Lec 31 Friday Nov 13

The Hopf algebra $GL_q(2)$

Motivation: Recall Hopf alg $GL(2)$

Gens: $a = x_{11}$ $b = x_{12}$ t (mut commuting)

$c = x_{21}$ $d = x_{22}$

Rels: $(ad-bc)t = 1$

Alg hom

Δ : $GL(2) \to GL(2)$

$x_{ij} \to \sum_{l} x_{il} \otimes x_{lj}$ $\forall i,j$

$t \to t \otimes t$

Alg hom ε : $GL(2) \to k$

$x_{ij} \to \delta_{ij}$ $\forall i,j$

$t \to 1$

Alg hom

S : $GL(2) \to GL(2)$

$a \to dt$

$b \to -bt$

$c \to -ct$

$d \to at$

$t \to t^{-1} = ad-bc$

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DEF 12 The algebra $GL_q(2)$ has gens

$a = x_{11}$ $b = x_{12}$ t

$c = x_{21}$ $d = x_{22}$

and rels

$ba = q ab$ $dc = q cd$

$ca = q ac$ $db = q bd$

$bc = cb$ $ad - da = (q^{-1} - q) bc$

$ta = at$ $tb = bt$

$tc = ct$ $td = dt$

$$t(ad - q^{-1} bc) = 1$$

Write

$$\det_q = ad - q^{-1} bc$$
$$= da - q bc$$

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LEM 13 $\exists$ algebra iso $GL_q(2)^{op} \to GL_{q^{-1}}(2)$

that sends

$a \to a \quad b \to b \quad c \to c \quad d \to d \quad t \to t$

pf Similar to the proof of LEM 9 $\square$

LEM 14 $\exists$ algebra morphisms

$$\Delta : GL_q(2) \to GL_q(2) \otimes GL_q(2)$$

$$x_{ij} \to \sum_{l=1}^{2} x_{il} \otimes x_{lj} \qquad 1 \le i,j \le 2$$

$$t \to t \otimes t$$

$$\varepsilon : GL_q(2) \to k$$

$$x_{ij} \to \delta_{ij} \qquad 1 \le i,j \le 2$$

$$t \to 1$$

pf Δ: Show Δ respects the defining rels for $GL_q(2)$. For the relations that do not involve t, done by LEM 5

Consider the relation $ta = at$

$$\underset{\overset{\shortparallel}{t\otimes t}}{\Delta(t)}\ \underset{\overset{\shortparallel}{a\otimes a + b\otimes c}}{\Delta(a)} \overset{?}{=} \underset{\overset{\shortparallel}{a\otimes a + b\otimes c}}{\Delta(a)}\ \underset{\overset{\shortparallel}{t\otimes t}}{\Delta(t)}$$

$$ta \otimes ta + tb \otimes tc \qquad at \otimes at + bt \otimes ct$$

$$=$$

since t is central in $GL_q(2)$

Consider the relation

$$t(ad - q^{-1}bc) = 1$$

$$\Delta(t)\left(\underbrace{\Delta(a)\,\Delta(d) - q^{-1}\Delta(b)}_{\shortparallel}\,\Delta(c)\right) \overset{?}{=} 1\otimes 1$$

$$\shortparallel$$

$$\underbrace{t\otimes t \qquad (ad - q^{-1}bc)\otimes(ad - q^{-1}bc)}_{\shortparallel}$$

$$\underbrace{t(ad - q^{-1}bc)}_{\substack{\shortparallel\\ 1}} \otimes \underbrace{t(ad - q^{-1}bc)}_{\substack{\shortparallel\\ 1}}$$

ok

ε: $\quad ta = at$

$$\underset{\substack{\shortparallel\\ 1}}{\varepsilon(t)}\,\underset{\substack{\shortparallel\\ 1}}{\varepsilon(a)} \overset{?}{=} \varepsilon(a)\,\varepsilon(t) \quad \checkmark$$

$$t(ad - q^{-1}bc) = 1$$

$$\varepsilon(t)\left(\varepsilon(a)\,\varepsilon(d) - q^{-1}\varepsilon(b)\,\varepsilon(c)\right) \overset{?}{=} 1$$

$$\shortparallel$$

$$1\;(1\cdot 1 - q^{-1}\,0\cdot 0)$$

$$\shortparallel$$

$$1 \qquad \text{ok}$$

$\square$

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LEM 15 The maps Δ, ε in LEM 14 turn $GL_q(2)$ into a bialgebra.

pf Similar to pf of LEM 6 $\square$

Next goal: display an antipode S for $GL_q(2)$.

