Lec 34 Friday Nov 20 11/20/15

1

Given a vector space V with $\dim(V) = 2$

Recall Lie alg $\mathfrak{sl}(V)$

Lie alg $\mathfrak{sl}_2$ is matrix version of $\mathfrak{sl}(V)$

with basis

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and

$$[h, e] = 2e \qquad [h, f] = -2f \qquad [e, f] = h$$

LEM 12 Assume $\mathrm{char}(k) = 0$. $\exists$ family of f.d. irred $U(\mathfrak{sl}_2)$-modules

$$L(n) \qquad n \in \mathbb{N}$$

with the following properties: *

$\exists$ basis $\{v_i\}_{i=0}^{n}$ for $L(n)$ such that

$$h v_i = (n-2i) v_i \qquad (0 \le i \le n)$$

$$f v_i = (i+1) v_{i+1} \qquad (0 \le i \le n-1), \qquad f v_n = 0$$

$$e v_i = (n-i+1) v_{i-1} \qquad (1 \le i \le n), \qquad e v_0 = 0$$

Each f.d. irred $U(\mathfrak{sl}_2)$-module is iso to exactly one / *

pf (ex)

Next goal: turn the algebra $A = k[x, y]$ into a $U(\mathcal{L})$-module algebra for $\mathcal{L} = sl_2$.

Recall the grading

$$A = \sum_{n \in \mathbb{N}} A_n \qquad (ds)$$

$A_n =$ nth homog comp

So A_1 has basis x, y

A_1 is a $U(\mathcal{L})$-module with action

$ex = 0 \qquad ey = x$

$fx = y \qquad fy = 0$

$hx = x \qquad hy = -y$

For $d \in \mathcal{L}$ the action of d on A_1 extends uniquely to a derivation $\bar{d}$ of A via LEM 11:

$$\bar{d}(x^r y^s) = d(x)\, r x^{r-1} y^s + d(y)\, s x^r y^{s-1} \qquad r, s \in \mathbb{N}$$

we haves

d	$d(x)$	$d(y)$	$\bar{d}(x^r y^s)$	$\bar{d}$ desc
e	0	x	$s x^{r+1} y^{s-1}$	$x \frac{\partial}{\partial y}$
f	y	0	$r x^{r-1} y^{s+1}$	$y \frac{\partial}{\partial x}$
h	x	$-y$	$(r-s) x^r y^s$	$x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}$

Show the map

$$\mathcal{L} \longrightarrow Der(A)$$
$$d \longrightarrow \bar{d}$$

is a Lie algebra morphism.

For $d, \delta \in \mathcal{L}$ show

$$\overline{[d, \delta]} = [\bar{d}, \bar{\delta}] \qquad *$$

Both sides of $*$ are in $Der(A)$

Suf to show both sides of $*$ agree on A_1

$\overline{[d,\delta]}$ acts on A_1 as $[d, \delta] = d\delta - \delta d$

Also $\bar{d}$ acts on A_1 as d

$\bar{\delta}$ - - δ

So $[\bar{d}, \bar{\delta}]$ acts on A_1 as $d\delta - \delta d$

So both sides of $*$ agree on A_1 ✓

By above comments A becomes a $U(\mathcal{L})$-module on which each element of $\mathcal{L}$ acts as a derivation.

So A becomes a $U(\mathcal{L})$-module algebra.

For $n \in \mathbb{N}$, A_n has basis

$$x^r y^s \qquad r, s \in \mathbb{N} \qquad r+s=n$$

Basis is

$$x^n, \quad x^{n-1}y, \quad x^{n-2}y^2, \quad \dots, \quad y^n$$

Each of e, f, h leaves A_n invariant.

So A_n is $U(\mathcal{L})$-submodule of A.

Describe $U(\mathcal{L})$-module A_n.

For notational conv assume $\mathrm{char}(\mathbb{K})=0$.

