Lec 35 Monday Nov 30

Recall bialgebras

$U = U(\mathcal{L})$, $\quad \mathcal{L} = sl_2$ $\qquad$ char$(k) = 0$

$H = M(2)$

Recall a duality

$$\langle , \rangle : U \times H \to k$$

is bilinear and satisfies

$$\langle uv, x \rangle = \sum_{(x)} \langle u, x' \rangle \langle v, x'' \rangle \qquad u, v \in U$$

$$\langle u, xy \rangle = \sum_{(u)} \langle u', x \rangle \langle u'', y \rangle \qquad x, y \in H$$

$$\langle u, 1 \rangle = \varepsilon(u)$$

$$\langle 1, x \rangle = \varepsilon(x)$$

Next goal: Show $\exists$ a duality s.t

$$\begin{pmatrix} \langle e, a \rangle & \langle e, b \rangle \\ \langle e, c \rangle & \langle e, d \rangle \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} \langle f, a \rangle & \langle f, b \rangle \\ \langle f, c \rangle & \langle f, d \rangle \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \langle h, a \rangle & \langle h, b \rangle \\ \langle h, c \rangle & \langle h, d \rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

To show that the standard duality exists we work with another basis for $M(2)$

Recall the $M(2)$ - comodule alg $\underset{\parallel}{K[x,y]}$ A

Recall

$$\Delta_A : \begin{array}{ccc} A & \longrightarrow & M(2)\otimes A \\ x & \longrightarrow & a\otimes x + b\otimes y \\ y & \longrightarrow & c\otimes x + d\otimes y \end{array}$$

For $n \in \mathbb{N}$ and $0 \leq i,j \leq n$ define

$$X_{ij}^n \in M(2)$$

by

$$\Delta_A(x^{n-i}y^i) = \sum_{j=0}^{n} X_{ij}^n \otimes (x^{n-j}y^j) \qquad 0 \leq i \leq n$$

Using

$$\Delta_A(x^{n-i}y^i) = (\Delta_A(x))^{n-i}(\Delta_A(y))^i$$

$$= (a\otimes x + b\otimes y)^{n-i}(c\otimes x + d\otimes y)^i$$

we find

$$X^n_{ij} = \sum_{z} \binom{i}{z}\binom{n-i}{j-z} a^{n-i-j+z} b^{j-z} c^{i-z} d^{z}$$

sum over $z \in \mathbb{N}$ s.t.

$$i+j-n \le z \le i, j$$

For $n \in \mathbb{N}$ and $0 \le i, j \le n$ we see that

as a polynomial in a, b, c, d,

X^n_{ij} is homogeneous with total degree n

the elements

X^n_{ij} $\quad 0 \le i, j \le n$

are lin independent, since for

X^n_{ij} $\quad X^n_{i'j'}$ $\quad (i,j) \ne (i',j')$

the z-summands in X^n_{ij} are distinct from

the corresp summands in $X^n_{i'j'}$

n	$(x^n_{ij})_{0\le i,j\le n}$
0	1
1	$\begin{matrix} a & b \\ c & d \end{matrix}$
2	$\begin{matrix} a^2 & 2ab & b^2 \\ ac & ad+bc & bd \\ c^2 & 2cd & d^2 \end{matrix}$
3	$\begin{matrix} a^3 & 3a^2b & 3ab^2 & b^3 \\ a^2c & a(ad+2bc) & b(2ad+bc) & b^2d \\ ac^2 & c(2ad+bc) & d(ad+2bc) & bd^2 \\ c^3 & 3c^2d & 3cd^2 & d^3 \end{matrix}$
⋮	$\cdots$