<u>claim 1</u> $\exists$ alg morphism

$$S: GL_q(2) \to GL_q(2)^{op}$$

that sends

$a \to dt$ $\quad$ $b \to -qbt$

$c \to -q^{-1}ct$ $\quad$ $d \to at$

$$t \to t^{-1}$$

<u>pf cl1</u> Show S respects the defining rels for $GL_q(2)$

$ba = qab$

Require
$$S(b) \cdot S(a) \overset{?}{=} q\, S(a) \cdot S(b)$$

$$\underset{\overset{\|}{dt}}{S(a)}\ \underset{\overset{\|}{-qbt}}{S(b)} \overset{?}{=} q\, \underset{\overset{\|}{-qbt}}{S(b)}\ \underset{\overset{\|}{dt}}{S(a)}$$

ok since in $GL_q(2)$
t is central and $db = qbd$

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$dc = qcd$

$S(c)S(d) \overset{?}{=} qS(d)S(c)$

$ca \overset{?}{=} qac$

ok

$ca = qac$

$S(a)S(c) \overset{?}{=} qS(c)S(a)$

$dc \overset{?}{=} qcd$

ok

$db = qbd$

$S(b)S(d) \overset{?}{=} qS(d)S(b)$

$ba \overset{?}{=} qab$

ok

$bc = cb$

$S(c)S(b) \overset{?}{=} S(b)S(c)$

$cb = bc$ ✓

$ad - da = (q^{-1}-q)bc$

$$S(d)S(a) - S(a)S(d) \overset{?}{=} (q^{-1}-q)\underbrace{S(c)S(b)}_{\overset{\shortparallel}{bct^2}}$$

$at \quad dt \qquad dt \quad at$

$(ad-da)t^2 = (q^{-1}-q)bct^2$

✓

$ta = at$

$tb = bt$ sim

$tc = ct$ sim

$td = dt$ sim

$S(a)S(t) \overset{?}{=} S(t)S(a)$

$dt\,t^{-1} = t^{-1}dt$

✓

$(ad - q^{-1}bc)t = 1$

$$S(t)\big(S(d)S(a) - q^{-1}S(c)S(b)\big) \overset{?}{=} 1$$

$$t^{-1}\big(at\,dt - q^{-1}\,bct^2\big)$$

$$t^{-1}t^2(ad - q^{-1}bc)$$

$$t(ad - q^{-1}bc) = 1$$ ✓

claim proved

<u>claim 2</u> $\exists$ algebra morphism

$$\tilde{S} : GL_q(2)^{op} \to GL_q(2)$$

that sends

$a \to dt$ $b \to -q^{-1}bt$

$c \to -qct$ $d \to at$

$t \to t^{-1}$

<u>pf cl 2</u> Apply claim 1 to $GL_{q^{-1}}(2)$

and use LEM 13

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claim 3 the maps $S, \tilde{S}$ are inverses.

Moreover they are bijections.

pf cl3 check

$$\begin{tikzcd} GL_q(2) \arrow[r, "S"] & GL_q(2)^{op} \arrow[r, "\tilde{S}"] & GL_q(2) \end{tikzcd}$$

a	dt	$att^{-1} = a$
b	$-qbt$	$-q(-q^{-1}bt)t^{-1} = b$
c	$-q^{-1}ct$	$-q^{-1}(-ct)\,t^{-1} = c$
d	at	$dtt^{-1} = d$
t	t^{-1}	t

$$\begin{tikzcd} GL_q(2)^{op} \arrow[r, "\tilde{S}"] & GL_q(2) \arrow[r, "S"] & GL_q(2)^{op} \end{tikzcd}$$

a	dt	$att^{-1} = a$
b	$-q^{-1}bt$	$-q^{-1}(-qbt)t^{-1} = b$
c	$-qct$	$-q(-q^{-1}ct)t^{-1} = c$
d	at	$dtt^{-1} = d$
t	t^{-1}	t

claim 3 proved ✓

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We have shown that $S : GL_q(2) \to GL_q(2)^{op}$ is an algebra isomorphism. Therefore

