VIII.1 The Yang-Baxter Equation (Zhaochen Wang, Nov 23, 2015)

Definition VIII.1.1: Let V be a vector space over a field k. A linear automorphism c of $V\otimes V$ is said to be an R-matrix if it is a solution of the Yang-Baxter equation
$$(c\otimes id_V)(id_V\otimes c)(c\otimes id_V) = (id_V\otimes c)(c\otimes id_V)(id_V\otimes c)$$
that holds in the automorphism group of $V\otimes V\otimes V$.

(Finding all solutions of the Yang-Baxter equation is a difficult task, as will appear from the examples given below.)

Let $\{v_i\}_i$ be a basis of the vector space V. An automorphism c of $V\otimes V$ is defined by the family $(c_{ij}^{kl})_{i,j,k,l}$ of scalars determined by
$$c(v_i\otimes v_j) = \sum_{k,l} c_{ij}^{kl}\, v_k\otimes v_l$$
$$c: V\otimes V \longrightarrow V\otimes V$$
Then c is a solution of the Yang-Baxter equation iff for all i,j,k,l,m,n, we have
$$\sum_{p,q,r,x,y,z} (c_{ij}^{pq}\delta_{kr})(\delta_{px} c_{qr}^{yz})(c_{xy}^{lm}\delta_{zn}) = \sum_{p,q,r,x,y,z} (\delta_{ip} c_{jk}^{qr})(c_{pq}^{xy}\delta_{rz})(\delta_{xl} c_{yz}^{mn}),$$
which is equivalent to
$$\sum_{p,q,y} c_{ij}^{pq} c_{qk}^{yn} c_{py}^{lm} = \sum_{j,q,r} c_{jk}^{qr} c_{iq}^{ly} c_{yr}^{mn}. \qquad (1.1)$$
for all i,j,k,l,m,n.

(Solving the non-linear equations (1.1) is a highly non-trivial problem. Nevertheless, numerous solutions of the Yang-Baxter equation have been discovered in the 1980's)

Let us list a few examples.

Ex 1. For any vector space V we denote by $\tau_{V,V}\in Aut(V\otimes V)$ the flip switching the two copies of V. It is defined by
$$\tau_{V,V}(v_1\otimes v_2) = v_2\otimes v_1$$
for any $v_1,v_2\in V$.

The flip satisfies the Yang-Baxter equation because of the Coxeter relation $(12)(23)(12)=(23)(12)(23)$ in the symmetry group S_3.
$$(\tau_{V,V}\otimes id_V)(id_V\otimes \tau_{V,V})(\tau_{V,V}\otimes id_V) = (id_V\otimes \tau_{V,V})(\tau_{V,V}\otimes id_V)(id_V\otimes \tau_{V,V})$$

Here is a way to generate new R-matrices from old ones.

Lem VIII.1.2. If $c \in \mathrm{Aut}(V\otimes V)$ is an R-matrix, then so are λc, c^{-1} and $T_{V,V}\circ c\circ T_{V,V}$, where λ is any non-zero scalar.

Pf: We have identities

① $(\lambda c\otimes id_V)=\lambda(c\otimes id_V)$, $(id_V\otimes \lambda c)=\lambda(id_V\otimes c)$

$$\Rightarrow (\lambda c\otimes id_V)(id_V\otimes\lambda c)(\lambda c\otimes id_V)=\lambda^3(c\otimes id_V)(id_V\otimes c)(c\otimes id_V)$$
$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \shortparallel$$
$$(id_V\otimes\lambda c)(\lambda c\otimes id_V)(id_V\otimes\lambda c)=\lambda^3(id_V\otimes c)(c\otimes id_V)(id_V\otimes c)$$