For the standard duality $\langle\, ,\, \rangle$,

$$\left(\langle e, x_{ij}^n \rangle \right)_{0 \le i,j \le n} = \begin{pmatrix} 0 & n & & & & \\ & 0 & n-1 & & & \\ & & 0 & \ddots & & \\ & & & \ddots & 2 & \\ & & & & 0 & 1 \\ & & & & & 0 \end{pmatrix}$$

$$\left(\langle f, x_{ij}^n \rangle \right)_{0 \le i,j \le n} = \begin{pmatrix} 0 & & & & & \\ 1 & 0 & & & & \\ & 2 & 0 & & & \\ & & \ddots & \ddots & & \\ & & & n-1 & 0 & \\ & & & & n & 0 \end{pmatrix}$$

$$\left(\langle h, x_{ij}^n \rangle \right)_{0 \le i,j \le n} = \mathrm{diag}\,(n, n-2, n-4, \ldots, -n)$$

Recall

$$\det = ad - bc$$

Prop 17 The following is a basis for $M(2)$:

$$x_{ij}^{n} \det^{t} \qquad i, j, n, t \in \mathbb{N} \qquad 0 \le i, j \le n$$

pf For $H = M(2)$ consider the grading

$$H = \sum_{n \in \mathbb{N}} \underset{\uparrow \text{ nth homog comp}}{H_n} \qquad \text{ds}$$

H_n has basis

$$a^i b^j c^k d^l \qquad i + j + k + l = n$$

Note $\dim H_n = \binom{n+3}{3}$

Define

$$\Phi = \frac{\partial}{\partial a} \frac{\partial}{\partial d} - \frac{\partial}{\partial b} \frac{\partial}{\partial c}$$

$\Phi : H \to H$ is linear

$$\Phi(H_0) = 0, \qquad \Phi(H_1) = 0$$

For $n \geq 2$,

$$\Phi(H_n) \subseteq H_{n-2}$$

Define

$$\widetilde{H_n} = \text{kernel of } \Phi \text{ on } H_n$$
$$= \{ h \in H_n \mid \Phi(h) = 0 \}$$

Φ sends

$$a^i b^j c^k d^l \longrightarrow il\, a^{i-1} b^j c^k d^{l-1} - jk\, a^i b^{j-1} c^{k-1} d^l \qquad i,j,k,l \in \mathbb{N}$$

Using this one checks

$$\Phi(x_{\hat{ij}}) = 0 \qquad 0 \leq i,j \leq n$$

so

$$x_{\hat{ij}} \in \widetilde{H_n} \qquad 0 \leq i,j \leq n$$

Also $\forall\, h \in H_n$,

$$\Phi(h \det) = \Phi(h)\det + (n+2)h \qquad (*)$$

Iterating this we get

$$\Phi(h \det^t) = \Phi(h)\det^t + t(n+t+1)\, h \det^{t-1} \qquad t \in \mathbb{N}$$

So for $n \in \mathbb{N}$ and $t \geq 1$,

$$\Phi(h \det{}^t) = t(n+t+1)\, h \det{}^{t-1} \qquad \forall h \in \tilde{H}_n$$

So the action of Φ on $\tilde{H}_n \det^t$ gives a bijection

$$\tilde{H}_n \det{}^t \longrightarrow \tilde{H}_n \det{}^{t-1}$$

By this and induction on n we find

$$H_n = \tilde{H}_n + \tilde{H}_{n-2} \det + \tilde{H}_{n-4} \det{}^2 + \cdots \qquad (ds) \quad *$$

So

$$H_n = \tilde{H}_n + H_{n-2} \det \qquad (ds)$$

So
$$\dim \tilde{H}_n = \dim H_n - \dim H_{n-2}$$
$$= \binom{n+3}{3} - \binom{n+1}{3}$$
$$= (n+1)^2$$

Now $\tilde{H}_n$ has basis

$$x^n_{ij} \qquad 0 \leq i, j \leq n$$

Result follows from this and $*$ $\square$

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We now compute the action of Δ, ε on the basis in Prop 17.