$$S : GL_q(2) \to GL_q(2)$$

is vector space iso, and

$$S(xy) = S(y) S(x) \quad \forall x,y \in GL_q(2)$$

LEM 16 the above S is an antipode for $GL_q(2)$

pf We require that $\forall h \in GL_q(2)$ $(= H)$

$$\varepsilon(h) 1_H = \sum_{(h)} h' S(h'') = \sum_{(h)} S(h') h'' \quad ☆$$

By LEM 44 in Ch III, suffices to show ☆ for

$h \in \{a, b, c, d, t\}$

The requirement becomes

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} S(a) & S(b) \\ S(c) & S(d) \end{pmatrix} \overset{?}{=} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \overset{?}{=} \begin{pmatrix} S(a) & S(b) \\ S(c) & S(d) \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\begin{pmatrix} dt & -qbt \\ -q^{-1}ct & at \end{pmatrix} \qquad \begin{pmatrix} dt & -qbt \\ -q^{-1}ct & at \end{pmatrix}$$

$$\left(\begin{array}{c|c} \begin{matrix} adt - q^{-1}bct \\ =1 \end{matrix} & \begin{matrix} -qabt + bat \\ = 0 \end{matrix} \\ \hline \begin{matrix} cdt - q^{-1}dct \\ = 0 \end{matrix} & \begin{matrix} -qcbt + dat \\ = 1 \end{matrix} \end{array}\right) \qquad \text{ok} \qquad \left(\begin{array}{c|c} \begin{matrix} dat - qbct \\ =1 \end{matrix} & \begin{matrix} dbt - qbdt \\ = 0 \end{matrix} \\ \hline \begin{matrix} -q^{-1}cat - act \\ = 0 \end{matrix} & \begin{matrix} -q^{-1}cbt + adt \\ = 1 \end{matrix} \end{array}\right)$$

and

$$t \underset{t^{-1}}{\underset{\shortparallel}{S(t)}} \overset{?}{=} 1 \overset{?}{=} \underset{t^{-1}}{\underset{\shortparallel}{S(t)}}\, t$$

ok $\square$

We have shown that $GL_q(2)$ is a Hopf algebra with antipode S. Note that $S^2 \neq id$. We have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \xrightarrow{S} \begin{pmatrix} dt & -qbt \\ -q^{-1}ct & at \end{pmatrix} \xrightarrow{S} \begin{pmatrix} a & q^2 b \\ q^{-2}c & d \end{pmatrix}$$

$$= \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}^{-1}$$

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The Hopf algebra $SL_q(2)$

The construction is similar to $GL_q(2)$.

Here is a summary.

DEF 17 The algebra $SL_q(2)$ has generators

$a = x_{11}$ $b = x_{12}$

$c = x_{21}$ $d = x_{22}$

and relations

$ba = q\,ab$ $dc = q\,cd$

$ca = q\,ac$ $db = q\,bd$

$bc = cb$ $ad - da = (q^{-1} - q)\,bc$

$$ad - q^{-1}bc = 1$$

Note ∃ alg morphism $GL_q(2) \to SL_q(2)$

that sends

$x_{ij} \longrightarrow x_{ij}$ $1 \le i,j \le 2$

$t \to 1$

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LEM 18 ∃ alg iso $SL_q(2)^{op} \to SL_{q^{-1}}(2)$

that sends

$a \to a \qquad b \to b \qquad c \to c \qquad d \to d$

pf Sim to pf of Lem 13

(detail) Compare the def rels for $SL_q(2)^{op}$, $SL_{q^{-1}}(2)$

$SL_q(2)^{op}$:

$$ab = qba \qquad cd = qdc$$

$$ac = qca \qquad bd = qdb$$

$$cb = bc \qquad da - ad = (q^{-1} - q)bc$$

$$da - q^{-1}cb = 1$$

$SL_{q^{-1}}(2)$:

$$ba = q^{-1}ab \qquad dc = q^{-1}cd$$

$$ca = q^{-1}ac \qquad db = q^{-1}bd$$

$$bc = cb \qquad ad - da = (q - q^{-1})bc$$

$$ad - qbc = 1$$

OK

□

LEM 19 $\exists$ algebra morphisms

$$\Delta : SL_q(2) \rightarrow SL_q(2) \otimes SL_q(2)$$

$$x_{ij} \rightarrow \sum_{l=1}^{2} x_{il} \otimes x_{lj} \qquad 1 \le i,j \le 2$$

$$\varepsilon : SL_q(2) \longrightarrow K$$

$$x_{ij} \rightarrow \delta_{ij} \qquad 1 \le i,j \le 2$$

pf Sim to pf of LEM 14

(detail) Show Δ, ε respect the defining rels for $SL_q(2)$.