② $(c^{-1}\otimes id_V)=(c\otimes id_V)^{-1}$, $(id_V\otimes c^{-1})=(id_V\otimes c)^{-1}$

$$\Rightarrow (c^{-1}\otimes id_V)(id_V\otimes c^{-1})(c^{-1}\otimes id_V)=(c\otimes id_V)^{-1}(id_V\otimes c)^{-1}(c\otimes id_V)^{-1}=((c\otimes id_V)(id_V\otimes c)(c\otimes id_V))^{-1}$$
$$\shortparallel$$
$$(id_V\otimes c^{-1})(c^{-1}\otimes id_V)(id_V\otimes c^{-1})=(id_V\otimes c)^{-1}(c\otimes id_V)^{-1}(id_V\otimes c)^{-1}=((id_V\otimes c)(c\otimes id_V)(id_V\otimes c))^{-1}$$

③ $(c'\otimes id_V)=\sigma(id_V\otimes c)\sigma^{-1}$, $(id_V\otimes c')=\sigma(c\otimes id_V)\sigma^{-1}$

where $c'=T_{V,V}\circ c\circ T_{V,V}$ and $\sigma\in \mathrm{Aut}(V\otimes V\otimes V)$ defined by $\sigma(v_1\otimes v_2\otimes v_3)=v_3\otimes v_2\otimes v_1$ for $v_1,v_2,v_3\in V$.

$$\Rightarrow (c'\otimes id_V)(id_V\otimes c')(c'\otimes id_V)=\sigma(id_V\otimes c)\sigma^{-1}\sigma(c\otimes id_V)\sigma^{-1}\sigma(id_V\otimes c)\sigma^{-1}$$
$$\shortparallel$$
$$\sigma(id_V\otimes c)(c\otimes id_V)(id_V\otimes c)\sigma^{-1}$$
$$\shortparallel$$
$$\sigma(c\otimes id_V)(id_V\otimes c)(c\otimes id_V)\sigma^{-1}$$
$$\shortparallel$$
$$(id_V\otimes c')(c'\otimes id_V)(id_V\otimes c')=\sigma(c\otimes id_V)\sigma^{-1}\sigma(id_V\otimes c)\sigma^{-1}\sigma(c\otimes id_V)\sigma^{-1}$$

∎

Ex 2. Let us solve the Yang-Baxter equation when $V=V_1=V_{1,1}$ is the 2-dimensional simple module over the Hopf algebra $U_q = U_q(sl_2)$. (We have Chevalley generators $E, F, K^{\pm}$)

(Thm VI 3.5 shows that, up to isomorphism, $\exists!$ simple U_q-module of dimension $n+1$ and generated by a highest weight vector of weight εq^n, denoted by $V_{\varepsilon, n}$, where $\varepsilon = \pm 1$, $\dim(V) = n+1$.
For $V_{1,n}$, we simplify it by V_n.)

(More precisely, let us find all U_q-automorphisms of $V_1 \otimes V_1$ that are R-matrices. We freely use the notation of the previous chapters).

Recall that if $v = v_0$ is a highest weight vector of V_1, then the set $\{v_0, v_1 = Fv\}$ is a basis of V_1 by Thm VI 3.5 (ii). (Here $v_p = \frac{F^p v}{[p]!}$, $v_p > 0$ for $p > n$, $[n] = \frac{q^n - q^{-n}}{q - q^{-1}}$ for an invertible element q of k different from ± 1)

(Thm VII 7.1. Let $n \geq m$ be two nonnegative integers. There exists an isomorphism of U_q-modules
$$V_n \otimes V_m \cong V_{n+m} \oplus V_{n+m-2} \oplus \cdots \oplus V_{n-m}$$
)

By the Clebsch-Gordan Thm VII 7.1, we have $V_1 \otimes V_1 \cong V_{1+1} \oplus V_{1-1} = V_2 \oplus V_0$

(Lem VII 7.2: Let $v^{(n)}$ be a highest weight vector of weight q^n in V_n and $v^{(m)}$ be a highest weight vector of weight q^m in V_m. Let us define $v_p^{(n)} = \frac{1}{[p]!} F^p v^{(n)}$ and $v_p^{(m)} = \frac{1}{[p]!} F^p v^{(m)}$ for all $p \geq 0$.
Then, $v^{(n+m-2p)} = \sum_{i=0}^{p} (-1)^i \frac{[m-p+i]!\,[n-i]!}{[m-p]!\,[n]!} \cdot q^{-i(m-2p+i+1)} v_i^{(n)} \otimes v_{p-i}^{(m)}$
is a highest weight vector of weight q^{n+m-2p} in $V_n \otimes V_m$)

