Nov 24.

§ 6.3. Representations of U_q.

Let q be a complex number which is not a root of unity. (So all the previous notations $[n]=\frac{q^n-q^{-n}}{q-q^{-1}}\neq 0$). We want to determine all simple U_q-modules.

First let V be a U_q-mod, $\lambda\neq 0$. Then we define $V^\lambda=\{v\in V: Kv=\lambda v\}$

We say λ is a weight of V if $V^\lambda\neq 0$.

Lemma 1. $EV^\lambda\subset V^{q^2\lambda}$, $FV^\lambda\subset V^{q^{-2}\lambda}$

proof: Easy: $KEv=q^2EKv=\lambda q^2 Ev$. $KFv=q^{-2}FKv=\lambda q^{-2}Fv$.

$\square$.

Def'n 1: V: a U_q-mod. $\lambda\neq 0$. If $v\in V$, s.t. $Ev=0$ and $Kv=\lambda v$. Then v is a highest weight vector of weight λ.

If V is generated by a highest weight vector of weight λ, it is called of highest weight λ.

Prop 2: Any non-zero f.d. U_q-mod V contains a highest weight vector. And E, F act on V nilpotently.

proof: Start with a eigenvector w, s.t. $Kw=\alpha w$. If $Ew=0$, ✓.

Otherwise, consider $\{E^n w\}$, these are eigenvectors for K with eigenvalue $q^{2n}\alpha$. Since V is f.d, we cannot have infinite many eigenvalues for K. Therefore $E^m w=0$ for some m. Then $E^{m-1}w$ is a highest weight vector.

To show E acts on V as a nilpotent, we need all its eigenvalues to be 0. Suppose it has non-zero eigenvalue λ with eigenvector v. Then $EK^n v = q^{-2n} K^n E v = q^{-2n} \lambda K^n v$

So $K^n v$ is eigenvector with eigenvalue $q^{-2n}\lambda$. This will produce infinitely many eigenvalues. Contradiction. $\square$.

<u>Lemma 4</u>: Let v be a highest weight vector of weight λ, $v_0 = v$. $v_p = \frac{1}{[p]!} F^p v$.

Then $K v_p = \frac{1}{[p]!} K F^p v = \frac{1}{[p]!} q^{-2p} F^p K v = \lambda \cdot q^{-2p} \frac{F^p v}{[p]!} = \lambda \cdot q^{-2p} v_p$

$$E v_p = [E, F^p] v = [p] F^{p-1} \frac{q^{-(p-1)} K - q^{p-1} K^{-1}}{q - q^{-1}} v = [p] \cdot \frac{q^{-(p-1)}\lambda - q^{p-1}\lambda^{-1}}{q-q^{-1}} F^{p-1} v.$$

$F v_{p-1} = [p] v_p$.

$$= \frac{q^{-(p-1)}\lambda - q^{p-1}\lambda^{-1}}{q - q^{-1}} v_{p-1}.$$

Now we are ready to determine all simple U_q-modules.

<u>Theorem 5</u>: (a) Let V be a f.d. U_q-mod generated by a highest weight vector v of weight λ. Then:

(i). $\lambda = \varepsilon q^n$, $\varepsilon = \pm 1$. $\dim V = n+1$.

(ii). $v_p = 0$ for $p > n$, and $\{v_0, v_1, \dots, v_n\}$ is a basis for V.

(iii). K is diagonalizable on V with $(n+1)$ distinct eigenvalues $\{\varepsilon q^n, \varepsilon q^{n-2}, \dots, \varepsilon q^{-n}\}$

(iv). Any highest weight vector in V is a scaler multiple of v of weight λ.

(v). V is simple.