Recall
$$\Delta(\det) = \det \otimes \det$$
$$\varepsilon(\det) = 1$$

LEM 18 For $n \in \mathbb{N}$ and $0 \le i, j \le n$,

$$\Delta(x^n_{ij}) = \sum_{l=0}^{n} x^n_{il} \otimes x^n_{lj}$$

pf This diag commutes

$$\begin{tikzcd}
H \otimes H \otimes A & H \otimes A \arrow[l, "id \otimes \Delta_A"'] \\
H \otimes A \arrow[u, "\Delta \otimes id"] & A \arrow[l, "\Delta_A"] \arrow[u, "\Delta_A"']
\end{tikzcd}
\qquad
\begin{array}{l} A = k[x, y] \\ H = M(2) \end{array}$$

$$\begin{tikzcd}
\displaystyle\sum_{l=0}^{n} x^n_{il} \sum_{j=0}^{n} x^n_{lj} \otimes x^{n-j}y^j & \displaystyle\sum_{l=0}^{n} x^n_{il} \otimes x^{n-l}y^l \arrow[l] \\
\displaystyle\sum_{j=0}^{n} \Big(\sum_{l=0}^{n} x^n_{il} \otimes x^n_{lj} \Big) \otimes x^{n-j}y^j \arrow[u, equal] & \\
\displaystyle\sum_{j=0}^{n} \Delta(x^n_{ij}) \otimes x^{n-j}y^j \arrow[u, equal, "\text{Require}"] & \\
\displaystyle\sum_{j=0}^{n} x^n_{ij} \otimes x^{n-j}y^j \arrow[u] & x^{n-i}y^i \arrow[uuu]
\end{tikzcd}$$

Result follows $\square$

COR 19 For $n, t \in \mathbb{N}$ and $0 \le i, j \le n$

$$\Delta\left(x^n_{ij} \det^t\right) = \sum_{l=0}^{n} x^n_{il} \det^t \otimes x^n_{lj} \det^t$$

pf

Δ is alg morphism so

$$\Delta\left(x^n_{ij} \det^t\right) = \Delta\left(x^n_{ij}\right) \Delta(\det)^t$$

$$= \left(\sum_{l=0}^{n} x^n_{il} \otimes x^n_{lj}\right) \det^t \otimes \det^t$$

$$= \sum_{l=0}^{n} x^n_{il} \det^t \otimes x^n_{lj} \det^t \qquad \square$$

LEM 20 For $n, t \in \mathbb{N}$ and $0 \le i, j \le n$

$$\varepsilon(X^n_{ij} \det^t) = \delta_{ij}$$

pf ε is alg morphism so

$$\varepsilon(X^n_{ij} \det^t) = \varepsilon(X^n_{ij}) \; \varepsilon(\det)^t$$

$$\varepsilon(\det) = 1$$

$$\varepsilon(X^n_{ij}) = \varepsilon \sum_z \binom{i}{z}\binom{n-i}{j-z} a^{n-i-j+z} b^{j-z} c^{i-z} d^z$$

$$\left[\varepsilon(a) = 1 \quad \varepsilon(b) = 0 \quad \begin{matrix} \varepsilon(c) = 0 \\ \varepsilon(d) = 1 \end{matrix} \right]$$

$$\forall z \quad \varepsilon(z\text{-summand}) = \begin{matrix} 0 \text{ unless} \\ j = z = i \end{matrix}$$

$$\varepsilon(X^n_{ij}) = \binom{i}{i}\binom{n-i}{0} \delta_{ij}$$

$$= \delta_{ij}$$

$\square$

LEM 21 For the standard duality, for $\phi \in \mathcal{L}$,

(i) $\langle \phi, \det \rangle = 0$

(ii) $\langle \phi, x \det^t \rangle = \langle \phi, x \rangle$ $\quad \forall x \in M(2)$, $\forall t \in \mathbb{N}$

pf (i) Check it for $\phi = e, f, h$

(ii) $\langle \phi, x \det^t \rangle = \langle \phi, x \rangle \underbrace{\varepsilon(\det^t)}_{=1} + t \underbrace{\langle \phi, \det \rangle}_{=0} \varepsilon(x) \varepsilon(\det)^{t-1}$

$= \langle \phi, x \rangle$ $\square$

Given any duality $\langle\ ,\ \rangle : \mathcal{U} \times H \to K$

$\mathcal{U} = \mathcal{U}(\mathfrak{L}) \quad H = M(2)$

Given $u, v \in \mathcal{U}$

Given $n, t \in \mathbb{N}$ and $0 \le i, j \le n$,

by Cor 19,

$$\langle uv, x_{ij}^n \det^t \rangle = \sum_{l=0}^{n} \langle u, x_{il}^n \det^t \rangle \langle v, x_{lj}^n \det^t \rangle$$