Lem 7.2 implies that the vector $w_0 = v_0 \otimes v_0$, $t = v_0 \otimes v_1 - q^{-1} v_1 \otimes v_0$ are highest weight vectors of respective weights q^2 and 1. (q^2: $m=n=1, p=0$, 1: $m=n=p=1$)

We complete the set of linearly independent vectors $\{w_0, t\}$ into a basis for $V_1 \otimes V_1$ by setting
$$w_1 = Fw_0 = \Delta(F)(w_0) = (K^{-1} \otimes F + F \otimes 1)(v_0 \otimes v_0) = (K^{-1} v_0) \otimes (Fv_0) + (Fv_0) \otimes (\mathrm{id}(v_0))$$
$$= q^{-1} v_0 \otimes v_1 + v_1 \otimes v_0$$
$$w_2 = \frac{1}{[2]} F^2 w_0 = \frac{1}{[2]} (K^{-1} \otimes F + F \otimes 1)(q^{-1} v_0 \otimes v_1 + v_1 \otimes v_0)$$
$$= \frac{1}{[2]} (q^{-2} v_0 \otimes \underset{\overset{\|}{0}}{[2] v_2} + q^{-1} v_1 \otimes v_1 + q v_1 \otimes v_1 + \underset{\overset{\|}{0}}{[2] v_2} \otimes v_0) = \frac{q + q^{-1}}{[2]} \cdot v_1 \otimes v_1 = v_1 \otimes v_1$$

where $[2] = \frac{q^2 - q^{-2}}{q - q^{-1}} = q + q^{-1}$

Prop VIII.1.3. Any U_q-linear automorphism φ of $V_1\otimes V_1$ is diagonalizable and of the form $\varphi(w_i)=\lambda w_i$ $(i=0,1,2)$ and $\varphi(t)=\mu t$ where λ and μ are non-zero scalars. The automorphism φ is an R-matrix iff $(\lambda-\mu)(q\lambda+q^{-1}\mu)(q^{-1}\lambda+q\mu)=0$.

pf (1) Since φ is U_q-linear, then $\forall a\in U_q$, $u\otimes v\in V_1\otimes V_1$, $\varphi(a\, u\otimes v)=a\,\varphi(u\otimes v)$.

Hence the image under φ of a highest weight vector is a highest weight vector of the same weight.

$$\begin{tikzcd} V_1\otimes V_1 \arrow[r, "a"] \arrow[d, "\varphi"'] & V_1\otimes V_1 \arrow[d, "\varphi"] \\ V_1\otimes V_1 \arrow[r, "a"] & V_1\otimes V_1 \end{tikzcd}$$

diagram commutes.

Since w_0, t has weight q^2, 1, respectively (we assume $q^2\neq 1$ here), then $\exists\lambda,\mu$. s.t $\varphi(w_0)=\lambda w_0$, $\varphi(t)=\mu t)$, $\lambda,\mu\neq 0$

Since $\varphi(w_i)=\varphi\left(\frac{F^i w_0}{[i]}\right)=\frac{1}{[i]}F^i\varphi(w_0)=\frac{\lambda}{[i]}F^i w_0=\lambda w_i$ for $i=1,2$,

then $\varphi\begin{pmatrix} w_0\\ w_1\\ w_2\\ t\end{pmatrix}=\begin{pmatrix}\lambda&&&\\&\lambda&&\\&&\lambda&\\&&&\mu\end{pmatrix}\begin{pmatrix} w_0\\ w_1\\ w_2\\ t\end{pmatrix}$. Therefore: $\varphi\in \mathrm{Aut}(V_1\otimes V_1)$ is diagonalizable.

