Show θ is iso:

Recall M_2 has basis

$$a^i b^j \qquad i,j \in \mathbb{N}$$

Recall the sum

$$M_2[t,\alpha,\delta] = M_2 + M_2 t + M_2 t^2 + \cdots$$

is direct. Therefore $M_2[t,\alpha,\delta]$ has basis

$$a^i b^j t^k \qquad i,j,k \in \mathbb{N}$$

By constr M_3 is spanned by

$$a^i b^j c^k \qquad i,j,k \in \mathbb{N} \qquad ☆$$

θ sends

$$a^i b^j c^k \to a^i b^j t^k \qquad i,j,k \in \mathbb{N}$$

Therefore ☆ is a basis for M_3 and θ is an iso.

We have shown $M_3 \simeq M_2[t,\alpha,\delta]$ is an Ore ext of M_2

Show M_4 is an Ore ext of M_3

$\exists$ alg morphism $\alpha : M_3 \to M_3$ that sends

$a \to a, \quad b \to qb, \quad c \to qc$

(since $\bar{a} = a, \quad \bar{b} = qb, \quad \bar{c} = qc$ satisfy the defining relations for M_3)

Obs α is a bijection.

Show $\exists$ α-derivation $\delta : M_3 \to M_3$ that sends

$a \to (q - q^{-1}) bc, \quad b \to 0, \quad c \to 0$

$\exists$ lin map $\delta : M_3 \to M_3$ that sends

$a^i b^j c^k \to q^{i-1} (q^i - q^{-i}) a^{i-1} b^{j+1} c^{k+1}$

for $i, j, k \in \mathbb{N}$.

Show δ is α-der.

Show

$\delta(xy) = \delta(x) y + \alpha(x) \delta(y) \quad \forall x, y \in M_3$

WLOG

$$x = a^i b^j c^k \qquad y = a^r b^s c^t$$

obs

$$xy = a^{i+r} b^{j+s} c^{k+t} q^{rj+rk}$$

$$\delta(xy) = q^{rj+rk+i+r-1}(q^{i+r} - q^{-i-r}) a^{i+r-1} b^{j+s+1} c^{k+t+1}$$

$$\delta(x) = q^{i-1}(q^i - q^{-i}) a^{i-1} b^{j+1} c^{k+1}$$

$$\delta(x)y = q^{rj+rk+2r+i-1}(q^i - q^{-i}) a^{i+r-1} b^{j+s+1} c^{k+t+1}$$

$$\alpha(x) = q^{j+k} a^i b^j c^k$$

$$\delta(y) = q^{r-1}(q^r - q^{-r}) a^{r-1} b^{s+1} c^{t+1}$$

$$\alpha(x)\delta(y) = q^{rj+rk+r-1}(q^r - q^{-r}) a^{i+r-1} b^{j+s+1} c^{k+t+1}$$

By these comments

$$\delta(xy) = \delta(x)y + \alpha(x)\delta(y) \quad \checkmark$$

So δ is an α-der of M_3

For the Ore ext $M_3[t, \alpha, \delta]$

$$ta = \delta(a) + \alpha(a)t = (q-q^{-1})bc + at$$

$$tb = \delta(b) + \alpha(b)t = qbt$$

$$tc = \delta(c) + \alpha(c)t = qct$$

So $\exists$ alg morph

$$\varphi : M_4 \to M_3[t, \alpha, \delta]$$

that sends

$$a \to a, \quad b \to b, \quad c \to c, \quad d \to t$$

Show φ is iso

Recall M_3 has a basis

$$a^i b^j c^k \qquad i, j, k \in \mathbb{N}$$

Recall the sum

$$M_3[t, \alpha, \delta] = M_3 + M_3 t + M_3 t^2 + \cdots$$

is directo

So $M_3[t,\alpha,\delta]$ has a basis

$$a^i b^j c^k t^l \qquad i,j,k,l \in \mathbb{N}$$

Using the rels $*$ we find M_4 is spanned by

$$a^i b^j c^k d^l \qquad i,j,k,l \in \mathbb{N} \qquad ☆☆$$

φ sends

$$a^i b^j c^k d^l \to a^i b^j c^k t^l \qquad i,j,k,l \in \mathbb{N}$$

Therefore ☆☆ is a basis for M_4 and φ is an iso.