In other words the matrix

$$\left(\langle uv, x_{ij}^n \det^t \rangle \right)_{0 \le i,j \le n} = \left(\langle u, x_{ij}^n \det^t \rangle \right)_{0 \le i,j \le n} \left(\langle v, x_{ij}^n \det^t \rangle \right)_{0 \le i,j \le n}$$

So the map

$$\begin{aligned} \mathcal{U} &\longrightarrow \mathrm{Mat}_{n+1}(K) \\ u &\longrightarrow \left(\langle u, x_{ij}^n \det^t \rangle \right)_{0 \le i,j \le n} \end{aligned}$$

is an alg morphism

For the standard duality we get

$$\left(\langle e, x_{ij}^n \det^t \rangle \right)_{0 \le i,j \le n} = \begin{pmatrix} 0 & n & & & & \\ & 0 & n-1 & & & 0 \\ & & 0 & \ddots & & \\ & & & \ddots & \ddots & \\ & 0 & & & 0 & 2 \\ & & & & & 0 & 1 \\ & & & & & & 0 \end{pmatrix}$$

$$\left(\langle f, x_{ij}^n \det^t \rangle \right)_{0 \le i,j \le n} = \begin{pmatrix} 0 & & & & & \\ 1 & 0 & & & & 0 \\ & 2 & 0 & & & \\ & & \ddots & \ddots & & \\ & 0 & & \ddots & 0 & \\ & & & & n-1 & 0 \\ & & & & & n & 0 \end{pmatrix}$$

$$\left(\langle h, x_{ij}^n \det^t \rangle \right)_{0 \le i,j \le n} = \operatorname{diag}(n, n-2, n-4, \ldots, -n)$$

The above matrices give a module str for U that is iso $L(n)$

We now show the standard duality exists

Thm 22 For the bialgebras

$U = U(\mathcal{L})$, $\mathcal{L} = sl_2$ char$(K) = 2$

$H = M(2)$

$\exists$ unique duality $\langle , \rangle : U \times H \to K$

s.t.

$\langle , \rangle$	a	b	c	d
e	0	1	0	0
f	0	0	1	0
h	1	0	0	-1

☆

pf Recall a basis for $H = M(2)$:

$x_{ij}^n \det^t$ $\quad i, j, n, t \in \mathbb{N} \quad 0 \le i, j \le n$

Define $e^\vee, f^\vee, h^\vee \in H^*$ as follows:

For $n, t \in \mathbb{N}$

$$\left(e^\vee (x_{ij}^n \det^t) \right)_{0 \le i, j \le n} = \begin{pmatrix} 0 & n & & & 0 \\ & 0 & n-1 & & \\ & & 0 & \ddots & \\ & & & \ddots & 1 \\ 0 & & & & 0 \end{pmatrix}$$

$$\left(f^\vee (x_{ij}^n \det^t) \right)_{0 \le i, j \le n} = \begin{pmatrix} 0 & & & & 0 \\ 1 & 0 & & & \\ & 2 & 0 & & \\ & & \ddots & \ddots & \\ 0 & & & n & 0 \end{pmatrix}$$

$$\left(h^\vee (x_{ij}^n \det^t) \right)_{0 \le i, j \le n} = \operatorname{diag}(n, n-2, n-4, \dots, -n)$$

In the algebra H^*,

$$h^\vee e^\vee - e^\vee h^\vee = 2 e^\vee, \qquad h^\vee f^\vee - f^\vee h^\vee = -2 f^\vee$$

$$e^\vee f^\vee - f^\vee e^\vee = h^\vee$$

So $\exists$ alg morphism $\varphi : U \to H^*$ that sends

$e \to e^\vee \qquad f \to f^\vee \qquad h \to h^\vee$

Define

$$\langle\, , \rangle : \begin{array}{ccc} U \times H & \to & k \\ u, x & \to & \varphi(u)(x) \end{array}$$