$$ad - q^{-1}bc = 1$$

Require

$$\underbrace{\Delta(a)\,\Delta(d) - q^{-1}\,\Delta(b)\,\Delta(c)}_{\overset{\shortparallel}{(ad - q^{-1}bc) \otimes (ad - q^{-1}bc)}} \overset{?}{=} \underset{\underset{1\otimes 1}{\shortparallel}}{\Delta(1)}$$

$$\shortparallel$$

$$1 \otimes 1 \qquad \text{ok}$$

$$\underset{\underset{1}{\shortparallel}}{\varepsilon(a)}\,\underset{\underset{1}{\shortparallel}}{\varepsilon(d)} - q^{-1}\,\underset{\underset{0}{\shortparallel}}{\varepsilon(b)}\,\underset{\underset{0}{\shortparallel}}{\varepsilon(c)} \overset{?}{=} \underset{\underset{1}{\shortparallel}}{\varepsilon(1)} \qquad \text{ok}$$

LEM 20 The maps Δ, ε turn $SL_q(2)$ into a bialgebra.

pf Sim to pf of LEM 15

(detail) One checks these diagrams commute $\left(C = SL_q(2)\right)$

$$\begin{tikzcd}
C\otimes C\otimes C & C\otimes C \arrow[l, "\Delta\otimes id"'] \\
C\otimes C \arrow[u, "id\otimes\Delta"] & C \arrow[l, "\Delta"] \arrow[u, "\Delta"']
\end{tikzcd}$$

$$\begin{tikzcd}
k\otimes C & C\otimes C \arrow[l, "\varepsilon\otimes id"'] \arrow[r, "id\otimes\varepsilon"] & C\otimes k \\
& C \arrow[ul, "1\otimes c \leftarrow c"] \arrow[u, "\Delta"'] \arrow[ur, "c \to c\otimes 1"'] &
\end{tikzcd}$$

$\square$

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We now display an antipode S for $SL_q(2)$

- $\exists$ alg morphism

$$S : SL_q(2) \longrightarrow SL_q(2)^{op}$$

that sends

$$a \to d \qquad b \to -qb$$

$$c \to -q^{-1}c \qquad d \to a$$

Check:

$$ad - q^{-1}bc = 1$$

$$S(a)\cdot S(d) - q^{-1}S(b)\cdot S(c) \overset{?}{=} 1$$

$$\|$$

$$S(d)\,S(a) - q^{-1}S(c)\,S(b)$$

$$\|$$

$$ad - q^{-1}\underset{\overset{\|}{bc}}{cb} = 1 \checkmark$$

- $\exists$ alg morphism

$$\tilde{S} : SL_q(2)^{op} \longrightarrow SL_q(2)$$

that sends

$$a \to d \qquad b \to -q^{-1}b$$

$$c \to -qc \qquad d \to a$$

- The maps $S, \tilde{S}$ are inverses. Moreover they are bijections.

We have shown that $S : SL_q(2) \to SL_q(2)^{op}$ is an algebra isomorphism.

Therefore

$$S : SL_q(2) \to SL_q(2)$$

is a VS iso, and

$$S(xy) = S(y)\,S(x) \qquad \forall\, x, y \in SL_q(2)$$

LEM 21 The above S is an antipode for $SL_q(2)$

pf Similar to pf of LEM 16

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} S(a) & S(b) \\ S(c) & S(d) \end{pmatrix} \overset{?}{=} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \overset{?}{=} \begin{pmatrix} S(a) & S(b) \\ S(c) & S(d) \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\begin{pmatrix} S(a) & S(b) \\ S(c) & S(d) \end{pmatrix} = \begin{pmatrix} d & -qb \\ -q^{-1}c & a \end{pmatrix} \qquad \begin{pmatrix} d & -qb \\ -q^{-1}c & a \end{pmatrix}$$

$$\begin{pmatrix} ad-q^{-1}bc & ba-qab \\ cd-q^{-1}dc & da-qcb \end{pmatrix} \qquad \begin{pmatrix} da-qbc & db-qbd \\ ac-q^{-1}ca & ad-q^{-1}cb \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \checkmark \qquad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \checkmark$$

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Earlier we saw that

$\underset{\overset{\shortparallel}{A}}{K_q[x,y]}$ becomes a (left or right) comodule alg for $M_q(2)$

Similarly

$K_q[x,y]$ --- -- $SL_q(2)$

In both cases, (in left comodule)