Define

$$v_i = \frac{x^{n-i}}{(n-i)!}\frac{y^i}{i!} \qquad 0 \le i \le n$$

Then $\{v_i\}_{i=0}^{n}$ is a basis for A_n

$$h v_i = (n-2i) v_i \qquad (0 \le i \le n)$$

$$f v_i = (i+1) v_{i+1} \qquad (0 \le i \le n-1), \qquad f v_n = 0$$

$$e v_i = (n-i+1) v_{i-1} \qquad (1 \le i \le n), \qquad e v_0 = 0$$

So the $U(\mathcal{L})$-module A_n is iso to $L(n)$.

In particular the $U(\mathcal{L})$-module A_n is irred.

We have shown how $K[x,y]$ is a

comodule algebra for $M(2)$

and module algebra for $U(sl_2)$

We now consider how $U(sl_2)$, $M(2)$ are related

DEF 13 Given bialgebras U, H

A <u>duality</u> between U, H is a bilinear map

$$\langle\, , \rangle \quad U \times H \longrightarrow K$$

such that for $u, v \in U$ and $x, y \in H$,

$$\text{(i)} \quad \langle uv, x \rangle = \sum_{(x)} \langle u, x' \rangle \langle v, x'' \rangle$$

$$\text{(ii)} \quad \langle u, xy \rangle = \sum_{(u)} \langle u', x \rangle \langle u'', y \rangle$$

$$\text{(iii)} \quad \langle u, 1 \rangle = \underset{\varepsilon_U}{\underset{\uparrow}{\varepsilon}}(u)$$

$$\text{(iv)} \quad \langle 1, x \rangle = \underset{\varepsilon_H}{\underset{\uparrow}{\varepsilon}}(x)$$

Given bialgebras U, H

Given any bilin map

$$\langle\, ,\, \rangle : U \times H \to K$$

Define lin maps

$$\varphi : U \to H^* \qquad \varphi(u)(x) = \langle u, x \rangle$$
$$u \to \varphi(u) \qquad \forall x \in H$$

$$\psi : H \to U^* \qquad \psi(x)(u) = \langle u, x \rangle$$
$$x \to \psi(x) \qquad \forall u \in U$$

LEM 14 With above notation TFAE

(i) $\langle\, ,\, \rangle$ is a duality

(ii) Each of φ, ψ is an algebra morphism

$(i') \to (ii')$ Show φ is alg morph

Recall product in H^*

$\forall f, g \in H^*$

$$(fg)(x) = \sum_{(x)} f(x')g(x'') \qquad x \in H$$

For $u, v \in U$ show

$$\varphi(uv) \overset{?}{=} \varphi(u)\varphi(v)$$

Fa $x \in H$ show

$$\varphi(uv)(x) \overset{?}{=} \varphi(u)\varphi(v)(x)$$

$$\varphi(uv)(x) = \langle uv, x\rangle$$

$$\varphi(u)\varphi(v)(x) = \sum_{(x)} \varphi(u)(x')\,\varphi(v)(x'') = \sum_{(x)} \langle u, x'\rangle\langle v, x''\rangle$$

$$\langle uv, x\rangle \overset{ok}{=} \sum_{(x)} \langle u, x'\rangle\langle v, x''\rangle$$

Show

$$\varphi(1_U) \overset{?}{=} 1_{H^*} = \varepsilon_H$$

Fa $x \in H$

$$\varphi(1_U)(x) \overset{?}{=} \varepsilon_H(x)$$

$$\varphi(1_U)(x) = \langle 1_U, x\rangle = \varepsilon_H(x) \quad \text{ok}$$

We have shown φ is alg morph.

Similarly ψ is alg morph.

$(ii) \to (i)$ Reverse direction above $\square$

Here is a variation on LEM 14

LEM 15 Given bialgebras U, H with H fin dim'l (so bialg H^* exists)

TFAE

(i) $\langle , \rangle$ is a duality

(ii) φ is bialgebra morphism

pf (i) $\to$ (ii) By LEM 14 φ is alg morph.