We have shown that $M_4 \cong M_3[t,\alpha,\delta]$ is an Ore ext of M_3.

— ∘ —

COR 8

(i) $M_q(2)$ has a basis

$$a^i b^j c^k d^l \qquad i,j,k,l \in \mathbb{N}$$

(ii) $M_q(2)$ has no 0-divisors

(iii) $M_q(2)$ is Noetherian

LEM 9 $\exists$ algebra iso $M_q(2)^{op} \to M_{q^{-1}}(2)$

that sends

$$a \to a, \quad b \to b, \quad c \to c, \quad d \to d$$

pf Compare the defining relations for $M_q(2)^{op}$, $M_{q^{-1}}(2)$

$M_q(2)^{op}$:

$$ab = q\,ba \qquad cd = q\,dc$$

$$ac = q\,ca \qquad bd = q\,db$$

$$cb = bc \qquad da - ad = (q^{-1} - q)\,cb$$

$M_{q^{-1}}(2)$:

$$ba = q^{-1}ab \qquad dc = q^{-1}cd$$

$$ca = q^{-1}ac \qquad db = q^{-1}bd$$

$$bc = cb \qquad ad - da = (q - q^{-1})\,bc$$

$\square$

the q-det

Motivation: In $M(2)$ recall

$$\delta = x_{11}x_{22} - x_{12}x_{21}$$
$$= ad - bc$$

satisfies $\Delta(\delta) = \delta \otimes \delta$

For $M_q(2)$, pick $0 \neq \alpha \in k$ and consider

$$\delta = ad - \alpha bc \quad \in M_q(2)$$

Find α s.t.

$$\Delta(\delta) = \delta \otimes \delta$$

Recall Δ

$$\Delta : \quad M_q(2) \longrightarrow M_q(2) \otimes M_q(2)$$
$$a \to a \otimes a + b \otimes c = A$$
$$b \to a \otimes b + b \otimes d = B$$
$$c \to c \otimes a + d \otimes c = C$$
$$d \to c \otimes b + d \otimes d = D$$

$$\Delta(\delta) = AD - \alpha BC$$

$=$

	$c\otimes b \;+\; d\otimes d$	
$a\otimes a$	$ac \otimes ab$	$ad \otimes ad$
$+$ $b\otimes c$	$bc \otimes cb$	$bd \otimes cd$

$-\ \alpha$

	$c\otimes a \;+\; d\otimes c$	
$a\otimes b$	$ac \otimes ba$	$ad \otimes bc$
$+$ $b\otimes d$	$bc \otimes da$	$bd \otimes dc$

$=$

term $\otimes$	coef		
ac	ab	$-\alpha ba$	$= ab(1-\alpha q)$
ad	ad	$-\alpha bc$	$= \delta$
bc	cb	$-\alpha da$	$= -\alpha\delta + bc(1-\alpha q)(1+\alpha q^{-1})$
bd	cd	$-\alpha dc$	$= cd(1-\alpha q)$

[take $\alpha = q^{-1}$]

$= \delta \otimes \delta$ provided $\alpha = q^{-1}$

Obs $\quad ad - q^{-1}bc = da - qbc$

Call this common value $\det_q$

we have $\Delta(\det_q) = \det_q \otimes \det_q$

Obs $\quad \varepsilon(\det_q) = 1$

LEM 10 The element $\det_q$ is in the center of $M_q(2)$.

pf Using the defining rels for $M_q(2)$ one checks that $ad - q^{-1}bc$ commutes with each of a, b, c, d. $\square$

Given an algebra R,

an <u>R-point</u> of $M_q(2)$ is a matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \qquad A, B, C, D \in R$$

s.t.

$$BA = qAB, \qquad DC = qCD$$

$$CA = qAC, \qquad DB = qBD$$

$$BC = CB, \qquad AD - DA = (q^{-1} - q)BC$$

So for $A, B, C, D \in R$

$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is an R-point of $M_q(2)$

iff

$\exists$ alg morphism $M_q(2) \to R$ that sends

$a \to A$, $b \to B$, $c \to C$, $d \to D$

Given an R-pt of $M_q(2)$:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

define

$$\mathrm{Det}_q \begin{pmatrix} A & B \\ C & D \end{pmatrix} = AD - q^{-1} BC$$

LEM 11 Given algebra R

Given R-points X, Y of $M_q(2)$:

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \qquad Y = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix}$$

Assume each entry of X commutes with each entry of Y.

