Lec 32 Monday Nov 16

Motivation Assume $*$ exists

Consider composition

$$\varphi : \quad H \xrightarrow{*} H \xrightarrow{S} H$$

$$\begin{array}{lclcl}
a & \rightarrow & dt & \rightarrow & (at)t^{-1} = a \\
b & & -qct & & -q(-q^{-1}ct)t^{-1} = c \\
c & & -q^{-1}bt & & -q^{-1}(-qbt)t^{-1} = b \\
d & & at & & (dt)t^{-1} = d \\
t & & t^{-1} & & t
\end{array}$$

So φ sends

$$x_{ij} \rightarrow x_{ji} \qquad 1 \le i, j \le 2$$
$$t \rightarrow t$$

Recall for $x, y \in H$

$$S(xy) = S(y)S(x)$$
$$(xy)^* = y^* x^*$$

So $\varphi(xy) = \varphi(x)\varphi(y)$

$\varphi : H \to H$ is $\mathbb{R}$-algebra iso

By constr $\varphi^2 = id$

$$\varphi(\alpha x) = \bar{\alpha}\,\varphi(x) \qquad \forall \alpha \in \mathbb{C} \quad \forall x \in H$$

No longer assume * exists.

We prove * exists by first showing
$\exists$ variation on φ called ϕ

<u>claim 1</u> $\exists$ $\mathbb{C}$-alg morphism $\phi : H \to H$ that sends

$a \to a$ $\quad$ $b \to c$

$c \to b$ $\quad$ $d \to d$

$t \to t$

<u>pf cl 1</u> Since $\bar{a} = a$, $\bar{b} = c$, $\bar{c} = b$, $\bar{d} = d$, $\bar{t} = t$
satisfy the defining rels for $H = GL_q(2)$

<u>claim 2</u> $\phi^2 = id$. Moreover ϕ is a bijection.

<u>pf cl2</u> clear

<u>claim 3</u> these diagrams commute:

$$\begin{tikzcd} H \arrow[r, "\phi"] \arrow[d, "\Delta"'] & H \arrow[d, "\Delta^{op}"] \\ H \otimes H \arrow[r, "\phi \otimes \phi"'] & H \otimes H \end{tikzcd} \qquad \begin{tikzcd} H \arrow[r, "\phi"] \arrow[d, "\varepsilon"'] & H \arrow[d, "\varepsilon"] \\ \mathbb{C} \arrow[r, "id"'] & \mathbb{C} \end{tikzcd}$$

<u>pf d3</u> For $h \in H$ chase h around diagram.

Since all the maps are $\mathbb{C}$-alg morphisms

WLOG $h = x_{ij}$ or $h = t$

$$\begin{tikzcd} x_{ij} \arrow[r] \arrow[d] & x_{ji} \arrow[d] \\ \sum_{l=1}^{2} x_{il} \otimes x_{lj} \arrow[r] & \sum_{l=1}^{2} x_{li} \otimes x_{jl} \end{tikzcd}$$

ok

$$\begin{tikzcd} x_{ij} \arrow[r] \arrow[d] & x_{ji} \arrow[d] \\ \delta_{ij} \arrow[r] & \delta_{ij} \end{tikzcd}$$

$$\begin{tikzcd} t \arrow[r] \arrow[d] & t \arrow[d] \\ t \otimes t \arrow[r] & t \otimes t \end{tikzcd}$$

ok

$$\begin{tikzcd} t \arrow[r] \arrow[d] & t \arrow[d] \\ 1 \arrow[r] & 1 \end{tikzcd}$$

ok

Define

$$\star : H \xrightarrow[\phi]{} H \xrightarrow[S^{-1}]{} H$$

$\star$ is $\mathbb{C}$-vector space iso

$$(xy)^\star = y^\star x^\star \qquad \forall x, y \in H$$

$$1^\star = 1$$

So $\star : H \to H^{op}$ is $\mathbb{C}$-alg iso

By constr $\star$ sends

$a \to dt \qquad b \to -qct$

$c \to -q^{-1}bt \qquad d \to at$

$$t \to t^{-1}$$

$\star^2 : H \to H$ is $\mathbb{C}$-alg morphism that fixes each of a, b, c, d, t so

$$(x^\star)^\star = x \qquad \forall x \in H$$

these diagrams commute:

$$
\begin{tikzcd}
H \arrow[r, "\star"] \arrow[d, "\Delta"'] & H \arrow[d, "\Delta"] \\
H\otimes H \arrow[r, "\star\otimes\star"'] & H\otimes H
\end{tikzcd}
\qquad
\begin{tikzcd}
H \arrow[r, "\star"] \arrow[d, "\varepsilon"'] & H \arrow[d, "\varepsilon"] \\
\mathbb{C} \arrow[r, "id"'] & \mathbb{C}
\end{tikzcd}
$$

