

LEM 6 Given a Lie algebra \mathcal{L}
and consider $U(\mathcal{L})$.

For $n \in \mathbb{N}$ and $x_1, x_2, \dots, x_n \in \mathcal{L}$,

$$\Delta(x_1 x_2 \dots x_n) = \sum_{r=0}^n \sum_{\substack{\sigma \in S_n \\ \sigma \text{ is } (r, n-r)\text{-shuffle}}} x_{\sigma(1)} x_{\sigma(2)} \dots x_{\sigma(r)} \otimes x_{\sigma(r+1)} \dots x_{\sigma(n)}$$

pf Use $\Delta(x_1 x_2 \dots x_n) = \Delta(x_1) \Delta(x_2) \dots \Delta(x_n)$.

and thm 5 (i)

□

Given a vector space V ,

Recall the tensor algebra $T(V)$ is a Hopf algebra

with

$$\Delta(x) = x \otimes 1 + 1 \otimes x \quad x \in V$$

$$\varepsilon(x) = 0 \quad "$$

$$S(x) = -x \quad "$$

Recall the symmetric algebra $S(V) = T(V)/I$

$I = 2$ -sided ideal of $T(V)$ gen by $xy - yx \quad x, y \in V$

Recall $S(V)$ inherits the Hopf alg str:

$$\begin{aligned} \Delta: S(V) &\rightarrow S(V) \otimes S(V) \\ x &\rightarrow x \otimes 1 + 1 \otimes x \quad x \in V \end{aligned}$$

$$\begin{aligned} \varepsilon: S(V) &\rightarrow k \\ x &\rightarrow 0 \quad x \in V \end{aligned}$$

$$\begin{aligned} S: S(V) &\rightarrow S(V) \\ x &\rightarrow -x \quad x \in V \end{aligned}$$

Given a Lie algebra \mathcal{L} .

We cite (without pt) a few facts about \mathcal{L} .

Compare:

the univ env algebra $U(\mathcal{L})$

the sym algebra $S(\mathcal{L})$

Start with tensor algebra

$$T(\mathcal{L}) = \sum_{i \in \mathbb{N}} T^i \quad (ds)$$

where

$$T^i = \underbrace{\mathcal{L} \otimes \mathcal{L} \otimes \dots \otimes \mathcal{L}}_i$$

Recall

$\{T^0 + T^1 + \dots + T^i\}_{i \in \mathbb{N}}$ is a filtration of $T(\mathcal{L})$

Recall the canm alg morphism

$$\text{can: } T(\mathcal{L}) \rightarrow U(\mathcal{L})$$

For $i \in \mathbb{N}$ let

$$U_i = \text{image of } T^0 + T^1 + \dots + T^i \text{ under can}$$

So

$\{u_i\}_{i \in \mathbb{N}}$ is a filter of $U(L)$.

Fact: the corresp graded alg is iso $S(L)$.

For notational convenience assume $\dim L < \infty$

Pick a basis for L :

$$x_1, x_2, \dots, x_n$$

Fact (PBW Thm): the following is a basis for $U(L)$:

$$x_1^{r_1} x_2^{r_2} \dots x_n^{r_n}$$

$$r_1, r_2, \dots, r_n \in \mathbb{N}$$

Given a Lie algebra \mathcal{L}

Assume $\text{char}(k) = 0$

Fact: \exists vectn space iso

$$\gamma: S(\mathcal{L}) \rightarrow U(\mathcal{L})$$

that sends

$$x_1 x_2 \dots x_n \rightarrow \frac{1}{n!} \sum_{\sigma \in S_n} x_{\sigma(1)} x_{\sigma(2)} \dots x_{\sigma(n)}$$

$$\forall n \in \mathbb{N} \quad \forall x_1, x_2, \dots, x_n \in \mathcal{L}$$

Prop 7 the above map γ is a

coalgebra iso

pf Show this diag commutes

$$\begin{array}{ccc}
 S(\mathcal{L}) & \xrightarrow{\gamma} & U(\mathcal{L}) \\
 \Delta \downarrow & & \downarrow \Delta \\
 S(\mathcal{L}) \otimes S(\mathcal{L}) & \xrightarrow{\gamma \otimes \gamma} & U(\mathcal{L}) \otimes U(\mathcal{L})
 \end{array}$$

$\forall n \quad x \in S(\mathcal{L})$ chase x around diag.