Then

(i) Matrix product XY is an R-pt of $M_q(2)$

(ii) $\mathrm{Det}_q(XY) = \mathrm{Det}_q(X)\,\mathrm{Det}_q(Y)$

pf We have

$$XY = \left(\begin{array}{c|c} AA' + BC' & AB' + BD' \\ \hline CA' + DC' & CB' + DD' \end{array}\right)$$

Recall

$$\Delta : M_q(2) \to M_q(2) \otimes M_q(2)$$

is alg morphism.

Also, by assumption $\exists$ alg morphism

$$M_q(2) \otimes M_q(2) \longrightarrow R \qquad *$$

that sends

$$a\otimes 1 \to A, \qquad b\otimes 1 \to B, \qquad c\otimes 1 \to C, \qquad d\otimes 1 \to D$$

$$1\otimes a \to A', \qquad 1\otimes b \to B', \qquad 1\otimes c \to C', \qquad 1\otimes d \to D'$$

Consider alg morphism

$$\theta : \quad M_q(2) \xrightarrow{\Delta} M_q(2)\otimes M_q(2) \xrightarrow{*} R$$

(i) θ sends

$$\begin{aligned}
a &\xrightarrow{\Delta} a\otimes a + b\otimes c \xrightarrow{*} AA' + BC' &&= (1,1)\text{-entry of } XY\\
b &\to a\otimes b + b\otimes d \to AB' + BD' &&= (1,2) \;..\\
c &\to c\otimes a + d\otimes c \to CA' + DC' &&= (2,1) \;..\\
d &\to c\otimes b + d\otimes d \to CB' + DD' &&= (2,2) \;..
\end{aligned}$$

So XY is an R-point of $M_q(2)$

(ii) θ is alg morphism so

$$\theta(\det_q) = \theta(ad - q^{-1}bc) = \theta(a)\theta(d) - q^{-1}\theta(b)\theta(c)$$
$$= \mathrm{Det}_q(XY)$$

But θ sends

$$\det_q \xrightarrow{\Delta} \det_q \otimes \det_q \xrightarrow{*} \mathrm{Det}_q(X)\,\mathrm{Det}_q(Y)$$

Result follows. $\square$

Given algebra R

We have some comments about R-points.

- $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is an R-point in $M_q(2)$

Given R-point in $M_q(2)$:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

- For $\alpha, \beta, \gamma, \delta \in k$ s.t. $\alpha\delta = \beta\gamma$,
  $\begin{pmatrix} \alpha A & \beta B \\ \gamma C & \delta D \end{pmatrix}$ is an R-pt in $M_q(2)$

- the transpose $\begin{pmatrix} A & C \\ B & D \end{pmatrix}$
  is an R-pt in $M_q(2)$

- the matrix $\begin{pmatrix} D & B \\ C & A \end{pmatrix}$
  is an R-point in $M_{q^{-1}}(2)$

- the matrix $\begin{pmatrix} D & B \\ C & A \end{pmatrix}$
  is an R^{op}-point in $M_q(2)$

- The matrix

$$\begin{pmatrix} D & -qB \\ -q^{-1}C & A \end{pmatrix}$$

is an R-pt in $M_{q^{-1}}(2)$ and an R^{op}-pt in $M_q(2)$

- $$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} D & -qB \\ -q^{-1}C & A \end{pmatrix} = \begin{pmatrix} AD-q^{-1}BC & 0 \\ 0 & AD-q^{-1}BC \end{pmatrix}$$

$$= \begin{pmatrix} D & -qB \\ -q^{-1}C & A \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$