Show φ is coalg morph.

Show this diag commutes:

$$\begin{array}{ccc} U & \xrightarrow{\varphi} & H^* \\ \Delta \downarrow & ? & \downarrow \Delta \\ U \otimes U & \xrightarrow[\varphi \otimes \varphi]{} & H^* \otimes H^* \end{array}$$

Recall Δ for H^*:

$\forall f \in H^*$

$$f(xy) = \sum_{(f)} f'(x)\, f''(y) \qquad x, y \in H$$

For $u \in U$,

$$\begin{tikzcd} u \arrow[r] \arrow[d] & \varphi(u) \arrow[d, "?"] \\ \sum_{(u)} u' \otimes u'' \arrow[r] & \sum_{(u)} \varphi(u') \otimes \varphi(u'') \end{tikzcd}$$

Require

$$\varphi(u)(xy) \overset{?}{=} \sum_{(u)} \underbrace{\varphi(u')(x)}_{\langle u', x\rangle} \underbrace{\varphi(u'')(y)}_{\langle u'', y\rangle}$$

$$\begin{aligned} &\| \\ &\langle u, xy \rangle \\ &\| \\ &\sum_{(u)} \langle u', x\rangle \langle u'', y\rangle \end{aligned}$$

ok

Show this diag commutes:

$$\begin{tikzcd} U \arrow[r, "\varphi"] \arrow[d, "\varepsilon"'] \arrow[dr, phantom, "?"] & H^* \arrow[d, "\varepsilon"] \\ K \arrow[r, "id"'] & K \end{tikzcd}$$

$$\begin{tikzcd} u \arrow[r] \arrow[d] & \varphi(u) \arrow[d] \\ \varepsilon(u) \arrow[r, "?"'] & \varepsilon(\varphi(u)) \end{tikzcd}$$

$$\begin{aligned} \varepsilon(\varphi(u)) &= \varphi(u)(1) \\ &= \langle u, 1 \rangle \\ &= \varepsilon(u) \end{aligned}$$

ok

$\square$

(iii) → (i) Retrace the above steps

LEM 16 Given bialgebras U, H s.t U is fin dim'l. TFAE:

(i) $\langle , \rangle$ is a duality

(ii) φ is bialgebra morphism.

pf Swap roles of U, H in LEM 15 $\square$

Until further notice assume

$U = U(\mathcal{L})$ $\qquad \mathcal{L} = sl_2$

$H = M(2)$

$char(k) = 0$

Recall

	$U(\mathcal{L})$	$M(2)$
gens	e, f, h	a, b, c, d
Δ	$\Delta(e) = e\otimes 1 + 1\otimes e$ $\Delta(f) = f\otimes 1 + 1\otimes f$ $\Delta(h) = h\otimes 1 + 1\otimes h$ $\Delta(1) = 1\otimes 1$	$\Delta(a) = a\otimes a + b\otimes c$ $\Delta(b) = a\otimes b + b\otimes d$ $\Delta(c) = c\otimes a + d\otimes c$ $\Delta(d) = c\otimes b + d\otimes d$ $\Delta(1) = 1\otimes 1$
ε	$\varepsilon(e) = 0$ $\varepsilon(f) = 0$ $\varepsilon(h) = 0$ $\varepsilon(1) = 1$	$\varepsilon(a) = 1$ $\quad \varepsilon(b) = 0$ $\varepsilon(c) = 0$ $\quad \varepsilon(d) = 1$ $\varepsilon(1) = 1$

Next general goal: display a duality

$$\langle\, ,\rangle : \mathcal{U} \times H \to K$$

For the time being assume $\langle\, ,\rangle$ exists and consider implications.