(By claim 3 and since $S: H \to H^{cop}$ is $\mathbb{C}$-coalg morph)

$$
\begin{tikzcd}
H \arrow[r, "S"] & H \arrow[d, "\star"] \\
H \arrow[u, "\star"] & H \arrow[l, "S"]
\end{tikzcd}
$$

(since $\phi^2 = id$)

We now get $*$ from $\star$:

View the $\mathbb{C}$-alg H as an $\mathbb{R}$-alg

Obs $\star : H \to H^{op}$ is $\mathbb{R}$-alg iso

Let $A = \mathbb{R}$-subalg of H gen by a, b, c, d, t

Then $H = A + Ai$ (ds) "$\mathbb{Z}_2$-grading"

Each A, Ai is invar under $S, \star$

$\exists$ $\mathbb{R}$-vector space iso $* : H \to H$

s.t

on A	on Ai
$* = \star$	$* = -\star$

By constr $*$ sends $1 \mapsto 1$ and

$a \to dt$ $\qquad b \to -q\, ct$

$c \to -q^{-1} bt$ $\qquad d \to at$

$t \to t^{-1}$

By constr $(x^*)^* = x$ $\quad \forall x \in H$

Remains to show H is $*$- Hopf alg.

- show $(xy)^* = y^* x^*$ $\quad x, y \in H$

Case	reason
$x, y \in A$	$xy \in A$ so $(xy)^* = (xy)^{\star} = y^{\star} x^{\star} = y^* x^*$
$x \in A,\ y \in Ai$	$xy \in Ai$ so $(xy)^* = -(xy)^{\star} = -y^{\star} x^{\star} = y^* x^*$
$x \in Ai,\ y \in A$	sim
$x, y \in Ai$	$xy \in A$ so $(xy)^* = (xy)^{\star} = y^{\star} x^{\star} = (-y^*)(-x^*) = y^* x^*$

- show $(\alpha x)^* = \bar{\alpha}\, x^*$ $\quad \alpha \in \mathbb{C} \quad x \in H$

case	reason
$\alpha \in \mathbb{R},\ x \in A$	$\alpha x \in A$ so $(\alpha x)^* = (\alpha x)^\star = \alpha x^\star = \bar{\alpha}\, x^*$
$\alpha \in i\mathbb{R} \quad x \in A$	$\alpha x \in Ai \qquad \bar{\alpha} = -\alpha$ so $(\alpha x)^* = -(\alpha x)^\star = -\alpha x^\star = \bar{\alpha}\, x^*$
$\alpha \in \mathbb{R},\ x \in Ai$	$\alpha x \in Ai \qquad \bar{\alpha} = \alpha$ so $(\alpha x)^* = -(\alpha x)^\star = -\alpha x^\star = \bar{\alpha} x^*$
$\alpha \in i\mathbb{R},\ x \in Ai$	$\alpha x \in A \qquad \bar{\alpha} = -\alpha$ so $(\alpha x)^* = (\alpha x)^\star = \alpha x^\star = -\alpha x^* = \bar{\alpha}\, x^*$

- Show these diagrams commute:

$$\begin{tikzcd} H \arrow[r, "*"] \arrow[d, "\Delta"'] & H \arrow[d, "\Delta"] \\ H\otimes H \arrow[r, "*\otimes *"'] & H\otimes H \end{tikzcd} \qquad \begin{tikzcd} H \arrow[r, "*"] \arrow[d, "\varepsilon"'] & H \arrow[d, "\varepsilon"] \\ \mathbb{C} \arrow[r, "z \to \bar{z}"'] & \mathbb{C} \end{tikzcd}$$

For $x \in H$ chase x around diag

WLOG $x \in A$ or $x \in A_i$

Each case is routine

- Show this diag commutes:

$$\begin{tikzcd} H \arrow[r, "S"] & H \arrow[d, "*"] \\ H \arrow[u, "*"] & H \arrow[l, "S"] \end{tikzcd}$$

For $x \in H$ chase x around diag

WLOG $x \in A$ or $x \in A_i$

Each case is routine

$\square$

We have turned $GL_q(2)$ into a $*$-Hopf alg.

