

MotivationAssume $*$ exists

Consider composition

$$\varphi: \begin{array}{ccc} H & \xrightarrow{\quad} & H & \xrightarrow{\quad} & H \\ & & * & & S \\ a & \xrightarrow{\quad} & dt & \xrightarrow{\quad} & (at)t^{-1} = a \\ b & & -qct & & -q(-q^{-1}ct)t^{-1} = c \\ c & & -q^{-1}bt & & -q^{-1}(-q^{-1}bt)t^{-1} = b \\ d & & at & & (dt)t^{-1} = d \\ t & & t^{-1} & & t \end{array}$$

So φ sends

$$\begin{array}{ccc} x_{ij} & \rightarrow & x_{ji} \\ t & \rightarrow & t \end{array} \quad |z, j \leq 2$$

Recall for $x, y \in H$

$$S(xy) = S(y)S(x)$$

$$(xy)^* = y^* x^*$$

So

$$\varphi(xy) = \varphi(x)\varphi(y)$$

 $\varphi: H \rightarrow H$ is \mathbb{R} -algebra iso

By Lemma

$$\varphi^2 = \text{id}$$

$$\varphi(\alpha x) = \bar{\alpha} \varphi(x) \quad \forall \alpha \in \mathbb{C} \quad \forall x \in H$$

No longer assume \ast exists.

We prove \ast exists by first showing \exists variation in φ called ϕ

claim 1 $\exists \mathbb{C}$ -alg morphism $\phi: H \rightarrow H$ that sends

$$\begin{array}{ll} a \rightarrow a & b \rightarrow c \\ c \rightarrow b & d \rightarrow d \\ & t \rightarrow t \end{array}$$

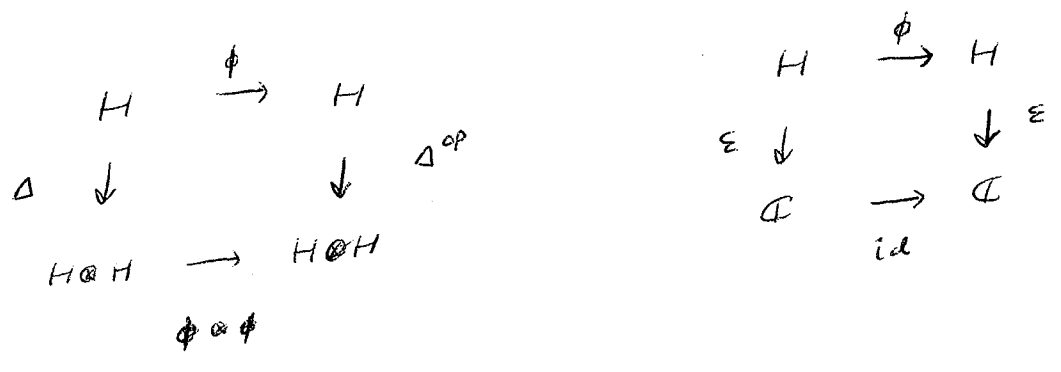
pf cl 1 Since $\bar{a}=a, \bar{b}=c, \bar{c}=b, \bar{d}=d, \bar{t}=t$

satisfy the defining rels for $H = GL_9(\mathbb{Z})$

claim 2 $\phi^2 = id.$ Moreover ϕ is a bijection.