So $\langle\, , \rangle$ is bilinear

By constr $\langle\, , \rangle$ satisfies ☆

Since φ is an alg morphism,

$$\langle uv, x \rangle = \sum_{(x)} \langle u, x' \rangle \langle v, x'' \rangle \qquad u, v \in U \quad x \in H$$

$$\langle 1, x \rangle = \varepsilon(x)$$

Show $\langle u, 1 \rangle = \varepsilon(u) \qquad u \in U$

By constr
$$\langle e, 1 \rangle = 0 = \varepsilon(e)$$
$$\langle f, 1 \rangle = 0 = \varepsilon(f)$$
$$\langle h, 1 \rangle = 0 = \varepsilon(h)$$

Also for $u, v \in U$
$$\langle uv, 1 \rangle = \langle u, 1 \rangle \langle v, 1 \rangle$$
Since $\Delta(1) = 1 \otimes 1$

So the map
$$\begin{array}{ccc} U & \longrightarrow & K \\ u & \longrightarrow & \langle u, 1 \rangle \end{array}$$

is an alg morphism that agrees with ε at $u = e, f, h$. So this alg morphism is equal to ε.

For $u \in U$ show

$$\langle u, xy \rangle = \sum_{(u)} \langle u', x \rangle \langle u'', y \rangle \quad \forall x, y \in H$$

" Condition $C(u)$ "

By cnstr $C(e)$, $C(f)$, $C(h)$ hold

For $u, v \in U$ s.t. $C(u)$, $C(v)$ hold, show $C(uv)$ holds.

$$\langle uv, xy \rangle \overset{?}{=} \sum_{(uv)} \langle (uv)', x \rangle \langle (uv)'', y \rangle$$

$$= \quad [\Delta(uv) = \Delta(u)\Delta(v)]$$

$$\sum_{(u)} \sum_{(v)} \langle u'v', x \rangle \langle u''v'', y \rangle$$

$$=$$

$$\sum_{(u)} \sum_{(v)} \sum_{(x)} \sum_{(y)} \langle u', x' \rangle \langle v', x'' \rangle \langle u'', y' \rangle \langle v'', y'' \rangle$$

$$=$$

$$\sum_{(xy)} \langle u, (xy)' \rangle \langle v, (xy)'' \rangle$$

$$= \quad [\Delta(xy) = \Delta(x)\Delta(y)]$$

$$\sum_{(x)} \sum_{(y)} \langle u, x'y' \rangle \langle v, x''y'' \rangle$$

// ok

$$= \quad C(u) + C(v)$$

$$\sum_{(u)} \sum_{(v)} \sum_{(x)} \sum_{(y)} \langle u', x' \rangle \langle u'', y' \rangle \langle v', x'' \rangle \langle v'', y'' \rangle$$

We have shown $C(u)$ holds for all $u \in U$

We have shown that $\langle , \rangle$ is a duality that satisfies ☆

One checks $\langle , \rangle$ is unique. $\square$

let I = ideal of $M(2)$ gen by

$$ad-bc-1$$

LEM 23 For the duality $\langle\ ,\ \rangle$ in thm 22,

$$\langle U, I\rangle = 0$$

pf <u>claim 1</u>

$$\langle 1, I\rangle = 0$$

<u>pf cl 1</u> $\forall x \in H$,

$$\langle 1, (\det -1)x\rangle = \varepsilon((\det -1)x)$$

$$= \varepsilon(\det -1)\,\varepsilon(x)$$

$$= (\underbrace{\varepsilon(\det)}_{1} - \underbrace{\varepsilon(1)}_{1})\,\varepsilon(x)$$

$$= 0$$

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claim 2 $\quad \langle \phi, I \rangle = 0 \quad \forall \phi \in \mathcal{L}$