(2) (The second assertion results from tedious computation.)

By first assertion, we have $\varphi(w_i)=\lambda w_i$ $(i=0,1,2)$, $\varphi(t)=\mu t$

It is clear that $\varphi(v_0\otimes v_0)=\varphi(w_0)=\lambda w_0=\lambda v_0\otimes v_0$, $\varphi(v_1\otimes v_1)=\varphi(w_2)=\lambda w_2=\lambda v_1\otimes v_1$

$$\begin{cases} q^{-1}\varphi(v_0\otimes v_1)+\varphi(v_1\otimes v_0)=\varphi(q^{-1}v_0\otimes v_1+v_1\otimes v_0)=\varphi(w_1)=\lambda w_1=\lambda(q^{-1}v_0\otimes v_1+v_1\otimes v_0)\\ \varphi(v_0\otimes v_1)-q^{-1}\varphi(v_1\otimes v_0)=\varphi(v_0\otimes v_1-q^{-1}v_1\otimes v_0)=\varphi(t)=\mu t=\mu(v_0\otimes v_1-q^{-1}v_1\otimes v_0)\end{cases}$$

Then we have the matrix Φ of φ w.r.t the basis $\{v_0\otimes v_0, v_0\otimes v_1, v_1\otimes v_0, v_1\otimes v_1\}$ is given by

$$\varphi\begin{pmatrix} v_0\otimes v_0\\ v_0\otimes v_1\\ v_1\otimes v_0\\ v_1\otimes v_1\end{pmatrix}=\Phi\begin{pmatrix} v_0\otimes v_0\\ v_0\otimes v_1\\ v_1\otimes v_0\\ v_1\otimes v_1\end{pmatrix},\quad \Phi=\begin{pmatrix}\lambda&&&\\&\alpha&\gamma&\\&\gamma&\beta&\\&&&\lambda\end{pmatrix}$$

where $\alpha=\frac{q^{-1}\lambda+q\mu}{[2]}$, $\beta=\frac{q\lambda+q^{-1}\mu}{[2]}$, $\gamma=\frac{\lambda-\mu}{[2]}$

The automorphism $\varphi\otimes id$ and $id\otimes\varphi$ can be expressed, respectively, by the 8×8-matrices Φ_{12} and Φ_{23} in the basis consisting of the elements

$v_0\otimes v_0\otimes v_0$, $v_0\otimes v_0\otimes v_1$, $v_0\otimes v_1\otimes v_0$, $v_0\otimes v_1\otimes v_1$, $v_1\otimes v_0\otimes v_0$, $v_1\otimes v_0\otimes v_1$, $v_1\otimes v_1\otimes v_0$, $v_1\otimes v_1\otimes v_1$

of $V\otimes V\otimes V$ where

$$\Phi_{12}=\begin{pmatrix}\lambda&&&&&&&\\&\lambda&&&&&&\\&&\alpha&0&\gamma&&&\\&&0&\alpha&0&\gamma&&\\&&\gamma&0&\beta&0&&\\&&&\gamma&0&\beta&&\\&&&&&&\lambda&\\&&&&&&&\lambda\end{pmatrix}_{8\times 8},\quad \Phi_{23}=\begin{pmatrix}\lambda&&&&&&&\\&\alpha&\gamma&&&&&\\&\gamma&\beta&&&&&\\&&&\lambda&&&&\\&&&&\lambda&&&\\&&&&&\alpha&\gamma&\\&&&&&\gamma&\beta&\\&&&&&&&\lambda\end{pmatrix}_{8\times 8}$$