pf cl 2 clear

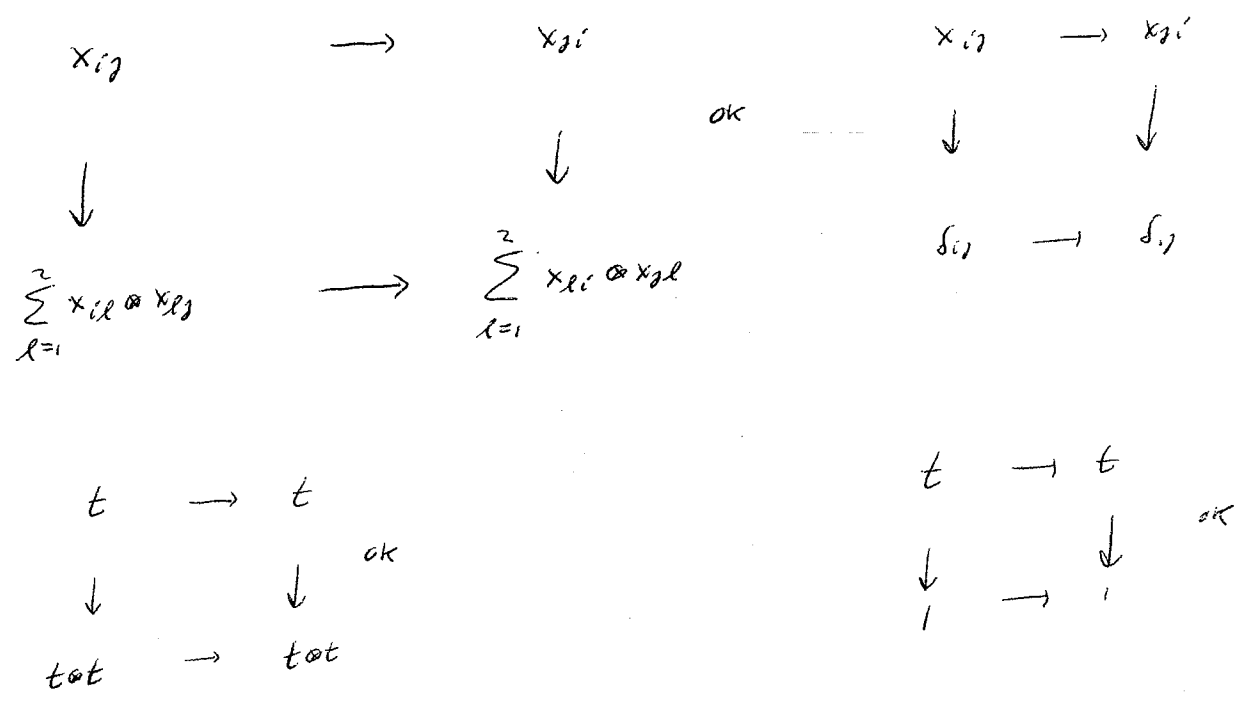
claim 3 these diagrams commute:



pt d3 For $h \in H$ chase h around diagram.

Since all the maps are \mathbb{Q} -alg morphisms

wlog $h = x_{ij}$ or $h = t$



Define

$$\star : H \rightarrow H \rightarrow H$$

$$\phi \quad \psi^{-1}$$

\star is \mathbb{C} -vector space iso

$$(xy)^\star = y^\star x^\star \quad \forall x, y \in H$$

$$1^\star = 1$$

So $\star : H \rightarrow H^{\text{op}}$ is \mathbb{C} -alg iso

By constr \star sends

$$a \rightarrow dt \quad b \rightarrow -qct$$

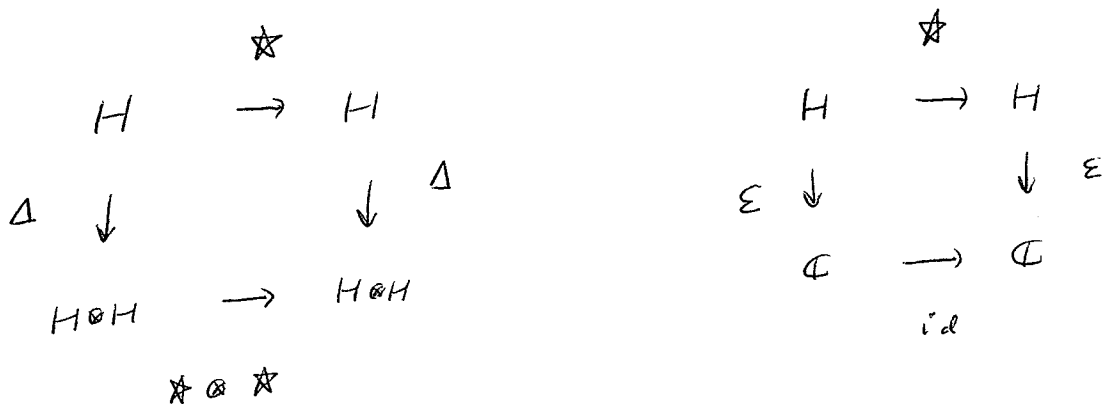
$$c \rightarrow -qbt \quad d \rightarrow at$$

