Ch VI

We will construct a Hopf alg $U_q(sl_2)$.

Notation: $[n] = \frac{q^n - q^{-n}}{q - q^{-1}}$; $\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!}$, $[n]! = [1][2]\cdots[n]$

It's easy to check if x, y satisfies ~~x~~ $yx = q^2 xy$, then

$$(x+y)^n = \sum_{k=0}^{n} q^{k(n-k)} \begin{bmatrix} n \\ k \end{bmatrix} x^k y^{n-k}$$ (Use the fact that

$$\begin{bmatrix} n+1 \\ k \end{bmatrix} = q^{-(n+1-k)} \begin{bmatrix} n \\ k-1 \end{bmatrix} + q^k \begin{bmatrix} n \\ k \end{bmatrix}$$)

Definition: U_q is the algebra generated by symbols ~~satisfying:~~ E, F, K, K^{-1} satisfying: $KK^{-1} = K^{-1}K = 1$

$$KE = q^2 EK, \quad KF = q^{-2} FK, \quad EF - FE = \frac{K - K^{-1}}{q - q^{-1}}$$

Prop 1: $E^m K^n = q^{-2mn} K^n E^m$, $F^m K^n = q^{2mn} K^n F^m$,

$$[E, F^m] = [m] F^{m-1} \frac{q^{-(m-1)}K - q^{(m-1)}K^{-1}}{q - q^{-1}} = [m] \frac{q^{m-1}K - q^{-(m-1)}K^{-1}}{q - q^{-1}} F^{m-1}$$

$$[E^m, F] = [m] \frac{q^{-(m-1)}K - q^{m-1}K^{-1}}{q - q^{-1}} E^{m-1} = [m] E^{m-1} \frac{q^{m-1}K - q^{-(m-1)}K^{-1}}{q - q^{-1}}$$

Pf: The first two are direct consequences of the fact that $KE = q^2 EK$ and $KF = q^{-2} FK$. For $[E, F^m]$, we can use induction:

$$[E, F^{m+1}] = EF^{m+1} - F^m EF + F^m EF - F^{m+1}E = [E, F^m]F + F^m [E, F]$$

$$= [m] \frac{q^{m-1}K - q^{-(m-1)}K^{-1}}{q - q^{-1}} F^m + F^m \frac{K - K^{-1}}{q - q^{-1}}$$

$$= [m] \frac{q^{m-1}K - q^{-(m-1)}K^{-1}}{q - q^{-1}} F^m + \frac{q^{2m}KF^m - q^{-2m}K^{-1}F^m}{q - q^{-1}} =$$

$$[m]\frac{q^{m-1}K-q^{-(m-1)}K^{-1}}{q-q^{-1}}F^m+\frac{q^{2m}K-q^{-2m}K^{-1}}{q-q^{-1}}F^m=\frac{(q^{m-1}[m]+q^{2m})K-(q^{-(m-1)}[m]+q^{-2m})K^{-1}}{q-q^{-1}}F^m$$

$$=\frac{q^m[m+1]K-q^{-m}[m+1]K^{-1}}{q-q^{-1}}F^m=[m+1]\frac{q^mK-q^{-m}K^{-1}}{q-q^{-1}}F^m, \text{ done.}$$

Prop: U_q is noetherian and has no zero divisor, and $\{E^iF^jK^l\}_{i,j\in\mathbb{N},\, l\in\mathbb{Z}}$ is a basis of U_q.

Pf: In fact, we can construct U_q as an Ore extension. Set $A=k[K,K^{-1}]$, and consider the algebra automorphism α of A defined by $\alpha(K)=q^2K$, and set $A'=A[F,\alpha,0]$. Then it's obvious that A' is an algebra generated by F, K, K^{-1} satisfying the ~~ret~~ relation $FK=q^2KF$.
Next we show that there is an α-derivation δ of A' such that $\delta(K)=0$, $\delta(F)=\frac{K-K^{-1}}{q-q^{-1}}$, and that U_q is the Ore extension $A'[E,\alpha_1,\delta]$. $\xrightarrow{} \alpha_1(F^jK^l)=q^{-2l}F^jK^l$ In fact, we know that A' has basis $\{F^jK^l\}$, and we define δ to be the linear map given by $\delta(F^jK^l)=\sum_{i=0}^{j-1}F^{j-1}\delta(F)(q^{-2i}K)K^l$, where $\delta(F)(q^{-2i}K)=\frac{q^{-2i}K-q^{2i}K^{-1}}{q-q^{-1}}$. Let's check this.
We only need to show $\delta(F^mK^n\cdot F^pK^q)=\delta(F^mK^n)F^pK^q+\alpha_1(F^mK^n)\delta(F^pK^q)$.
First, $F^mK^nF^pK^q=(q^{-2})^{np}F^{m+p}K^{n+q}$, so by definition