Similarly $SL_q(2)$ is a $*$-Hopf alg s.t.

$*$ sends

$$a \longrightarrow d \qquad b \longrightarrow -qc$$

$$c \longrightarrow -q^{-1}b \qquad d \longrightarrow a$$

(ex).

— o —

Note $\forall\, 0 \neq \gamma \in \mathbb{R}$,

$GL_q(2)$ becomes a $*$-Hopf alg s.t.

$*$ sends

$a \to dt$ $\qquad$ $b \to \gamma c t$

$c \to \gamma^{-1} b t$ $\qquad$ $d \to a t$

$t \to t^{-1}$

V Lie algebras

DEF 1 A <u>Lie algebra</u> is a vector space $\mathcal{L}$ together with a map

$$[\,,\,] : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$$

s.t.

(i) $[\,,\,]$ is bilinear

(ii) $[x,x] = 0 \quad \forall x \in \mathcal{L}$

(iii) For $x, y, z \in \mathcal{L}$,

$$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$$

"Jacobi identity"

Call $[\,,\,]$ the <u>Lie bracket</u>

Obs

$$[x,y] = -[y,x] \qquad x, y \in \mathcal{L}$$

Ex 2 Given (assoc) alg A

the Lie alg $L(A)$ consists of the vs A

and

$$[x, y] = xy - yx \qquad x, y \in A$$

Ex 3 For a vector space V

recall alg $End(V)$

write $gl(V) = L(End(V))$

$sl(V) = \{x \in gl(V) \mid tr(x) = 0\}$

$= $ Lie subalgebra of $gl(V)$

DEF 4 Given a Lie algebra $\mathcal{L}$

$\mathcal{U}(\mathcal{L})$ denotes the (assoc) algebra

with gens $\mathcal{L}$ and relations

$$xy - yx = [x,y] \qquad x,y \in \mathcal{L}.$$

Call $\mathcal{U}(\mathcal{L})$ the <u>universal enveloping algebra</u> for $\mathcal{L}$

We will use the fact that $\exists$ injective linear map

$$\begin{aligned} \mathcal{L} &\longrightarrow \mathcal{U}(\mathcal{L}) \qquad \qquad * \\ x &\longmapsto \hat{x} \end{aligned}$$

s.t

$$\widehat{[x,y]} = \hat{x}\hat{y} - \hat{y}\hat{x} \qquad x,y \in \mathcal{L}$$

Obs $*$ is a Lie algebra morphism

$$\mathcal{L} \longrightarrow L(\mathcal{U}(\mathcal{L}))$$

We identify $\mathcal{L}$ with its image under $*$.

We now turn $\mathcal{U}(\mathcal{L})$ into a Hopf algebra.

Thm 5 Given a Lie algebra $\mathcal{L}$, write $\mathcal{U} = \mathcal{U}(\mathcal{L})$. Then:

(i) $\exists$ alg morphism

$$\Delta : \mathcal{U} \to \mathcal{U} \otimes \mathcal{U}$$

that sends $x \to x \otimes 1 + 1 \otimes x$ $\quad \forall x \in \mathcal{L}$

(ii) $\exists$ alg morphism

$$\varepsilon : \mathcal{U} \to k$$

that sends $x \to 0$ $\quad \forall x \in \mathcal{L}$

(iii) Δ, ε turn $\mathcal{U}$ into a bialgebra

(iv) $\exists$ alg morphism

$$S : \mathcal{U} \to \mathcal{U}^{op}$$

that sends $x \to -x$ $\quad \forall x \in \mathcal{L}$

(v) $S^2 = id$. Moreover S is a bijection

(vi) S is an antipode for bialgebra $\mathcal{U}$.