$$t \rightarrow t^\star$$

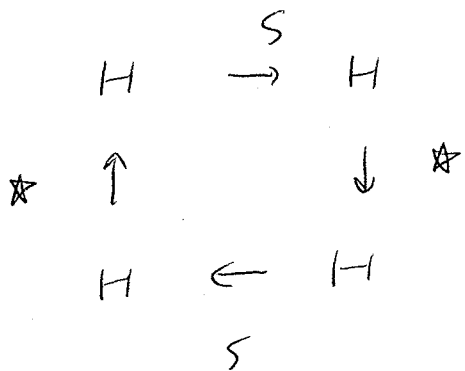
$\star^2 : H \rightarrow H$ is \mathbb{C} -alg morphism that
fixes each of a, b, c, d, t so

$$(x^\star)^\star = x \quad \forall x \in H$$

these diagrams commute:



(By claim 3 and since $S: H \rightarrow H^{\text{cop}}$ is \mathbb{C} -coalgebra morph)



(since $\phi^2 = \text{id}$)

We now get $*$ from \star :

View the \mathbb{C} -alg H as an \mathbb{R} -alg

obs $\star: H \rightarrow H^{op}$ is \mathbb{R} -alg iso

let $A = \mathbb{R}$ -subalg of H gen by a, b, c, d, t

then $H = A + Ai$ (ds) "Z₂-grading"

Each A, Ai is invar under S, \star

\exists \mathbb{R} -vector space iso $\star: H \rightarrow H$

s.t	on A	on Ai
	$\star = \star$	$\star = -\star$

By constr \star sends $1 \rightarrow 1$ and

$a \rightarrow dt$ $b \rightarrow -qct$
 $c \rightarrow -i^2bt$ $d \rightarrow at$
 $t \rightarrow t$

By constr $(x^*)^* = x \quad \forall x \in H$

Remains to show H is \ast -Hopf alg.

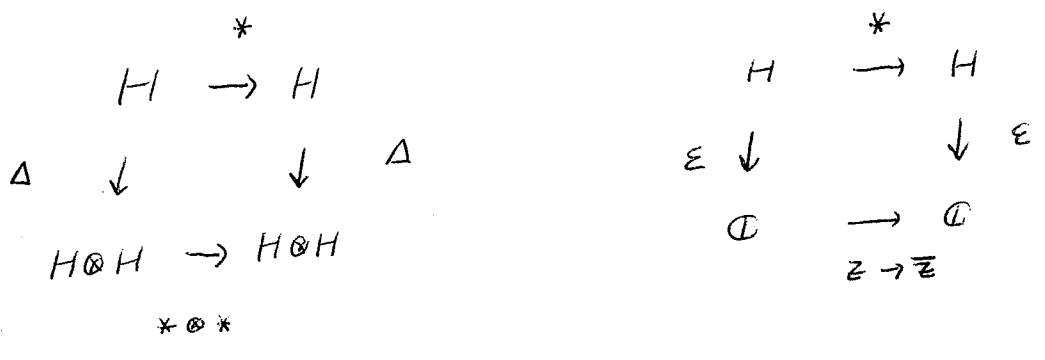
• Show $(xy)^\ast = y^\ast x^\ast$ $xy \in H$