For $x, y \in M(2)$

$$\langle e, xy\rangle = \langle e, x\rangle \,\varepsilon(y) + \langle e, y\rangle\, \varepsilon(x)$$

$$\langle f, xy\rangle = \langle f, x\rangle \,\varepsilon(y) + \langle f, y\rangle\, \varepsilon(x)$$

$$\langle h, xy\rangle = \langle h, x\rangle \,\varepsilon(y) + \langle h, y\rangle\, \varepsilon(x)$$

Iterating we find that for $\phi \in \mathcal{L}$ and $x_1, x_2, \ldots x_n \in M(2)$,

$$\langle \phi, x_1 x_2 \cdots x_n\rangle = \sum_{i=1}^{n} \langle \phi, x_i\rangle\, \varepsilon(x_1)\cdots\varepsilon(x_{i-1})\,\varepsilon(x_{i+1})\cdots\varepsilon(x_n)$$

So for $i, j, k, l \in \mathbb{N}$

$$\langle \phi, a^i b^j c^k d^l\rangle$$

As given in the table below:

j	k	$\langle \phi, a^i b^j c^k d^l \rangle$
0	0	$i \langle \phi, a \rangle + l \langle \phi, d \rangle$
1	0	$\langle \phi, b \rangle$
0	1	$\langle \phi, c \rangle$
$j+k \geq 2$		0

For $u, v \in \mathcal{U} = \mathcal{U}(\mathcal{L})$

$$\langle uv, a\rangle = \langle u, a\rangle\langle v, a\rangle + \langle u, b\rangle\langle v, c\rangle$$

$$\langle uv, b\rangle = \langle u, a\rangle\langle v, b\rangle + \langle u, b\rangle\langle v, d\rangle$$

$$\langle uv, c\rangle = \langle u, c\rangle\langle v, a\rangle + \langle u, d\rangle\langle v, c\rangle$$

$$\langle uv, d\rangle = \langle u, c\rangle\langle v, b\rangle + \langle u, d\rangle\langle v, d\rangle$$

In other words

$$\begin{pmatrix} \langle uv, a\rangle & \langle uv, b\rangle \\ \langle uv, c\rangle & \langle uv, d\rangle \end{pmatrix} = \begin{pmatrix} \langle u, a\rangle & \langle u, b\rangle \\ \langle u, c\rangle & \langle u, d\rangle \end{pmatrix}\begin{pmatrix} \langle v, a\rangle & \langle v, b\rangle \\ \langle v, c\rangle & \langle v, d\rangle \end{pmatrix}$$

So the map

$$\mathcal{U} \longrightarrow Mat_2(K)$$

$$u \longrightarrow \begin{pmatrix} \langle u, a\rangle & \langle u, b\rangle \\ \langle u, c\rangle & \langle u, d\rangle \end{pmatrix}$$

is an algebra morphism

De hne

$$E = \begin{pmatrix} \langle e, a \rangle & \langle e, b \rangle \\ \langle e, c \rangle & \langle e, d \rangle \end{pmatrix}$$

$$F = \begin{pmatrix} \langle f, a \rangle & \langle f, b \rangle \\ \langle f, c \rangle & \langle f, d \rangle \end{pmatrix}$$

$$H = \begin{pmatrix} \langle h, a \rangle & \langle h, b \rangle \\ \langle h, c \rangle & \langle h, d \rangle \end{pmatrix}$$

We have

$$HE - EH = 2E$$

$$HF - FH = -2F$$

$$EF - FE = H$$

This suggests that $\langle , \rangle$ can be chosen

s.t.

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

call this the standard duality

The standard duality satisfies

j	k	$\langle e, a^i b^j c^k d^l \rangle$	$\langle f, a^i b^j c^k d^l \rangle$	$\langle h, a^i b^j c^k d^l \rangle$
0	0	0	0	$i-l$
1	0	1	0	0
0	1	0	1	0
$j+k \geq 2$		0	0	0

still need to show standard duality exists.