(b). Any f.d. U_q-mod is generated by a highest weight vector.

proof: (a). Let $\{v_p\}_{p\geq 0}$ be a sequence of vectors defined as Lemma 4. Then since V is f.d, there must be some n, s.t. $v_{n+1} = 0$ $v_n \neq 0$. Then

$$0 = E v_{n+1} = \frac{q^{-n}\lambda - q^n \lambda^{-1}}{q - q^{-1}} v_n \Rightarrow \lambda^2 = q^{2n} \quad \lambda = \pm \varepsilon q^n.$$

$\{v_0, \dots, v_n\}$ form a basis for V. Therefore $\dim V = n+1$. This shows i, ii.

For $0 \le p \le n$, $K v_p = \varepsilon q^{n-2p} v_p$. Therefore K becomes diagonalizable with eigenvalues $\{\varepsilon q^n, \varepsilon q^{n-2}, \dots, \varepsilon q^{-n}\}$. This shows (iii)

If v' is another highest weight vector, then $v' = \lambda' v_i$ for some i. $Ev' = \lambda' E v_i$. Therefore $Ev' = 0$ iff $Ev_i = 0$ iff $i = 0$. Hence $v' = \lambda' v_0$. This shows (iv)

And finally, for any f.d. U_q-mod $V' \subseteq V$, V' has a highest weight vector ~~for~~ because of prop 2, hence V' contains v_0, which means $V' = V$. This shows (v)

(b). For any f.d. U_q-mod V, V has some highest weight vector v_0, then for $\{v_p\}_{p=1}^n$ defines a sub U_q-mod for V. If V is simple, then V coincides with this module generated by $\{v_p\}_{p=1}^n$. $\square$.

So by theorem above, there exists unique dim $n+1$ U_q-mod generated by highest weight vector of weight εq^n. We use $V_{\varepsilon,n}$ to denote this module. Then $\rho_{\varepsilon,n} : U_q \to \mathrm{End}(V_{\varepsilon,n})$ gives representations for U_q such that:

$$\rho_{\varepsilon,n}(K) = \mathrm{diag}(\varepsilon q^n, \varepsilon q^{n-2}, \dots, \varepsilon q^{-n})$$

$$\rho_{\varepsilon,n}(E) = \varepsilon \cdot \begin{pmatrix} 0 & [n] & & & \\ & 0 & [n-1] & & \\ & & 0 & \ddots & \\ & & & \ddots & 1 \\ & & & & 0 \end{pmatrix}$$

This is because

$$E v_p = \frac{q^{-(p-1)} \cdot \varepsilon q^n - q^{p-1} \cdot \varepsilon q^{-n}}{q - q^{-1}} v_{p-1} = \varepsilon \cdot \frac{q^{n-p+1} - q^{-(n-p+1)}}{q - q^{-1}} v_{p-1} = [1+n-p].$$

$$\rho_{\varepsilon,n}(F) = \varepsilon \begin{pmatrix} 0 & 0 & \cdots & & 0 \\ 1 & 0 & & & \\ & [2] & \ddots & & \\ & & \ddots & \ddots & \\ 0 & & & [n] & 0 \end{pmatrix}$$ because $F v_{p-1} = [p] v_p$

So now we have determined f.d. simple U_q-modules, they are modules generated by highest weight vectors of specied weights (εq^n). What if we choose λ to be arbitrary number?

<u>def'n + lemma 6</u>: For $\lambda \neq 0$, we let $V(\lambda)$ be a U_q-module generated by basis of vectors $\{v_i\}_{i\in\mathbb{N}}$, that satisfies:

$$K v_p = \lambda q^{-2p} v_p, \qquad E v_{p+1} = \frac{q^{-p}\lambda - q^{p}\lambda^{-1}}{q - q^{-1}} v_p.$$

$$F v_p = [p+1] v_{p+1}. \qquad K^{-1} v_p = \lambda^{-1} q^{2p} v_p$$

proof:

$$KK^{-1} v_p = v_p, \qquad K^{-1}K v_p = v_p.$$

$$KEK^{-1} v_p = q^2 E v_p. \qquad KFK^{-1} v_p = q^{-2} F v_p$$

$$[E,F] v_p = \left([p+1] \frac{q^{-p}\lambda - q^{p}\lambda^{-1}}{q - q^{-1}} - [p] \cdot \frac{q^{-(p-1)}\lambda - q^{p-1}\lambda^{-1}}{q - q^{-1}} \right) v_p$$

$$= \frac{K - K^{-1}}{q - q^{-1}} v_p. \qquad \square$$

Remark: $V(\lambda)$ is generated by v_0, which is a highest weight vector. Hence $V(\lambda)$ is also a highest weight U_q-mod, but with infinite dimension.