Case	reason
$xy \in A$	$xy \in A$ so $(xy)^\ast = (xy)^\star = y^\star x^\star = y^\ast x^\ast$
$x \in A, y \in A_i$	$xy \in A_i$ so $(xy)^\ast = -(xy)^\star = -y^\star x^\star = y^\ast x^\ast$
$x \in A_i, y \in A$	sim
$xy \in A_i$	$xy \in A$ so $(xy)^\ast = (xy)^\star = y^\star x^\star = (-y^\ast)(-x^\ast) = y^\ast x^\ast$

o show $(\alpha x)^* = \bar{\alpha} x^*$ $\alpha \in \mathbb{C}$ $x \in H$

case	reason
$\alpha \in \mathbb{R}, x \in A$	$\alpha x \in A$ so $(\alpha x)^* = (\alpha x)^{\star} = \alpha x^{\star} = \bar{\alpha} x^*$
$\alpha \in i\mathbb{R}, x \in A$	$\alpha x \in A_i$ $\bar{\alpha} = -\alpha$ so $(\alpha x)^* = -(\alpha x)^{\star} = -\alpha x^{\star} = \bar{\alpha} x^*$
$\alpha \in \mathbb{R}, x \in A_i$	$\alpha x \in A_i$ $\bar{\alpha} = \alpha$ so $(\alpha x)^* = -(\alpha x)^{\star} = -\alpha x^{\star} = \bar{\alpha} x^*$
$\alpha \in i\mathbb{R}, x \in A_i$	$\alpha x \in A$ $\bar{\alpha} = -\alpha$ so $(\alpha x)^* = (\alpha x)^{\star} = \alpha x^{\star} = -\alpha x^{\star} = \bar{\alpha} x^*$

• Show these diagrams commute:

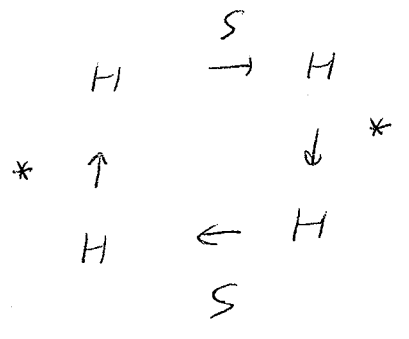


For $x \in H$ chase x around diag

wlog $x \in A \cap x \in A^c$

Each case is routine

• Show this diag commutes:



For $x \in H$ chase x around diag

wlog $x \in A \cap x \in A^c$

Each case is routine



We have turned $GL_q(\mathbb{Z})$ into a \ast -Hopf alg.

Similarly $SL_q(\mathbb{Z})$ is a \ast -Hopf alg site.

\ast sends

$$a \rightarrow d \qquad b \rightarrow -qc$$

$$c \rightarrow -q^{-1}b \qquad d \rightarrow a$$

(ex).

— 0 —

Note $\forall \gamma \neq 0 \in \mathbb{R}$,

$GL_q(2)$ becomes a γ -Hopf alg s.t.

γ sends

$$a \rightarrow dt$$

$$b \rightarrow \gamma ct$$

$$c \rightarrow \gamma^{-1} bt$$

$$d \rightarrow at$$

$$t \rightarrow t^{-1}$$

V Lie algebras

DEF 1 A Lie algebra is a vector space

\mathcal{L} together with a map

$$[\cdot, \cdot]: \mathcal{L} \times \mathcal{L} \rightarrow \mathcal{L}$$

s.t.