WLOG $x = x_1 x_2 \dots x_n \quad n \in \mathbb{N} \quad x_1, x_2, \dots, x_n \in \mathcal{L}$

$$x_1 x_2 \dots x_n$$

$$\Delta \downarrow$$

$$\prod_{i=1}^n (x_i \otimes 1 + 1 \otimes x_i)$$

$$= \sum_{r=0}^n \sum_{\substack{\theta \in S_n \\ \theta \text{ is } (r, n-r)\text{-shuffle}}} x_{\theta(1)} \dots x_{\theta(r)} \otimes x_{\theta(r+1)} \dots x_{\theta(n)} \xrightarrow{\gamma \otimes \gamma}$$

$$\sum_{r=0}^n \sum_{\substack{\theta \in S_n \\ \theta \text{ is } (r, n-r)\text{ shuf}}} \gamma(x_{\theta(1)} \dots x_{\theta(r)}) \otimes \gamma(x_{\theta(r+1)} \dots x_{\theta(n)})$$

$$= \sum_{r=0}^n \sum_{\sigma \in S_n} \frac{x_{\sigma(1)} \dots x_{\sigma(r)}}{r!} \otimes \frac{x_{\sigma(r+1)} \dots x_{\sigma(n)}}{(n-r)!}$$

Also

$$x_1 x_2 \dots x_n \xrightarrow{\gamma} \frac{1}{n!} \sum_{\tau \in S_n} x_{\tau(1)} x_{\tau(2)} \dots x_{\tau(n)}$$

↓ Δ

$$\frac{1}{n!} \sum_{\tau \in S_n} \prod_{i=1}^n (x_{\tau(i)} \otimes 1 + 1 \otimes x_{\tau(i)})$$

$$= \frac{1}{n!} \sum_{\tau \in S_n} \sum_{r=0}^n \sum_{\theta \in S_n} x_{\tau(\theta(1))} \dots x_{\tau(\theta(r))} \otimes x_{\tau(\theta(r+1))} \dots x_{\tau(\theta(n))}$$

θ is (r, n-r) shuf

$$= \frac{1}{n!} \sum_{r=0}^n \sum_{\theta \in S_n} \sum_{\tau \in S_n} x_{\tau(\theta(1))} \dots x_{\tau(\theta(r))} \otimes x_{\tau(\theta(r+1))} \dots x_{\tau(\theta(n))}$$

θ is (r, n-r) shuf

change vars: $\sigma(i) = \tau(\theta(i))$ for $1 \leq i \leq n$
As τ ranges over S_n , so does σ

$$= \frac{1}{n!} \sum_{r=0}^n \sum_{\theta \in S_n} \sum_{\sigma \in S_n} x_{\sigma(1)} \dots x_{\sigma(r)} \otimes x_{\sigma(r+1)} \dots x_{\sigma(n)}$$

θ is (r, n-r) shuf

indep of θ

$$= \frac{1}{n!} \sum_{r=0}^n \sum_{\sigma \in S_n} x_{\sigma(1)} \dots x_{\sigma(r)} \otimes x_{\sigma(r+1)} \dots x_{\sigma(n)} \binom{n}{r}$$

↑
(r, n-r) shuffles in S_n

$$= \sum_{r=0}^n \sum_{\sigma \in S_n} \frac{x_{\sigma(1)} \dots x_{\sigma(r)}}{r!} \otimes \frac{x_{\sigma(r+1)} \dots x_{\sigma(n)}}{(n-r)!}$$

OK

$E_x \quad n=2$

$x = x_1$

$y = y_2$

$x y$



$\frac{x_2 + y_2}{2}$



$$\frac{(x_1 + i x_2)(y_1 + i y_2) + (y_1 + i y_2)(x_1 + i x_2)}{2}$$

$(x_1 + i x_2)(y_1 + i y_2)$

||

=

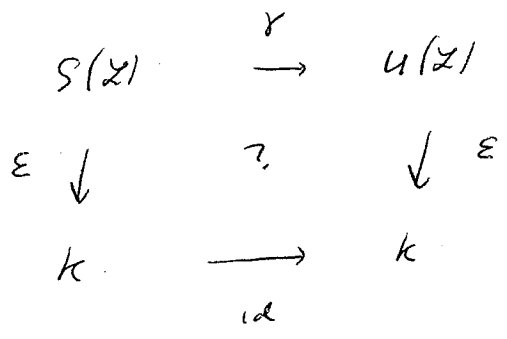
$x_1 y_1 + x_2 y_2 + i x_1 y_2 + i x_2 y_1$



$\frac{x_2 + y_2}{2} y_1 + x_1 y_2 + y_1 x_2$

$+ i \frac{x_2 + y_2}{2}$

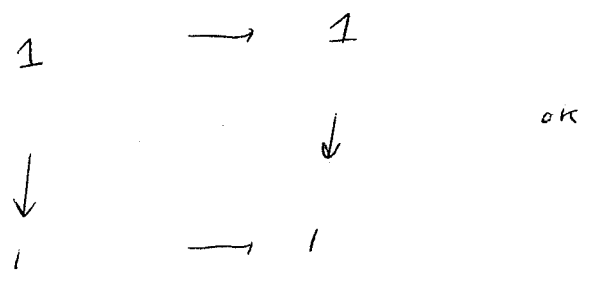
Show this diag commutes:



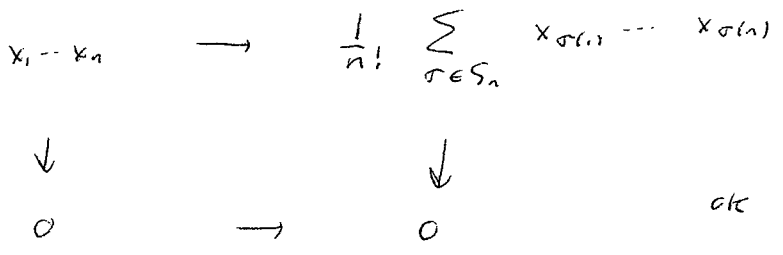
For $x \in S(\mathcal{L})$ chase x around diag

wlog $x = x_1 x_2 \dots x_n$ $x_1, x_2, \dots, x_n \in \mathcal{L}$

Case $n=0$:



Case $n \geq 1$:



Given a bialgebra H

We discuss H -module Algebras

Motivation: recall H -comodule algebra
is an algebra A together with an alg morphism

$\Delta_A : A \rightarrow H \otimes A$ that turns A
into an H -comodule.