$$\delta(F^mK^nF^pK^q)=q^{-2np}\delta(F^{m+p}K^{n+q})=q^{-2np}\Big(\sum_{i=0}^{m+p-1}F^{m+p-1}\delta(F)(q^{-2i}K)K^{n+q}\Big).$$

$$\delta(F^mK^n)F^pK^q=\sum_{i=0}^{m-1}F^{m-1}\delta(F)(q^{-2i}K)K^n\cdot F^p\cdot K^q=\sum_{i=0}^{m-1}q^{-2np}F^{m-1}\delta(F)(q^{-2i}K)F^pK^{n+q}$$

$$=\sum_{i=0}^{m-1}q^{-2np}F^{m-1}\frac{q^{-2i}KF^p-q^{2i}K^{-1}F^p}{q-q^{-1}}K^{n+q}=\sum_{i=0}^{m-1}q^{-2np}F^{m-1}\frac{q^{-2(i+p)}F^pK-q^{2(i+p)}F^pK^{-1}}{q-q^{-1}}K^{n+q}$$

$$\alpha_1(F^mK^n)\delta(F^pK^q) = q^{-2n}F^mK^n\cdot\sum_{i=0}^{p-1}F^{p-1}\delta(F)(q^{-2i}K)K^q$$
$$= q^{-2n}\sum_{i=0}^{p-1}F^mK^nF^{p-1}\delta(F)(q^{-2i}K)K^q = q^{-2n}\sum_{i=0}^{p-1}F^{m+p-1}\cdot q^{-2n(p-1)}K^n\delta(F)(q^{-2i}K)K^q$$
$$= \sum_{i=0}^{p-1}q^{-2np}F^{m+p-1}\delta(F)(q^{-2i}K)K^{n+q}$$

So we get $\delta(F^mK^n\cdot F^pK^q) = \delta(F^mK^n)F^pK^q + \alpha_1(F^mK^n)\delta(F^pK^q)$

So δ is a ~~derivation~~ α_1-derivation of A'. If we consider the Ore extension $A'[E,\alpha_1,\delta]$, then we have $KE = q^2EK$, $[E,F] = \dfrac{K-K^{-1}}{q-q^{-1}}$, so $U_q = A'[E,\alpha_1,\delta]$, and all the claim follows from this.

One can reconstruct the universal enveloping algebra $U = U(sl_2)$ from U_q by setting $q=1$, ~~For~~ this we need to rewrite U_q in the following way:

Prop: U_q is isomorphic to the algebra generated by 5 variables E, F, K, K^{-1}, L with relation:

$$KK^{-1} = K^{-1}K = 1,\quad KE = q^2EK,\quad KF = q^{-2}FK,\quad [E,F] = L,$$
$$(q-q^{-1})L = K-K^{-1},\quad [L,E] = q(EK+K^{-1}E),\quad [L,F] = -q^{-1}(FK+K$$

Pf: Denote the algebra generated by E, F, K, K^{-1}, L satisfying the above relations by U_q'. Then we have $U_q \xrightarrow{\phi} U_q'$ given by $\phi(K)=K$, $\phi(K^{-1}) = K^{-1}$, $\phi(E)=E$, $\phi(F)=F$. Also, there is another morphism $U_q' \xrightarrow{\varphi} U_q$ given by $\varphi(K)=K$, $\varphi(K^{-1})=K^{-1}$, $\varphi(E)=E$, $\varphi(F)=F$, $\varphi(L)=[E,F]$. Notice that now

U'_q is defined for all q, and φ, ϕ are inverses to each other, so $U'_q \simeq U_q$. ~~set $q=1$ we get $U'_q \simeq U(sl_2)$~~

Next denote the enveloping algebra of sl_2 by U, then we have: $U'_1 \simeq U[K]/(K^2-1)$, $U \simeq U'_1/(K-1)$.

In fact, recall that sl_2 is given by a vector space V with basis E, F, H satisfying $[H,E]=2E$, $[H,F]=$ $[E,F]=H$, and $U(sl_2)$ is generated by 3 variables X, Y, Z satisfying the following relation: $[X,Y]=Z$, $[Z,X]=2X$, $[Z,Y]=-2Y$. Now look at ~~the~~ $U'_1/(K-1)$, then it is generated by E, F, L with relation $[E,F]=L$, $[L,E]=2E$, $[L,F]=-2F$, so $U \simeq U'_1/(K-1)$.