(i) $[\cdot, \cdot]$ is bilinear

(ii) $[x, x] = 0 \quad \forall x \in \mathcal{L}$

(iii) For $x, y, z \in \mathcal{L}$,

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$$

"Jacobi identity"

Call $[\cdot, \cdot]$ the Lie bracket

obs

$$[x, y] = -[y, x]$$

$x, y \in \mathcal{L}$

Ex 2 Given (assoc) alg A
The Lie alg $L(A)$ consists of the vs A
and
 $[x, y] = xy - yx \quad x, y \in A.$

Ex 3 For a vectn space V
recall alg $\text{End}(V)$
write $\mathfrak{gl}(V) = L(\text{End}(V))$
 $\mathfrak{sl}(V) = \{x \in \mathfrak{gl}(V) \mid \text{tr}(x) = 0\}$
 $=$ Lie subalgebra of $\mathfrak{gl}(V)$

DEF 4 Given a Lie algebra \mathcal{L}

$U(\mathcal{L})$ denotes the (assoc) algebra with gens \mathcal{L} and relations

$$xy - yx = [x, y] \quad x, y \in \mathcal{L}$$

Call $U(\mathcal{L})$ the universal enveloping algebra for \mathcal{L}

We will use the fact that \exists injective linear map

$$\begin{array}{ccc} \mathcal{L} & \longrightarrow & U(\mathcal{L}) \\ x & \longmapsto & \hat{x} \end{array} \quad *$$

sit

$$\widehat{[x, y]} = \hat{x}\hat{y} - \hat{y}\hat{x} \quad x, y \in \mathcal{L}$$

Obs $*$ is a Lie algebra morphism

$$\mathcal{L} \longrightarrow L(U(\mathcal{L}))$$

We identify \mathcal{L} with its image under $*$.

We now turn $U(\mathcal{L})$ into a Hopf algebra.

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Thm 5 Given a Lie algebra \mathcal{L} , write $U = U(\mathcal{L})$. Then:

(i) \exists alg morphism

$$\Delta: U \rightarrow U \otimes U$$

that sends $x \rightarrow x \otimes 1 + 1 \otimes x \quad \forall x \in \mathcal{L}$

(ii) \exists alg morphism

$$\varepsilon: U \rightarrow K$$

that sends $x \rightarrow 0 \quad \forall x \in \mathcal{L}$

(iii) Δ, ε turn U into a bialgebra

(iv) \exists alg morphism

$$S: U \rightarrow U^{\text{op}}$$

that sends $x \rightarrow -x \quad \forall x \in \mathcal{L}$

(v) $S^2 = \text{id}$. Moreover S is a bijection

(vi) S is an antipode for bialgebra U .

pf (i), (ii) Show Δ, ε respect the defining relations for U :

$\forall x, y \in L,$

$$xy - yx = [x, y]$$

Require

$$\Delta(x) \Delta(y) - \Delta(y) \Delta(x) \stackrel{?}{=} \Delta([x, y])$$

"

$$(x \otimes 1 + 1 \otimes x)(y \otimes 1 + 1 \otimes y) - (y \otimes 1 + 1 \otimes y)(x \otimes 1 + 1 \otimes x)$$

"

$$(xy - yx) \otimes 1 + 1 \otimes (xy - yx)$$

"

$$[x, y] \otimes 1 + 1 \otimes [x, y]$$

"

$$\Delta([x, y])$$

✓

Require

$$\varepsilon(x) \varepsilon(y) - \varepsilon(y) \varepsilon(x) \stackrel{?}{=} \varepsilon([x, y])$$