Δ_A sends

$$x \longrightarrow a \otimes x + b \otimes y$$

$$y \longrightarrow c \otimes x + d \otimes y$$

For $i,j \in \mathbb{N}$,

$$\Delta_A(x^i y^j) = (\Delta_A(x))^i (\Delta_A(y))^j$$

$$= (a \otimes x + b \otimes y)^i (c \otimes x + d \otimes y)^j$$

[Note
$(b \otimes y)(a \otimes x) = q^2 (a \otimes x)(b \otimes y)$,
$(d \otimes y)(c \otimes x) = q^2 (c \otimes x)(d \otimes y)$]

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$$\overset{\substack{q^2\text{-binom}\\ \text{thm}}}{=} \left(\sum_{r=0}^{i} \binom{i}{r}_{q^2} (a\otimes x)^r (b\otimes y)^{i-r} \right)$$

$$\left(\sum_{s=0}^{j} \binom{j}{s}_{q^2} (c\otimes x)^s (d\otimes y)^{j-s} \right)$$

$$= \sum_{r=0}^{i} \sum_{s=0}^{j} \binom{i}{r}_{q^2} \binom{j}{s}_{q^2} a^r b^{i-r} c^s d^{j-s} \otimes \underbrace{x^r y^{i-r} x^s y^{j-s}}_{\substack{\parallel \\ x^{r+s} y^{i+j-r-s} q^{s(i-r)}}}$$

$$= \sum_{r=0}^{i} \sum_{s=0}^{j} q^{s(i-r)} \binom{i}{r}_{q^2} \binom{j}{s}_{q^2} a^r b^{i-r} c^s d^{j-s} \otimes x^{r+s} y^{i+j-r-s} \qquad (*)$$

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For the alg $A = k_q[x,y]$ the defining rel $yx = qxy$ is homogeneous. So A has a grading

$$A = \sum_{n \in \mathbb{N}} A_n \qquad \text{ds}$$

A_n has basis $\{x^i y^{n-i}\}_{i=0}^{n}$

LEM 22 For $H = M_q(2)$ or $SL_q(2)$ consider the H-comodule algebra $A = k_q[x,y]$

then for $n \in \mathbb{N}$

A_n is an H-co submodule of A

pf show

$$\Delta_A(A_n) \subseteq H \otimes A_n.$$

A_n has basis

$x^i y^j$ $\quad i+j=n.$

By $*$

$$\Delta_A(x^i y^j) \in H \otimes A_n \quad \text{if} \quad i+j=n$$

Result follows. □

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$*$- Hopf algebras

For this section $k = \mathbb{C}$

For $z \in \mathbb{C}$ write

$z = a+bi$ $\quad a,b \in \mathbb{R}$ $\quad i^2=-1$

$\bar{z} = a-bi$

Given a Hopf algebra H with antipode S

Call it a <u>$*$-Hopf algebra</u> whenever $*: H \to H$

is a bijection s.t.

$(x^*)^* = x$ $\quad \forall x \in H$

$(x+y)^* = x^* + y^*$ $\quad \forall x,y \in H$

$(\alpha x)^* = \bar{\alpha} x^*$ $\quad \alpha \in \mathbb{C}$

$(xy)^* = y^* x^*$

$1^* = 1$

(so $* : H \to H^{op}$ is an $\mathbb{R}$-alg iso)

and these diagrams commute:

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$$\begin{tikzcd}
H \arrow[r, "*"] \arrow[d, "\Delta"'] & H \arrow[d, "\Delta"] \\
H\otimes H \arrow[r, "*\otimes *"'] & H\otimes H
\end{tikzcd}$$

$$\begin{tikzcd}
H \arrow[r, "*"] \arrow[d, "\varepsilon"'] & H \arrow[d, "\varepsilon"] \\
\mathbb{C} \arrow[r, "z \to \bar{z}"'] & \mathbb{C}
\end{tikzcd}$$

$$\begin{tikzcd}
H \arrow[r, "S"] & H \arrow[d, "*"] \\
H \arrow[u, "*"] & H \arrow[l, "S"]
\end{tikzcd}$$

— o —

Next goal: For $H = GL_q(2)$ $\quad k = \mathbb{C}$

assume $q \in \mathbb{R}$ so $\bar{q} = q$

Turn H into a $*$-Hopf algebra s.t.

$*$ sends

$a \to dt \qquad b \to -q\,ct$

$c \to -q^{-1}bt \qquad d \to at$

$t \to t^{-1}$