pf cl 2 $\quad \forall x \in H$ recall

$$\langle \phi, x \det \rangle = \langle \phi, x \rangle$$

so $\langle \phi, (\det - 1)x \rangle = 0$

claim 3 $\quad \Delta(I) \subseteq I \otimes H + H \otimes I$

pf cl 3 $\quad \forall x \in H$

$$\Delta((\det - 1)x) = \Delta(\det - 1)\,\Delta(x)$$

$$= (\det \otimes \det - 1 \otimes 1) \sum_{(x)} x' \otimes x''$$

$$\det \otimes \det - 1 \otimes 1 = (\det - 1) \otimes \det + 1 \otimes (\det - 1)$$

$$= \sum_{(x)} \left((\det - 1)x' \otimes \det x'' + x' \otimes (\det - 1)x'' \right)$$

$$\in I \otimes H + H \otimes I \checkmark$$

claim 4 Given $u, v \in \mathcal{U}$ s.t

$$\langle u, I \rangle = 0 = \langle v, I \rangle$$

then

$$\langle uv, I \rangle = 0$$

pf cl 4 $\forall x \in I$

$$\langle uv, x \rangle = \sum_{(x)} \underbrace{\langle u, x' \rangle \langle v, x'' \rangle}_{\substack{\text{// by cl 3, } x' \in I \text{ or } x'' \in I \\ 0}}$$

$$= 0$$

Result follows $\square$

Recall

$I =$ ideal of $M(2)$ gen by $ad - bc - 1$

$$SL(2) = M(2)/I$$

By LEM 23 and Constr, the duality $\langle, \rangle$ in thm 22 induces a duality

$$\langle, \rangle : \mathcal{U} \times SL(2) \to k.$$

Recall $SL(2)$ is Hopf alg. It antipode

S satisfies

$$S(a) = d \qquad S(b) = -b$$
$$S(c) = -c \qquad S(d) = a.$$

Recall $\mathcal{U}$ is also a Hopf alg. Its antipode

S satisfies

$$S(\phi) = -\phi \qquad \forall \phi \in \mathcal{L}$$

LEM 24 For $u \in \mathcal{U}$ and $x \in SL(2)$,

$$\langle S(u), x \rangle = \langle u, S(x) \rangle \qquad *$$

pf One checks $*$ holds for $u \in \mathcal{L}$ and $x \in \{a, b, c, d\}$

Given $u \in \mathcal{L}$ and $x, y \in SL(2)$ s.t

$$\langle S(u), x \rangle = \langle u, S(x) \rangle$$
$$\langle S(u), y \rangle = \langle u, S(y) \rangle$$

Show $\langle S(u), xy \rangle \overset{?}{=} \langle u, S(xy) \rangle$

$$\begin{aligned}
\langle S(u), xy \rangle &= -\langle u, xy \rangle \\
&= -\sum_{(u)} \langle u', x \rangle \langle u'', y \rangle \\
&= -\langle u, x \rangle \varepsilon(y) - \langle u, y \rangle \varepsilon(x)
\end{aligned}$$

$$\begin{aligned}
\langle u, S(xy) \rangle &= \langle u, S(y) S(x) \rangle \\
&= \sum_{(u)} \langle u', S(y) \rangle \langle u'', S(x) \rangle \\
&= \langle u, S(y) \rangle \overbrace{\varepsilon(S(x))}^{\varepsilon(x)} + \langle u, S(x) \rangle \underbrace{\varepsilon(S(y))}_{= \varepsilon(y)} \\
&= \langle \underbrace{S(u)}_{= -u}, y \rangle \varepsilon(x) + \langle \underbrace{S(u)}_{= -u}, x \rangle \varepsilon(y)
\end{aligned}$$

= ok.

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So far, * holds $\forall u \in \mathcal{L}$ $\forall x \in SL(2)$.

