have $V = V' \oplus V''$.
- Case 2: $\varepsilon_1 = \varepsilon_2 = \varepsilon$. Then there exists a basis $\{v_1, v_2\}$ for V, with $V' = kv_1$, such that $Kv_1 = \varepsilon v_1$ and $Kv_2 \in \varepsilon v_2 + V'$. So we can write $Kv_2 = \varepsilon v_2 + \alpha v_1$ for some $\alpha \in k$. Again, $K(Ev_1) = q^2\varepsilon(Ev_1)$, so $Ev_1 = 0$. We will show that it is also the case that $Ev_2 = 0$. To see this, write $Ev_2 = \lambda v_1 + \mu v_2$. Then on the one hand,

$$KEv_2 = K(\lambda v_1 + \mu v_2) = \lambda\varepsilon v_1 + \mu\varepsilon v_2 + \mu\alpha v_1,$$

and on the other hand

$$KEv_2 = q^2EKv_2 = q^2E(\varepsilon v_2 + \alpha v_1) = q^2\varepsilon\lambda v_1 + q^2\varepsilon\mu v_2.$$

Comparing coefficients, we see $\mu\varepsilon = q^2\varepsilon\mu \implies \mu\varepsilon(q^2 - 1) = 0 \implies \mu = 0$, and $\lambda\varepsilon + \mu\alpha = q^2\varepsilon\lambda \implies \lambda\varepsilon(q^2 - 1) = 0 \implies \lambda = 0$. Therefore, $Ev_2 = 0$. A similar computation shows that $Fv_1 = Fv_2 = 0$. Thus $[E, F]$ acts as 0 on V. But $[E, F] = (K - K^{-1})/(q - q^{-1})$, and so K and K^{-1} have the same action on V. Thus

$$v_2 = KK^{-1}v_2 = K^2v_2 = K(\varepsilon v_2 + \alpha v_1) = \varepsilon(\varepsilon v_2 + \alpha v_1) + \alpha(\varepsilon v_1) = v_2 + 2\alpha\varepsilon v_1.$$

Therefore $2\alpha\varepsilon = 0$, and so $\alpha = 0$. Thus we have that the action of K is diagonalizable, and complete the proof as in case 1.

We now move on to the inductive step. Assume that $\dim(V') = p > 1$, and that the assertion is true for all smaller dimensions. We again have two cases:

- Case 1: Suppose that V' is not simple. Then there exists a submodule $V_1 \subseteq V'$ with $0 < \dim V_1 < \dim V'$. Let $\pi : V \mapsto \overline{V} := V/V_1$ be the canonical map. Then $\overline{V'} := \pi(V')$ is a submodule of $\overline{V}$ of codimension 1, and $\dim \overline{V'} < p$. By induction, there is a submodule $\overline{V''} \subseteq \overline{V}$ such that $\overline{V} = \overline{V'} \oplus \overline{V''}$. Lifting this to V, we have $V = V' + \pi^{-1}(\overline{V''})$, although the sum is no longer direct. Since $\dim \overline{V''} = 1$, we have that $V_1 \subseteq \pi^{-1}(\overline{V''})$ is a submodule of codimension 1. By induction again, there exists a submodule $V'' \subseteq \pi^{-1}(\overline{V''})$ such that $\pi^{-1}(\overline{V''}) = V_1 \oplus V''$. Thus $V = V' + V_1 + V''$, and since $V_1 \subseteq V'$, we have $V = V' + V''$. Since $\dim V = \dim V' + \dim V''$, we have $V = V' \oplus V''$.
- Case 2: Suppose that V' is simple. We know V/V' has dimension 1, hence $V/V' \simeq V_{\varepsilon,0}$ for some $\varepsilon \in \{\pm 1\}$. Take the element $C_\varepsilon \in U_q$ from the lemma. Then C_ε acts on V/V' as 0, so $C_\varepsilon V \subseteq V'$. Since $\dim V' > 1$ and V' is simple, C_ε acts as multiplication by some $\alpha \neq 0$ on V'. So C_ε/α acts as the identity on V'. Thus $C_\varepsilon/\alpha : V \to V'$ is a projection and is U_q-linear since C_ε/α is in the center of U_q. Thus $V = V' \oplus \ker(C_\varepsilon/\alpha)$ is a direct sum of submodules.

Step 2: We now consider the general case, where $V' \subseteq V$ is a subdimension of any codimension. Recall that, since U_q is a Hopf algebra, we can give $\mathrm{Hom}(V, V')$ the structure of a U_q-module as follows: given $x \in U_q$ and $f \in \mathrm{Hom}(V, V')$, we get $xf \in \mathrm{Hom}(V, V')$ with

$$(xf)(v) = \sum_{(x)} x' f(S(x'')v).$$

Let $W = \{\phi \in \mathrm{Hom}(V, V') \mid \exists\, \alpha \in k \text{ s.t. } \phi(v) = \alpha v \ \forall v \in V'\}$ and $W' = \{\phi \in \mathrm{Hom}(V, V') \mid \phi|_{V'} = 0\}$. Then $W' \subseteq W$ is a subspace of codimension 1. We show that W is a submodule of $\mathrm{Hom}(V, V')$. For any $x \in U_q$ and $f \in W$, we wish to show $xf \in W$. We have some $\alpha \in k$ such that $f(v) = \alpha v$ for all $v \in V'$. Then, for all $v \in V'$,

$$(xf)(v) = \sum_{(x)} x' f(S(x'')v) = \sum_{(x)} x'(\alpha S(x'')v) = \alpha \left(\sum_{(x)} x' S(x'') \right) v = \alpha \varepsilon(x) v,$$

and so $xf \in W$. Thus $W \subseteq \mathrm{Hom}(V, V')$ is a submodule. Replacing α by 0 in the above argument, we also have that $W' \subseteq W$ is a submodule.

Now, having proved the codimension 1 case, we know that there exists a dimension 1 submodule $W'' \subseteq W$ such that $W = W' \oplus W''$. Let f be a generator for W''. Then f is an element of W but not W', so there is some $\alpha \neq 0$ such that $f(v) = \alpha v$ for all $v \in V'$. Thus $f/\alpha : V \to V'$ is a projection. Letting $V'' = \ker(f/\alpha) = \ker(f)$, we have $V = V' \oplus V''$ as vector spaces. Thus it suffices to show that V'' is a submodule of V.

Since W'' is a dimension 1 U_q module, $W'' \simeq V_{\pm 1,0}$, and so $Ef = Ff = 0$ and $Kf = K^{-1}f = \pm f$. Fix $v \in V''$. Then we have

$$\pm f(Kv) = Kf(Kv) = Kf(S(K)Kv) = Kf(K^{-1}Kv) = Kf(v) = 0,$$

and thus $Kv \in V''$. Similarly $K^{-1}v \in V''$. In addition, we have

$$\begin{aligned} 0 = Ef(Kv) &= f(S(E)Kv) + Ef(S(K)Kv) \\ &= f((-EK^{-1})Kv) + Ef(K^{-1}Kv) \\ &= -f(Ev) + Ef(v) \\ &= -f(Ev), \end{aligned}$$

and thus $Ev \in V''$. Similarly, $Fv \in V''$. Therefore V'' is a submodule of V, completing the proof. □