We call such $V(\lambda)$ Verma module of highest degree λ.

Prop 7: Any highest weight U_q-module V of highest weight λ is a quotient of $V(\lambda)$.

Proof: Let V be a highest weight module generated by v.

Then we define $f: V(\lambda) \to V$. $f(v_p) = \frac{1}{[p]!} F^p v$. This is a linear map.

Then $f(v_0) = v$ generates V. Hence f is surjective. $\square$

Remark: f.d. simple U_q-modules can be realized as a quotient of verma module $V(\varepsilon q^n)$.

§6.4. The Harish-Chandra homomorphism and center of U_q.

In this section we describe the center $Z_q \subset U_q$.

Prop 8: $$C_q = EF + \frac{q^{-1}K + qK^{-1}}{(q-q^{-1})^2} = FE + \frac{qK + q^{-1}K^{-1}}{(q-q^{-1})^2} \text{ is in } Z_q.$$

Proof: $$EC_q = EFE + E \cdot \frac{qK + q^{-1}K^{-1}}{(q-q^{-1})^2} = EFE + \frac{q^{-1}K + qK^{-1}}{(q-q^{-1})^2} E = C_q E.$$

Similar for F. $\square$.

Let U_q^K be all elements that commute with K. This is a sub ~~alg~~ algebra of U_q.

Lemma 9: U_q^K has basis $\{F^i P_i E^i\}_{i \geq 0}$, where P_i's are Lauren polynomials.

proof: We know U_q is spanned by $\{F^i K^l E^j\}_{i,j \in \mathbb{N}, l \in \mathbb{Z}}$,

$K(F^i K^l E^j)K^{-1} = q^{2(j-i)} F^i K^l E^j$. Hence $j = i$. $\square$.

Let $I = U_q E \cap U_q^K \subseteq U_q^K$, left ideal of U_q^K.

Lemma 10: $I = FU_q \cap U_q^K$. And $U_q^K = k[K, K^{-1}] \oplus I$.

proof: $u \in I$ iff $u = \sum_{i \geq 1} F^i P_i E^i$, this u certainly lives in FU_q.

$U_q^K = k[K, K^{-1}] \oplus I$ is because for any $u = \sum_{i \geq 0} F^i P_i E^i$, u can be writen as $P_0 + \sum_{i \geq 1} F^i P_i E^i$. $\square$.

Hence we get I is a two-sided ideal and it gives a projection

$\varphi: U_q^k \to k[K, K^{-1}]$ which is also an algebra morphism.

This φ is called Harish-Chandra homomorphism.

Why do we need this φ?

Because $Z_q \subseteq U_q^k$, φ gives $Z_q \longrightarrow k[K, K^{-1}]$. Which can be used to express how Z_q acts on highest weight modules.

Prop 11: Let V be a highest weight U_q-module with highest weight λ. Then for any central element z and $v \in V$, we have $zv = \varphi(z)(\lambda)v$.

Where $\varphi(z)(\lambda)$ is the value fo $\varphi(z)$ at λ.

Proof: Let v_0 be the generator of V of weight λ. Then $z \in Z_q \subset U_q^k$ can be written as: $z = \varphi(z) + \sum_{i \geq 1} F^i P_i E^i$. Since $Ev_0 = 0$, $Kv_0 = \lambda v_0$

$zv_0 = \varphi(z)(\lambda)v_0$. For arbitrary $v \in V$, $v = xv_0$ for some $x \in U_q$.