Here we have

$(\varphi\otimes id)(v_i\otimes v_j\otimes v_k)$

$=\varphi(v_i\otimes v_j)\otimes id(v_k)$

$=\varphi(v_i\otimes v_j)\otimes v_k$ for $i,j,k\in\{0,1\}$

Now, $\Phi_{12}\Phi_{23}\Phi_{12}-\Phi_{23}\Phi_{12}\Phi_{23}=\begin{pmatrix} 0 &&&&&&& \\ & K & -\alpha\beta\gamma &&&&& \\ & -\alpha\beta\gamma & L & 0 & \alpha\beta\gamma &&& \\ && 0 & -K & 0 & \alpha\beta\gamma && \\ &&& \alpha\beta\gamma & 0 & M & 0 & \\ &&&& \alpha\beta\gamma & 0 & -L & \alpha\beta\gamma \\ &&&&&& \alpha\beta\gamma & -M \\ &&&&&&& 0 \end{pmatrix}_{8\times 8}$

where $K=\alpha((\lambda-\alpha)\lambda-\gamma^2)$, $L=\alpha\beta(\alpha-\beta)$ and $M=\beta(\gamma^2+\lambda(\beta-\lambda))$.

Let us check that: K, L, M are multiples of $\alpha\beta\gamma$.

$\lambda-\alpha=\frac{(q+q^{-1})\lambda-q^{-1}\lambda-q\mu}{[2]}=\frac{q(\lambda-\mu)}{[2]}=q\gamma$, $\lambda-\beta=\frac{(q+q^{-1})\lambda-q\lambda-q^{-1}\mu}{[2]}=\frac{q^{-1}(\lambda-\mu)}{[2]}=q^{-1}\gamma$

$q^{-1}\lambda-\gamma=\frac{q^{-1}(q+q^{-1})\lambda-(\lambda-\mu)}{[2]}=\frac{\lambda+q^{-2}\lambda-\lambda+\mu}{[2]}=\frac{q^{-1}(q^{-1}\lambda+q\mu)}{[2]}=q^{-1}\alpha$; also $\beta-\alpha=\frac{(q-q^{-1})(\lambda-\mu)}{[2]}$

$q\lambda-\gamma=\frac{q(q+q^{-1})\lambda-(\lambda-\mu)}{[2]}=\frac{q^2\lambda+\lambda-\lambda+\mu}{[2]}=\frac{q(q\lambda+q^{-1}\mu)}{[2]}=q\beta$ $\qquad =(q-q^{-1})\gamma$

Therefore, $K=\alpha(q\gamma\lambda-\gamma^2)=\alpha\gamma(q\lambda-\gamma)=\alpha\beta\gamma q$, $L=-(q-q^{-1})\alpha\beta\gamma$, $M=-q^{-1}\alpha\beta\gamma$.

Then $\Phi_{12}\Phi_{23}\Phi_{12}-\Phi_{23}\Phi_{12}\Phi_{23}=\alpha\beta\gamma\times\Psi$, where Ψ is a non-zero 8×8-matrix.

i.e. $((\varphi\otimes id)(id\otimes\varphi)(\varphi\otimes id)-(id\otimes\varphi)(\varphi\otimes id)(id\otimes\varphi))\begin{pmatrix} v_0\otimes v_0\otimes v_0 \\ \vdots \\ v_1\otimes v_1\otimes v_1\end{pmatrix}=\alpha\beta\gamma\times\Psi\begin{pmatrix} v_0\otimes v_0\otimes v_0 \\ \vdots \\ v_1\otimes v_1\otimes v_1\end{pmatrix}$

It implies φ is an R-matrix iff $\alpha\beta\gamma=0$.

Since $\alpha\beta\gamma=0 \iff (\lambda-\mu)(q\lambda+q^{-1}\mu)(q^{-1}\lambda+q\mu)=0$, then the automorphism φ is an R-matrix iff $(\lambda-\mu)(q\lambda+q^{-1}\mu)(q^{-1}\lambda+q\mu)=0$ ■

To sum up, the R-matrices of the U_q-module $V_1\otimes V_1$ belong to the following three types depending on a parameter $\lambda\neq 0$:

1. If $\lambda=\mu$, φ is a homothety. $\left(\lambda=\mu\Rightarrow\varphi\begin{pmatrix} v_0\otimes v_0 \\ v_0\otimes v_1 \\ v_1\otimes v_0 \\ v_1\otimes v_1\end{pmatrix}=\begin{pmatrix}\lambda &&& \\ &\lambda&& \\ &&\lambda& \\ &&&\lambda\end{pmatrix}\begin{pmatrix} \\ \\ \\ \end{pmatrix}=\lambda\begin{pmatrix} v_0\otimes v_0 \\ v_0\otimes v_1 \\ v_1\otimes v_0 \\ v_1\otimes v_1\end{pmatrix}\right)$

2. If $q\lambda+q^{-1}\mu=0$, then $\Phi=q\lambda\begin{pmatrix} q^{-1} &&& \\ & q^{-1}-q & 1 & \\ & 1 & 0 & \\ &&& q^{-1}\end{pmatrix}$

3. If $q^{-1}\lambda+q\mu=0$, then $\Phi=q^{-1}\lambda\begin{pmatrix} -q &&& \\ & 0 & 1 & \\ & 1 & q-q^{-1} & \\ &&& q\end{pmatrix}$

It is clear that Case 2 and 3 are equivalent within a change of basis after exchange q and q^{-1}.

(As we shall see in the next example, the minimal polynomial of Φ is of degree ≤ 2.)

Ex 3. We now give an important class of R-matrices with quadratic minimal polynomial.

(Such R-matrices will be used in Chapter XII to construct isotopy invariants of links in $\mathbb{R}^3$)

Let V be a finite-dimensional vector space with a basis $\{e_1, \ldots, e_N\}$. For two invertible scalars p, q and for any family $\{r_{ij}\}_{1\le i,j\le N}$ of scalars in k s.t. $r_{ii}=q$ and $r_{ij}r_{ji}=p$ when $i\neq j$, we define an automorphism c of $V\otimes V$ by

$$c(e_i\otimes e_j) = \begin{cases} r_{ii}\, e_i\otimes e_i & \text{if } i=j \\ r_{ji}\, e_j\otimes e_i & \text{if } i<j \\ r_{ji}\, e_j\otimes e_i + (q-pq^{-1})\, e_i\otimes e_j & \text{if } i>j \end{cases}$$

Prop VIII.1.4. The automorphism c is a solution of the Yang-Baxter equation. Moreover, we have $(c-q\,\mathrm{id}_{V\otimes V})(c+pq^{-1}\mathrm{id}_{V\otimes V})=0$ or, equivalently, $c^2-(q-pq^{-1})c-p\,\mathrm{id}_{V\otimes V}=0$.

pf: (a) We first show that c is an R-matrix.

In order to simplify the proof, let us introduce the following notation.

$(ijk) \triangleq e_i\otimes e_j\otimes e_k \in V\otimes V\otimes V$.

$$[i>j] \triangleq \begin{cases} 1 & \text{if } i>j \\ 0 & \text{otherwise.}\end{cases}$$

Then c can be redefined as $c(e_i\otimes e_j)=r_{ji}\, e_j\otimes e_i + [i>j]\beta\, e_i\otimes e_j$ where $\beta = q-pq^{-1}$

$$(c\otimes id)(id\otimes c)(c\otimes id)((ijk))$$
$$=(c\otimes id)(id\otimes c)(r_{ji}(jik)+[i>j]\beta(ijk)) = (c\otimes id)(r_{ji}(r_{ki}(jki)+[i>k]\beta(jik)) + [i>j]\beta(r_{kj}(ikj)+[j>k]\beta(ijk)))$$
$$=(c\otimes id)(r_{ji}r_{ki}(jki)+r_{ji}[i>k]\beta(jik)+r_{kj}[i>j]\beta(ikj)+[i>j][j>k]\beta^2(ijk))$$
$$=r_{ji}r_{ki}(r_{kj}(kji)+[j>k]\beta(jki))+r_{ji}[i>k]\beta(r_{ij}(ijk)+[j>i]\beta(jik))$$
$$+r_{kj}[i>j]\beta(r_{ki}(kij)+[i>k]\beta(ikj))+[i>j][j>k]\beta^2(r_{ji}(jik)+[i>j]\beta(ijk))$$
$$=r_{ji}r_{ki}r_{kj}(kji)+r_{ji}r_{ki}[j>k]\beta(jki)+r_{kj}r_{ki}[i>j]\beta(kij)+r_{kj}[i>j][i>k]\beta^2(ikj) \quad \text{typo?}$$
$$+r_{ji}([j>i][i>k]+[i>j][j>k])\beta^2(jik)+(r_{ji}r_{ij}[i>k]\beta+[i>j]^2[j>k]\beta^3)(ijk)$$