In this case both

(i) The mult map

$$\mu_A : A \otimes A \rightarrow A$$

is an H -comodule morph.

(ii) The map

$$\eta_A : k \rightarrow A$$

is an H -comodule morph.

DEF 8 Given a bialg H

An H -module algebra is an

H -module A that is also an algebra

Set

(i) the mult map

$$\mu_A : A \otimes A \rightarrow A$$

is an H -module morph

(ii) the map

$$\eta_A : k \rightarrow A$$

is an H -module morph.

Referring to def 8,

(i) detail:

H -module $A \otimes A$: H action is

$$h(a \otimes b) = \sum_{(h)} h'(a) \otimes h''(b)$$

$$h \in H$$

$$a, b \in A$$

\mathcal{M}_A is an H -module morph iff

$$h(a \otimes b) = \sum_{(h)} h'(a) \otimes h''(b)$$

$$h \in H$$

$$a, b \in A$$

(ii) detail:

H -module k : H action is

$$h(1) = \varepsilon(h) 1$$

$$h \in H$$

\mathcal{M}_A is an H -module morph iff

$$h(1_A) = \varepsilon(h) 1_A$$

$$h \in H$$

LEM 9 Given a bialg H .

Given a subset X of H that generates the algebra H .

Given an H -module A that is also an algebra.

TFAE:

(i) A is an H -module algebra

(ii) $\forall h \in X,$

$$h(ab) = \sum_{(h)} h'(a) h''(b)$$

$$\forall a, b \in A \quad *$$

**

$$h(1_A) = \varepsilon(h) 1_A$$

pf Use the fact that

$$\Delta : H \rightarrow H \otimes H$$

$$\varepsilon : H \rightarrow k$$

are algebra morphisms.

□

Given a Lie algebra \mathcal{L}

Recall the bialgebra $U(\mathcal{L})$:

$$\Delta(x) = x \otimes 1 + 1 \otimes x \quad x \in \mathcal{L}$$

$$\epsilon(x) = 0$$

Next goal: Describe the $U(\mathcal{L})$ -module algebras.

Given an algebra A ,

Recall a derivation of A is a linear map

$$\delta: A \rightarrow A$$

s.t.

$$\delta(ab) = \delta(a)b + a\delta(b) \quad \forall a, b \in A$$

A derivation δ satisfies

$$\delta(1_A) = 0$$

LEM 10 Given a Lie algebra \mathcal{L}
and a $U(\mathcal{L})$ -module A that is also
an algebra. Then TFAE:

- (i) A is a $U(\mathcal{L})$ -module algebra
- (ii) each element of \mathcal{L} acts on A as a derivation.

pf (i) \rightarrow (ii) $\forall h \in \mathcal{L}$ show h acts on A
as a derivation.

Given $a, b \in A$

$$h(ab) = \sum_{(h)} h'(a) h''(b)$$

$$= h(a)b + ah(b) \quad \checkmark$$

[$\Delta(h) = h \otimes 1 + 1 \otimes h$]

(ii) \rightarrow (i) Apply LEM 9 with $H = U(\mathcal{L})$ and
 $X = \mathcal{L}$.

$\forall h \in \mathcal{L}$ show h satisfies LEM 9 (ii)

$\forall a, b \in A$

$$h(ab) = \sum_{(h)} h'(a) h''(b)$$

$$= h(a)b + ah(b) \quad \checkmark$$

check

$$h(1_A) = \varepsilon(h) 1_A$$

Since h
acts as der

$$\begin{matrix} // & // \\ 0 & 0 \end{matrix}$$

OK



Given algebra A

Define

$$\text{Der}(A) = \text{set of derivations of } A$$

Obs: $\forall d, s \in \text{Der}(A)$,
 $ds - sd \in \text{Der}(A)$ (ex)

So $\text{Der}(A)$ is a Lie subalgebra of $\mathfrak{gl}(A)$

For example, take $A = k[x, y]$ (poly algebra)

LEM 11 For $A = k[x, y]$

For $a, b \in A$

For a lin map $d: A \rightarrow A$

TFAE:

(i) $d \in \text{Der}(A)$ and $d(x) = a, d(y) = b$

(ii) For $r, s \in \mathbb{N}$,

$$d(x^r y^s) = ar x^{r-1} y^s + b s x^r y^{s-1}$$

(iii) $d = a \frac{\partial}{\partial x} + b \frac{\partial}{\partial y}$

pf (ex)

