LEM 6 Given a Lie algebra $\mathcal{L}$ and consider $U(\mathcal{L})$.

For $n \in \mathbb{N}$ and $x_1, x_2, \dots, x_n \in \mathcal{L}$,

$$\Delta(x_1 x_2 \cdots x_n) = \sum_{r=0}^{n} \sum_{\substack{\sigma \in S_n \\ \sigma \text{ is } (r, n-r)\text{-shuffle}}} x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(r)} \otimes x_{\sigma(r+1)} \cdots x_{\sigma(n)}$$

pf Use

$$\Delta(x_1 x_2 \cdots x_n) = \Delta(x_1)\Delta(x_2) \cdots \Delta(x_n).$$

and thm 5 (i) $\square$

Given a vector space V,

Recall the tensor algebra $T(V)$ is a Hopf algebra

with

$$\Delta(x) = x \otimes 1 + 1 \otimes x \qquad x \in V$$

$$\varepsilon(x) = 0 \qquad \text{"}$$

$$S(x) = -x \qquad \text{"}$$

Recall the symmetric algebra $S(V) = T(V)/I$

I = 2-sided ideal of $T(V)$ gen by $xy - yx$ $\quad x, y \in V$

Recall $S(V)$ inherits the Hopf alg str:

$$\Delta : \begin{array}{ccc} S(V) & \to & S(V) \otimes S(V) \\ x & \mapsto & x \otimes 1 + 1 \otimes x \end{array} \qquad x \in V$$

$$\varepsilon : \begin{array}{ccc} S(V) & \to & k \\ x & \mapsto & 0 \end{array} \qquad x \in V$$

$$S : \begin{array}{ccc} S(V) & \to & S(V) \\ x & \mapsto & -x \end{array} \qquad x \in V$$

Given a Lie algebra $\mathcal{L}$.

We cite (without pf) a few facts about $\mathcal{L}$.

Compare:

the univ env algebra $\mathcal{U}(\mathcal{L})$

the sym algebra $S(\mathcal{L})$

Start with tensor algebra

$$T(\mathcal{L}) = \sum_{i \in \mathbb{N}} T^i \qquad (ds)$$

where

$$T^i = \underbrace{\mathcal{L} \otimes \mathcal{L} \otimes \cdots \otimes \mathcal{L}}_{i}$$

Recall

$\{T^0 + T^1 + \cdots + T^i\}_{i \in \mathbb{N}}$ is a filtration of $T(\mathcal{L})$

Recall the canon alg morphism

can: $T(\mathcal{L}) \to \mathcal{U}(\mathcal{L})$

For $i \in \mathbb{N}$ let

$\mathcal{U}_i$ = image of $T^0 + T^1 + \cdots + T^i$ under can

So

$\{U_i\}_{i \in \mathbb{N}}$ is a filtr of $U(\mathcal{L})$

Fact: the corresp graded alg is iso $S(\mathcal{L})$.

For notational convenience assume $\dim \mathcal{L} < \infty$

Pick a basis fr vs $\mathcal{L}$:

$x_1, x_2, \ldots, x_n$

Fact (PBW Thm): the following is a basis fr vs $U(\mathcal{L})$:

$x_1^{r_1} x_2^{r_2} \cdots x_n^{r_n}$ $\quad r_1, r_2, \ldots, r_n \in \mathbb{N}$

Given a Lie algebra $\mathcal{L}$

Assume $\operatorname{char}(k) = 0$

Fact: $\exists$ vector space iso

$$\gamma : S(\mathcal{L}) \longrightarrow U(\mathcal{L})$$

that sends

$$x_1 x_2 \cdots x_n \longrightarrow \frac{1}{n!} \sum_{\sigma \in S_n} x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(n)}$$

$\forall n \in \mathbb{N} \quad \forall \ x_1, x_2, \ldots, x_n \in \mathcal{L}$.

Prop 7 The above map γ is a

Coalgebra iso

pf Show this diag commutes

$$\begin{tikzcd} S(\mathcal{L}) \arrow[r, "\gamma"] \arrow[d, "\Delta"'] & U(\mathcal{L}) \arrow[d, "\Delta"] \\ S(\mathcal{L}) \otimes S(\mathcal{L}) \arrow[r, "\gamma \otimes \gamma"'] & U(\mathcal{L}) \otimes U(\mathcal{L}) \end{tikzcd}$$