pf (i), (ii) Show Δ, ε respect the defining relations for $\mathcal{U}$:

$\forall x, y \in \mathcal{L}$,

$$xy - yx = [x, y]$$

Require

$$\Delta(x)\,\Delta(y) - \Delta(y)\,\Delta(x) \overset{?}{=} \Delta([x, y])$$

$$\begin{aligned}
&\Delta(x)\,\Delta(y) - \Delta(y)\,\Delta(x) \\
&= (x\otimes 1 + 1\otimes x)(y\otimes 1 + 1\otimes y) - (y\otimes 1 + 1\otimes y)(x\otimes 1 + 1\otimes x) \\
&= (xy - yx)\otimes 1 + 1\otimes(xy - yx) \\
&= [x, y]\otimes 1 + 1\otimes[x, y] \\
&= \Delta([x, y])
\end{aligned}$$

✓

Require

$$\underset{0}{\varepsilon(x)}\,\underset{0}{\varepsilon(y)} - \varepsilon(y)\,\varepsilon(x) \overset{?}{=} \underset{0}{\varepsilon([x, y])}$$

ok

(iii) Show these diagrams commute:

$$\begin{tikzcd}
U\otimes U\otimes U & U\otimes U \arrow[l, "id\otimes\Delta"'] \\
U\otimes U \arrow[u, "\Delta\otimes id"] & U \arrow[l, "\Delta"] \arrow[u, "\Delta"']
\end{tikzcd}$$

$$\begin{tikzcd}
k\otimes U & U\otimes U \arrow[l, "\varepsilon\otimes id"'] \arrow[r, "id\otimes\varepsilon"] & U\otimes k \\
& U \arrow[u, "\Delta"] \arrow[ul, "1\otimes x \leftarrow x"] \arrow[ur, "x \rightarrow x\otimes 1"'] &
\end{tikzcd}$$

For $x \in U$ chase x around diags. WLOG $x \in \mathcal{L}$

All maps are alg morphisms.

$$\begin{tikzcd}
x\otimes1\otimes1 + 1\otimes x\otimes1 + 1\otimes1\otimes x & x\otimes1+1\otimes x \arrow[l] \\
x\otimes1+1\otimes x \arrow[u] & x \arrow[l] \arrow[u]
\end{tikzcd} \quad \checkmark$$

$$\begin{tikzcd}
\begin{matrix} 0\otimes1+1\otimes x \\ \parallel \\ 1\otimes x \end{matrix} & x\otimes1+1\otimes x \arrow[l] \arrow[r] & \begin{matrix} x\otimes1+1\otimes0 \\ \parallel \\ x\otimes1 \end{matrix} \\
& x \arrow[u] \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

$\checkmark \quad \checkmark$

(iv) Show S respects the defining rels for $\mathcal{U}$:

$\forall x, y \in \mathcal{L}$

$$xy - yx = [x, y]$$

Require

$$S(x) \circ S(y) - S(y) \circ S(x) \overset{?}{=} S([x, y])$$

$\uparrow$ op mult

$$\underbrace{\underset{\underset{-y}{\parallel}}{S(y)}\ \underset{\underset{-x}{\parallel}}{S(x)} - S(x)\,S(y)}_{\substack{\parallel \\ yx - xy \\ \parallel \\ -[x,y]}} \overset{?}{=} \underset{\underset{-[x,y]}{\parallel}}{S([x, y])}$$

ok

(v) $S^2 : \mathcal{U} \to \mathcal{U}$ is an alg morph that fixes each element of $\mathcal{L}$.

(vi') For $x \in U$ show

$$\varepsilon(x) 1_U \underset{\uparrow}{=} \sum_{(x)} x' S(x'') = \sum_{(x)} S(x') x'' \qquad *$$

Since $S(xy) = S(y) S(x) \quad \forall x, y \in U$

suf to check $*$ for $x \in \mathcal{L}$

Here $\Delta(x) = x \otimes 1 + 1 \otimes x$,

$\varepsilon(x) = 0$

$$0 \overset{?}{=} x \underbrace{S(1)}_{\substack{\parallel \\ 1}} + 1 \underbrace{S(x)}_{\substack{\parallel \\ -x}}$$

ok

$\square$