Similarly, $(id\otimes c)(c\otimes id)(id\otimes c)(ijk)$

$$=r_{ji}r_{ki}r_{kj}(kji)+r_{ji}r_{ki}[j>k]\beta(jki)+r_{kj}r_{ki}[i>j]\beta(kij)+r_{ji}[i>k][j>k]\beta^2(jik)$$
$$+r_{kj}([i>k][k>j]+[i>j][j>k])\beta^2(ikj)+(r_{jk}r_{kj}[i>k]\beta+[i>j][j>k]\beta^3)(ijk)$$

We have to show two expressions are equal for all i, j, k.

① If $i=j=k$, $c(c\otimes id)(id\otimes c)(c\otimes id)(ijk) = q^3(iii) = (id\otimes c)(c\otimes id)(id\otimes c)(ijk)$

② If i, j, k are distinct indices,

Since $i>j, i>k$ iff $i>j>k$ or $i>k>j$. then $[i>j][i>j] = [i>j][j>k] + [i>k][k>j]$.

It implies $r_{kj}[i>j][i>k]\beta^2(ikj) = r_{kj}([i>k][k>j] + [i>j][j>k])\beta^2(ikj)$

Since $r_{ji}([j>i][i>k] + [i>j][j>k])\beta^2(jik) = \begin{cases} r_{ji}\beta^2(jik) & \text{when } i>j>k \\ 0 & \text{when } i>k>j \end{cases}$

$r_{ji}[i>k][j>k]\beta^2(jik) = \begin{cases} r_{ji}\beta^2(jik) & \text{when } i>j>k \\ 0 & \text{when } i>k>j \end{cases}$,

then $r_{ji}([j>i][i>k] + [i>j][j>k])\beta^2(jik) = r_{ji}[i>k][j>k]\beta^2(jik)$

Also, $r_{ji}r_{ij} = p = r_{jk}r_{kj}$. for $i\neq j, j\neq k$.

Therefore, two expressions are equal.

③ If exactly two indices are equal, wlog say $i=j\neq k$.

Then $((c\otimes id)(id\otimes c)(c\otimes id) - (id\otimes c)(c\otimes id)(id\otimes c))(iik)$

$= r_{kj}([i\cancel{>j}][i>k] - [i>k][k>j] - \cancel{[i>j]}[j>k])\beta^2(ikj) + r_{ji}(\cancel{[j>i]}[i>k] + \cancel{[i>j]}[j>k] - [i>k][j>k])\beta^2(jik)$

$+ (r_{ij}r_{ji}[i>k]\beta + [i>j][j>k]\beta^3 - r_{jk}r_{kj}[i>k]\beta - [i>j][j>k]\beta^3)(ijk)$

$= r_{ki}(-[i>k]\underset{\overset{\|}{0}}{[k>i]})\beta^2(iki) + r_{ii}(-\underset{\|}{[i>k][i>k]})\beta^2(iik) + (r_{ii}^2[i>k]\beta - r_{ik}r_{ki}[i>k]\beta)(iik)$

$= \quad 0 \quad - \quad q[i>k]\beta^2(iik) \quad + (q^2[i>k]\beta - p[i>k]\beta)(iik)$

$= [i>k]\beta(q^2 - p - q\beta)(iik) = 0 \quad$ (where $\beta = q - pq^{-1}$)

Therefore, automorphism c is a solution of the Yang-Baxter equation.