Fn $x \in S(\mathcal{L})$ chase x around diag

WLOG $x = x_1 x_2 \cdots x_n$ $n \in \mathbb{N}$ $x_1, x_2, \ldots, x_n \in \mathcal{L}$

$$x_1 x_2 \cdots x_n$$

$$\Delta \downarrow$$

$$\prod_{i=1}^{n} (x_i \otimes 1 + 1 \otimes x_i)$$

$$= \sum_{r=0}^{n} \sum_{\substack{\theta \in S_n \\ \theta \text{ is } (r, n-r)\text{-shuffle}}} x_{\theta(1)} \cdots x_{\theta(r)} \otimes x_{\theta(r+1)} \cdots x_{\theta(n)} \xrightarrow[\gamma \otimes \gamma]{}$$

$$\sum_{r=0}^{n} \sum_{\substack{\theta \in S_n \\ \theta \text{ is } (r, n-r) \text{ shuf}}} \gamma(x_{\theta(1)} \cdots x_{\theta(r)}) \otimes \gamma(x_{\theta(r+1)} \cdots x_{\theta(n)})$$

$$= \sum_{r=0}^{n} \sum_{\sigma \in S_n} \frac{x_{\sigma(1)} \cdots x_{\sigma(r)}}{r!} \otimes \frac{x_{\sigma(r+1)} \cdots x_{\sigma(n)}}{(n-r)!}$$

Also

$$x_1 x_2 \cdots x_n \xrightarrow{\gamma} \frac{1}{n!} \sum_{\tau \in S_n} x_{\tau(1)} x_{\tau(2)} \cdots x_{\tau(n)}$$

$$\downarrow \Delta$$

$$\frac{1}{n!} \sum_{\tau \in S_n} \prod_{i=1}^{n} \left(x_{\tau(i)} \otimes 1 + 1 \otimes x_{\tau(i)} \right)$$

$$= \frac{1}{n!} \sum_{\tau \in S_n} \sum_{r=0}^{n} \sum_{\substack{\theta \in S_n \\ \theta \text{ is } (r,n-r) \text{ shuf}}} x_{\tau(\theta(1))} \cdots x_{\tau(\theta(r))} \otimes x_{\tau(\theta(r+1))} \cdots x_{\tau(\theta(n))}$$

$$= \frac{1}{n!} \sum_{r=0}^{n} \sum_{\substack{\theta \in S_n \\ \theta \text{ is } (r,n-r)\text{-shuf}}} \sum_{\tau \in S_n} x_{\tau(\theta(1))} \cdots x_{\tau(\theta(r))} \otimes x_{\tau(\theta(r+1))} \cdots x_{\tau(\theta(n))}$$

[change vars: $\sigma(i) = \tau(\theta(i))$ for $1 \le i \le n$
As τ ranges over S_n, so does σ]

$$= \frac{1}{n!} \sum_{r=0}^{n} \sum_{\substack{\theta \in S_n \\ \theta \text{ is } (r,n-r)\text{-shuf}}} \underbrace{\sum_{\sigma \in S_n} x_{\sigma(1)} \cdots x_{\sigma(r)} \otimes x_{\sigma(r+1)} \cdots x_{\sigma(n)}}_{\text{indep of } \theta}$$

$$= \frac{1}{n!} \sum_{r=0}^{n} \sum_{\sigma \in S_n} x_{\sigma(1)} \cdots x_{\sigma(r)} \otimes x_{\sigma(r+1)} \cdots x_{\sigma(n)} \binom{n}{r}$$

$\uparrow$
\# $(r,n-r)$ shuffles in S_n

$$= \sum_{r=0}^{n} \sum_{\sigma \in S_n} \frac{x_{\sigma(1)} \cdots x_{\sigma(r)}}{r!} \otimes \frac{x_{\sigma(r+1)} \cdots x_{\sigma(n)}}{(n-r)!}$$

ok

Ex $n=2$ $\quad x = x_1 \quad y = x_2$

$$
\begin{tikzcd}
xy \arrow[r] \arrow[d] & \frac{xy+yx}{2} \arrow[d] \\
(x\otimes 1+1\otimes x)(y\otimes 1+1\otimes y) & \frac{(x\otimes 1+1\otimes x)(y\otimes 1+1\otimes y)+(y\otimes 1+1\otimes y)(x\otimes 1+1\otimes x)}{2}
\end{tikzcd}
$$

$$= xy\otimes 1 + x\otimes y + y\otimes x + 1\otimes xy \longrightarrow \frac{xy+yx}{2}\otimes 1 + x\otimes y + y\otimes x + 1\otimes \frac{xy+yx}{2}$$

with $\parallel$ between the right-hand fraction and the last expression.

Show this diag commutes:

$$\begin{tikzcd} S(\mathcal{L}) \arrow[r, "\gamma"] \arrow[d, "\varepsilon"'] \arrow[dr, phantom, "?"] & U(\mathcal{L}) \arrow[d, "\varepsilon"] \\ k \arrow[r, "id"'] & k \end{tikzcd}$$

For $x \in S(\mathcal{L})$ chase x around diag

wlog $x = x_1 x_2 \cdots x_n$ $\quad x_1, x_2, \ldots, x_n \in \mathcal{L}$

Case $n=0$:

$$\begin{tikzcd} 1 \arrow[r] \arrow[d] & 1 \arrow[d] \\ 1 \arrow[r] & 1 \end{tikzcd} \qquad \text{ok}$$

Case $n \geq 1$:

$$\begin{tikzcd} x_1 \cdots x_n \arrow[r] \arrow[d] & \frac{1}{n!} \sum_{\sigma \in S_n} x_{\sigma(1)} \cdots x_{\sigma(n)} \arrow[d] \\ 0 \arrow[r] & 0 \end{tikzcd} \qquad \text{ok}$$

$\square$

Given a bialgebra H

We discuss <u>H-module Algebras</u>

Motivation: recall H-comodule algebra is an algebra A together with an alg morphism

$$\Delta_A : A \to H \otimes A$$

that turns A into an H-comodule.