"
0

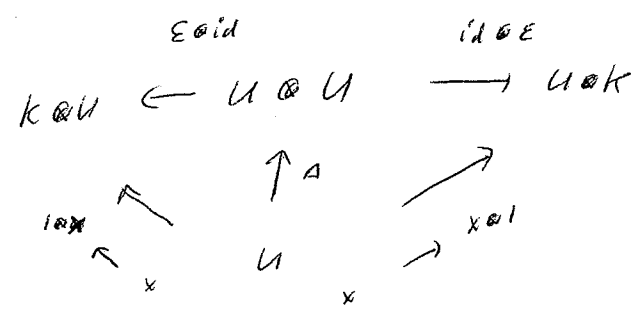
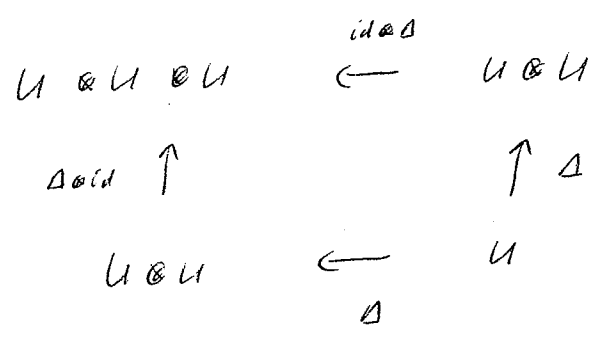
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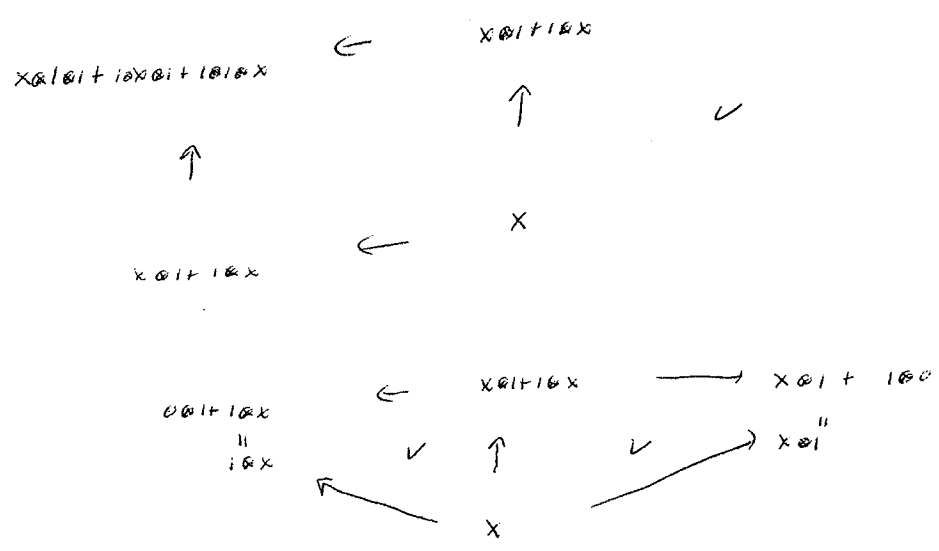
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(iii) Show these diagrams commute:



For $x \in U$ chase x around diag.

All maps are alg morphisms. $u \otimes v = x \otimes x$



(iv) Show S respects the defining rels for U :

$$\forall x, y \in L$$

$$xy - yx = [x, y]$$

Require

$$S(x) \circ S(y) - S(y) \circ S(x) \stackrel{?}{=} S([x, y])$$

↑
op mult

$$\underbrace{S(y)S(x) - S(x)S(y)}_{\substack{\text{"} \\ yx - xy \\ \text{"} \\ -[x, y]}} \stackrel{?}{=} S([x, y])$$

o.k.

(v) $S^2: U \rightarrow U$ is an alg morph that fixes each element of L .

(vi) $\forall x \in U$ show

$$\varepsilon(x) \perp_U = \sum_{(x)} x' S(x'') = \sum_{(x)} S(x') x'' \quad *$$

Since

$$S(xy) = S(y) S(x) \quad \forall x, y \in U$$

So to check * for $x \in L$

Here

$$\Delta(x) = x \otimes 1 + 1 \otimes x,$$

$$\varepsilon(x) = 0$$

$$0 \stackrel{?}{=} x \cdot S(1) + 1 \cdot S(x)$$

$$\quad \quad \quad \parallel \quad \quad \parallel$$

$$\quad \quad \quad 1 \quad \quad -x$$

OK