Then $zxv_0 = xzv_0 = x\varphi(z)(\lambda)v_0 = \varphi(z)(\lambda)(xv_0) = \varphi(z)(\lambda)v$. $\square$

Example: $\varphi(C_q) = \dfrac{q^{-1}K^{-1} + qK}{(q-q^{-1})^2}$. Hence C_q acts on V of highest weight λ

as multiplication with scalar $\dfrac{q^{-1}\lambda^{-1} + q\lambda}{(q-q^{-1})^2}$.

Now we show this $\varphi: Z_q \hookrightarrow k[K, K^{-1}]$ is an injection.

Lemma 12:

$$z \in Z_q. \quad \varphi(z) = 0 \iff z = 0$$

proof: Write $z = \sum_{i \geq k}^{l} F^i P_i E^i$. If $\varphi(z) = 0$, then $k > 0$.

Let $P_k, \dots, P_l$ be non-zero elements in $k[K, K^{-1}]$.

Let $V(\lambda)$ be the Verma module with highest weight λ not a power of q.
$V(\lambda) = \{v_p\}_{p \in \mathbb{N}}$.

Then $Ev_p \neq 0$ for $p \neq 0$. By prop 11, $zv_k = \varphi(z)(\lambda) z_k = 0$ since $\varphi(z) = 0$.

On the other hand, $zv_k = \sum_{i=k}^{l} F^i P_i E^i v_k = F^k P_k E^k v_k = c P_k(\lambda) v_k$.

(This is because $E^k v_k = c_1 \cdot v_0$. $P_k \cdot v_0 = P_k(\lambda) v_0$. $F^k v_0 = c_2 \cdot v_k$.)

Therefore $c P_k(\lambda) v_k = 0$. $\Rightarrow P_k(\lambda) = 0$. Choose infinitely many values for λ, we get $P_k = 0$. Contradiction.

$\square$.

Now for any element in $k[K, K^{-1}]$, we use notation: $\tilde{P}(\lambda) = P(q^{-1}\lambda)$.

<u>Lemma</u> 13: For any $z \in Z_q$, $\widetilde{\varphi(z)}(\lambda) = \widetilde{\varphi(z)}(\lambda^{-1})$

Proof:

~~Evn~~ Consider Verma module $V(q^{n-1})$. (with basis $\{v_p\}_{p \in \mathbb{N}}$)

Then $$E v_n = \frac{q^{-(n-1)} q^{n-1} - q^{n-1} q^{-(n-1)}}{q - q^{-1}} v_n = 0$$

Hence v_n is a highest weight module of weight $q^{n-1-2n} = q^{-n-1}$.

Then $z v_n = \varphi(z)(q^{-n-1}) v_n$.

But v_n is in $V(q^{n-1})$, hence $z v_n = \varphi(z)(q^{n-1}) v_n$

Therefore ~~$\varphi(z) = \varphi$~~ $\widetilde{\varphi(z)}(q^{n}) = \widetilde{\varphi(z)}(q^{-n})$. $\square$

<u>Lemma</u> 14: If $P(\lambda) = P(\lambda^{-1})$ for $P \in k[K, K^{-1}]$. Then P is a polynomial of $K + K^{-1}$.

proof: Easy. Use induction. $\square$.

Main Theorem 14:

Z_q is generated by C_q. And $\varphi: Z_q \to k[K, K^{-1}]$ has image the subalgebra of $k[K, K^{-1}]$ generated by $qK + q^{-1}K^{-1}$.

Proof: By lemma 14, we see $Z_q \cong \varphi(Z_q)$ which is the subalgebra generated by $qK + q^{-1}K^{-1}$. On the other hand,

$$\varphi(C_q) = \frac{qK + q^{-1}K^{-1}}{(q - q^{-1})^2}.$$ Hence $\varphi(C_q)$ generates $\varphi(Z_q)$.

Therefore C_q generates Z_q. $\square$.