In this case both

(i) The mult map

$$\mu_A : A \otimes A \to A$$

is an H-comodule morph

(ii) The map

$$\eta_A : k \to A$$

is an H-comodule morph.

DEF 8 Given a bialg H

An <u>H-module algebra</u> is an

H-module A that is also an algebra

s.t.

(i) the mult map

$$\mu_A : A \otimes A \to A$$

is an H-module morph

(ii) the map

$$\eta_A : k \to A$$

is an H-module morph.

Referring to Def 8,

(i) detail:

H-module $A \otimes A$: H action is

$$h(a \otimes b) = \sum_{(h)} h'(a) \otimes h''(b) \qquad \begin{matrix} h \in H \\ a, b \in A \end{matrix}$$

μ_A is an H-module morph iff

$$h(ab) = \sum_{(h)} h'(a)\, h''(b) \qquad \begin{matrix} h \in H \\ a, b \in A \end{matrix}$$

(ii) detail:

H-module k: H action is

$$h(1) = \varepsilon(h)\, 1 \qquad h \in H$$

η_A is an H-module morph iff

$$h(1_A) = \varepsilon(h)\, 1_A \qquad h \in H$$

LEM 9 Given a bialg H.

Given a subset X of H that generates the algebra H.

Given an H-module A that is also an algebra.

TFAE:

(i) A is an H-module algebra

(ii) $\forall\ h \in X$,

$$h(ab) = \sum_{(h)} h'(a)\, h''(b) \qquad \forall\, a, b \in A \quad *$$

$$h(1_A) = \varepsilon(h)\, 1_A \qquad **$$

pf Use the fact that

$\Delta : H \to H \otimes H$

$\varepsilon : H \to K$

are algebra morphisms. $\square$

Given a Lie algebra $\mathcal{L}$

Recall the bialgebra $U(\mathcal{L})$:

$$\Delta(x) = x \otimes 1 + 1 \otimes x \qquad x \in \mathcal{L},$$
$$\varepsilon(x) = 0$$

Next goal: Describe the $U(\mathcal{L})$-module algebras.

Given an algebra A,

Recall a derivation of A is a linear map

$$\delta : A \to A$$

s.t

$$\delta(ab) = \delta(a)b + a\delta(b) \qquad \forall a, b \in A$$

A derivation δ satisfies

$$\delta(1_A) = 0$$

LEM 10 Given a Lie algebra $\mathcal{L}$ and a $U(\mathcal{L})$-module A that is also an algebra. Then TFAE:

(i) A is a $U(\mathcal{L})$-module algebra

(ii) each element of $\mathcal{L}$ acts on A as a derivation.

pf (i) → (ii) $\forall h \in \mathcal{L}$ show h acts on A as a derivation.

Given $a, b \in A$

$$h(ab) = \sum_{(h)} h'(a)\, h''(b)$$

$$[\,\Delta(h) = h \otimes 1 + 1 \otimes h\,]$$

$$= h(a)\, b + a\, h(b) \quad \checkmark$$

(ii) → (i) Apply LEM 9 with $H = U(\mathcal{L})$ and $X = \mathcal{L}$.

For $h \in \mathcal{L}$ show h satisfies LEM 9 (ii)

$\forall a, b \in A$

$$h(ab) \overset{?}{=} \underbrace{\sum_{(h)} h'(a)\, h''(b)}_{\substack{\shortparallel \\ h(a)\,b + a\,h(b)}}$$

$$\checkmark =$$

Check

$$\begin{array}{ccc} h(1_A) & \overset{?}{=} & \varepsilon(h)\, 1_A \\ \| & & \| \\ 0 & & 0 \end{array}$$

since h acts as der

OK

$\square$

Given algebra A

Define

$Der(A)$ = set of derivations of A

Obs: $\forall\, d, \delta \in Der(A)$, (ex)

$d\delta - \delta d \in Der(A)$

So $Der(A)$ is a Lie subalgebra of $gl(A)$

For example, take $A = k[x,y]$ (poly algebra)

LEM 11 Fix $A = k[x,y]$
For $a, b \in A$
Fix a lin map $d : A \to A$ TFAE:

(i) $d \in Der(A)$ and
$d(x) = a$, $d(y) = b$

(ii) For $r, s \in \mathbb{N}$,
$d(x^r y^s) = a r x^{r-1} y^s + b s x^r y^{s-1}$

(iii) $d = a \dfrac{\partial}{\partial x} + b \dfrac{\partial}{\partial y}$ $\square$

pf (ex)