(b) For any $e_i\otimes e_j$.

If $i=j$, then $(c - q\,id_{V\otimes V})(c + pq^{-1}id_{V\otimes V})(e_i\otimes e_i)$

$= (c - q\,id_{V\otimes V})(q(e_i\otimes e_i) + pq^{-1}(e_i\otimes e_i)) = q^2(e_i\otimes e_i) - q^2(e_i\otimes e_i) + p(e_i\otimes e_i) - p(e_i\otimes e_i) = 0.$

If $i\neq j$, then $(c^2 - (q - pq^{-1})c - p\,id_{V\otimes V})(e_i\otimes e_j)$

$= c(r_{ji}e_j\otimes e_i + [i>j]\beta e_i\otimes e_j) - \beta(r_{ji}e_j\otimes e_i + [i>j]\beta e_i\otimes e_j) - pe_i\otimes e_j$

$= r_{ji}(r_{ij}e_i\otimes e_j + [j>i]\beta e_j\otimes e_i) + [i>j]\beta(r_{ji}e_j\otimes e_i + \beta e_i\otimes e_j)$

$- \beta(r_{ji}e_j\otimes e_i + [i>j]\beta e_i\otimes e_j) - pe_i\otimes e_j$

$= \cancel{pe_i\otimes e_j} + r_{ji}\cancel{[j>i]}\beta e_j\otimes e_i + r_{ji}\cancel{[i>j]}\beta e_j\otimes e_i + [i>j]\beta^2 e_i\otimes e_j$

$- r_{ji}\beta e_j\otimes e_i - [i>j]\beta^2 e_i\otimes e_j - \cancel{pe_i\otimes e_j} = 0$

Consider the following two special cases:

(i) If $p = q^2$ and $r_{ij} = q$ for all i, j, then c is homothety.

(ii) Take $p = 1$ and $r_{ij} = 1$ for $i \neq j$. Then c takes the form shown in Case 3 of Example 2 when V is two-dimensional. Thus, Example 2 turns out to be a special case of Ex 3.

The relation between Yang-Baxter Equation of the form $R_{12} \circ R_{13} \circ R_{23} = R_{23} \circ R_{13} \circ R_{12}$ and the form $(c \otimes id_V)(id_V \otimes c)(c \otimes id_V) = (id_V \otimes c)(c \otimes id_V)(id_V \otimes c)$.

The linear maps $R: V \otimes V \to V \otimes V$, with V a vector space. R is a solution of Y-B Equation if

$$R_{12} \circ R_{13} \circ R_{23} = R_{23} \circ R_{13} \circ R_{12}$$

where R_{ij} denote the map $R_{ij}: V \otimes V \otimes V \to V \otimes V \otimes V$ acting as R on (i,j) tensor factors and as the identity on the remaining factor. (i.e. $R_{12} = R \otimes id_V$.)

Let $\tau_{V,V}: V \otimes V \to V \otimes V$ be flip switching and $c = \tau_{V,V} \circ R$, then
$v_1 \otimes v_2 \mapsto v_2 \otimes v_1$

$(c \otimes id_V)(id_V \otimes c)(c \otimes id_V) = (id_V \otimes c)(c \otimes id_V)(id_V \otimes c)$ iff $R_{12} \circ R_{13} \circ R_{23} = R_{23} \circ R_{13} \circ R_{12}$.

The form $R_{12} \circ R_{13} \circ R_{23} = R_{23} \circ R_{13} \circ R_{12}$ can also be the form of product of elements in $A \otimes A \otimes A$ where A is a Hopf algebra.

Here, $R = \sum R^{(1)} \otimes R^{(2)} \in A \otimes A$. ($R_{12} = \sum R^{(1)} \otimes R^{(2)} \otimes 1$, $R_{13} = \sum R^{(1)} \otimes 1 \otimes R^{(2)}$, $R_{23} = \sum 1 \otimes R^{(1)} \otimes R^{(2)}$)

More details are in Section VIII.2, VIII.3. and related papers.