Given $u, v \in \mathcal{U}$ s.t.

$$\langle S(u), x \rangle = \langle u, S(x) \rangle \quad \forall x \in SL(2)$$

$$\langle S(v), x \rangle = \langle v, S(x) \rangle \quad \forall x \in SL(2)$$

show $\langle S(uv), x \rangle \overset{?}{=} \langle uv, S(x) \rangle \quad \forall x \in SL(2)$

$$\langle S(uv), x \rangle = \langle S(v) S(u), x \rangle = \sum_{(x)} \langle S(v), x' \rangle \langle S(u), x'' \rangle = \sum_{(x)} \langle v, S(x') \rangle \langle u, S(x'') \rangle$$

$$\langle uv, S(x) \rangle = \sum_{(S(x))} \langle u, S(x)' \rangle \langle v, S(x)'' \rangle \overset{[\Delta^{op} \circ S \otimes S = S \circ \Delta]}{=} \sum_{(x)} \langle u, S(x'') \rangle \langle v, S(x') \rangle$$

$=$ ok.

Result follows. $\square$

$U = U(\mathcal{L})$ $\quad \mathcal{L} = sl_2$ $\quad H = SL(2)$

$A = K[x,y]$

Recall grading $\quad A = \sum_{n \in \mathbb{N}} A_n \quad$ ds.

Recall A_n is H-comodule

$$\Delta_A : \begin{array}{ccc} A_n & \longrightarrow & H \otimes A_n \\ x^{n-i} y^i & \longrightarrow & \sum_{j=0}^{n} x^n_{ij} \otimes x^{n-j} y^j \\ \parallel & & \\ u_i & & \end{array} \qquad 0 \le i \le n$$

So A_n^* becomes an H^*-module:

$$H^* \otimes A_n^* \longrightarrow A_n^* \otimes H^* \simeq (H \otimes A_n)^* \xrightarrow[\Delta_A^*]{} A_n^*$$

The duality $\langle \, , \rangle : U \times H \to K$ gives an algmorphism

$$\varphi : U \longrightarrow H^*$$

Now A_n^* becomes a U-module:

$$U \otimes A_n^* \xrightarrow[\varphi \otimes id]{} H^* \otimes A_n^* \xrightarrow[T]{} A_n^* \otimes H^* \simeq (H \otimes A_n)^* \xrightarrow[\Delta_A^*]{} A_n^*$$

Describe the U-module A_n^*.

Consider basis $\{u^i\}_{i=0}^n$ for A_n^* dual to $\{u_i\}_{i=0}^n$

Prop 25 With above notation,

$$h u^i = (n-2i) u^i \qquad (0 \le i \le n),$$
$$f u^i = (i+1) u^{i+1} \qquad (0 \le i \le n-1), \qquad f u^n = 0,$$
$$e u^i = (n-i+1) u^{i-1} \qquad (1 \le i \le n), \qquad e u^0 = 0.$$

Moreover the U-module A_n^* is iso $L(n)$.

pf For $\phi \in \mathcal{L}$ and $0 \le j \le n$ show ϕ sends

$$u^j \longrightarrow \sum_{i=0}^n \langle \phi, x_{ij}^n \rangle u^i$$

check:

$$\phi \otimes u^j \longrightarrow \varphi(\phi) \otimes u^j \to u^j \otimes \varphi(\phi) \simeq F \longrightarrow \Delta_A^* F \overset{?}{=} \sum_{i=0}^n \langle \phi, x_{ij}^n \rangle u^i$$

$$\Delta_A^* F \overset{?}{=} \sum_{i=0}^{n} \langle \phi, x^n_{ij} \rangle u^i$$

For $0 \le \ell \le n$,

$$\langle \Delta_A^* F, u_\ell \rangle \overset{?}{=} \left\langle \sum_{i=0}^{n} \langle \phi, x^n_{ij} \rangle u^i, u_\ell \right\rangle$$

$$= \langle \phi, x^n_{\ell j} \rangle$$

$$\langle \Delta_A^* F, u_\ell \rangle = \langle F, \Delta_A u_\ell \rangle$$

$$= \left\langle F, \sum_{r=0}^{n} x^n_{\ell r} \otimes u_r \right\rangle$$

$$= \sum_{r=0}^{n} \underbrace{\varphi(\phi)(x^n_{\ell r})}_{= \langle \phi, x^n_{\ell r} \rangle} \underbrace{u^j(u_r)}_{= \delta_{jr}}$$

$$= \langle \phi, x^n_{\ell j} \rangle$$

ok

$\square$