

We saw how the algebra morphisms

$$\Delta : k[x] \rightarrow k[x', x'']$$
$$x \rightarrow x + x''$$

$$S : k[x] \rightarrow k[x]$$
$$x \rightarrow -x$$

$$\varepsilon : k[x] \rightarrow k$$
$$x \rightarrow 0$$

encode the additive abelian group structure of a commutative algebra A .

Δ, S, ε satisfy some relations that express associativity etc.

this turns $k[x]$ into a Hopf algebra
(to be defined shortly)

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For an algebra A define

$$A^{\times} = \{ a \in A \mid a^{-1} \text{ exists} \}$$

Next goal: For A commutative, model the
multiplicative group str on A^{\times}

Consider

$$\text{Mult: } (A^{\times})^2 \rightarrow A^{\times}$$

$$(a, b) \rightarrow ab$$

$$\text{Inv: } A^{\times} \rightarrow A^{\times}$$

$$a \rightarrow a^{-1}$$

$$\text{One: } (A^{\times})^0 \rightarrow A^{\times}$$

$$\emptyset \rightarrow 1$$

Recall Laurent polynomial algebra $k[x, x^{-1}]$

consists of

$$\sum_{i \in \mathbb{Z}} \alpha_i x^i \quad \alpha_i \in k$$

fin. many $\alpha_i \neq 0$

Algebra $k[x, x^{-1}]$ iso to

$$k[x, y] / I$$

$k[x, y]$ com. indets

$$I = \text{ideal of } k[x, y] \text{ gen by } xy - 1$$

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We have bijections

$$\begin{array}{ccc} \text{Hom}_{\text{alg}}(k, A) & \longrightarrow & (A^{\times})^0 \\ \text{"} & & \text{"} \\ \cong & & \cong \end{array}$$

$$\begin{array}{ccc} \text{Hom}_{\text{alg}}(k[x_1, x_2], A) & \longrightarrow & A^{\times} \\ \cong & & \cong \\ F & \longrightarrow & F(x) \end{array}$$

$$\begin{array}{ccc} \text{Hom}_{\text{alg}}(k[x_1^{\pm 1}, x_2^{\pm 1}], A) & \longrightarrow & (A^{\times})^2 \\ \cong & & \cong \\ F & \longrightarrow & (F(x_1), F(x_2)) \end{array}$$

Via above bijections,

Mult, inv, one

induce maps

$$\overline{\text{Mult}} : \text{Homalg}(k[(x')^{\pm 1}, (x'')^{\pm 1}], A) \rightarrow \text{Homalg}(k[x, x^{-1}], A)$$

$$\overline{\text{Inv}} : \text{Homalg}(k[x, x^{-1}], A) \rightarrow \text{Homalg}(k[x, x^{-1}], A)$$

$$\overline{\text{one}} : \text{Homalg}(k, A) \rightarrow \text{Homalg}(k[x, x^{-1}], A)$$

Define algebra horns

$$\Delta: k[x, x^{-1}] \rightarrow k[(x')^{\pm 1}, (x'')^{\pm 1}]$$
$$x \rightarrow x' x''$$

$$S: k[x, x^{-1}] \rightarrow k[x, x^{-1}]$$
$$x \rightarrow x^{-1}$$

$$\varepsilon: k[x, x^{-1}] \rightarrow k$$
$$x \rightarrow 1$$

Prop 15 the following diagrams commute

(i)

$$\begin{array}{ccc}
 & F & \rightarrow & F \circ \Delta \\
 \text{Hom}_{\text{alg}}(k[(x')^{\pm 1}, (x'')^{\pm 1}], A) & \rightarrow & \text{Hom}_{\text{alg}}(k[x, x^{-1}], A) \\
 \downarrow F & & \downarrow & \downarrow F \\
 (F(x'), F(x'')) & (A^x)^2 & \xrightarrow{\text{Mult}} & A^x & F(x)
 \end{array}$$

(ii)

$$\begin{array}{ccc}
 & F & \rightarrow & F \circ S \\
 \text{Hom}_{\text{alg}}(k[x, x^{-1}], A) & \rightarrow & \text{Hom}_{\text{alg}}(k[x, x^{-1}], A) \\
 \downarrow F & \downarrow & & \downarrow F \\
 F(x) & A^x & \xrightarrow{\text{Inv}} & A^x & F(x)
 \end{array}$$

(iii)

$$\begin{array}{ccc}
 & F & \rightarrow & F \circ \epsilon \\
 \text{Hom}_{\text{alg}}(k, A) & \rightarrow & \text{Hom}_{\text{alg}}(k[x, x^{-1}], A) \\
 \downarrow \eta_A & \downarrow & & \downarrow F \\
 \emptyset & (A^x)^0 & \xrightarrow{\text{one}} & A^x & F(x)
 \end{array}$$

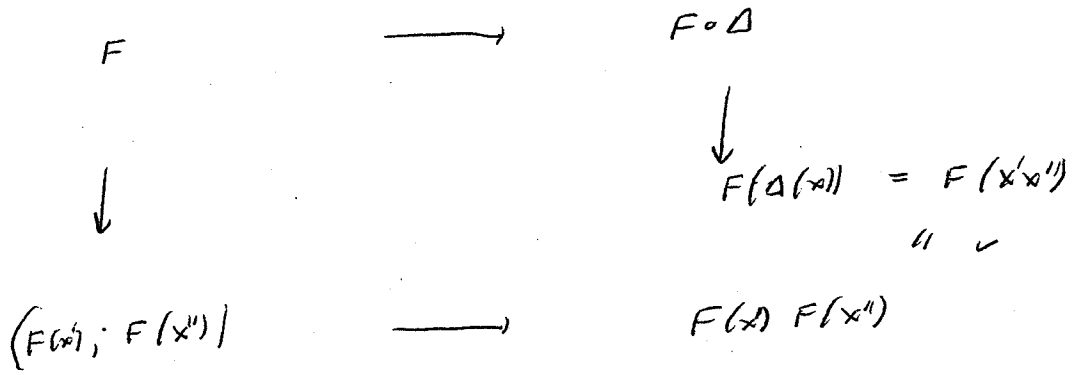
pf chase F around diagrams

pf detail:

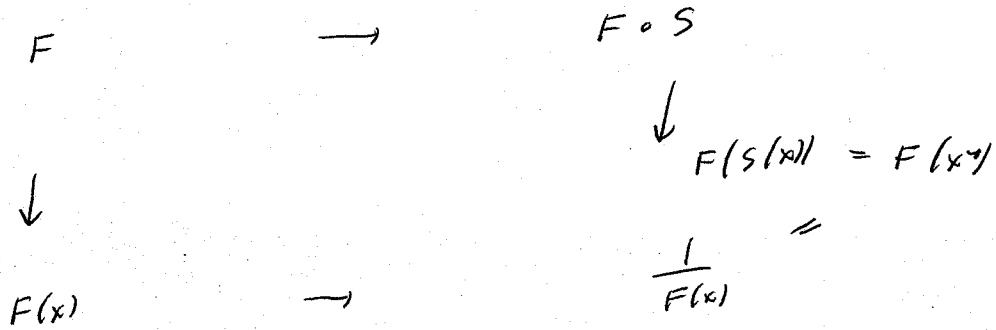
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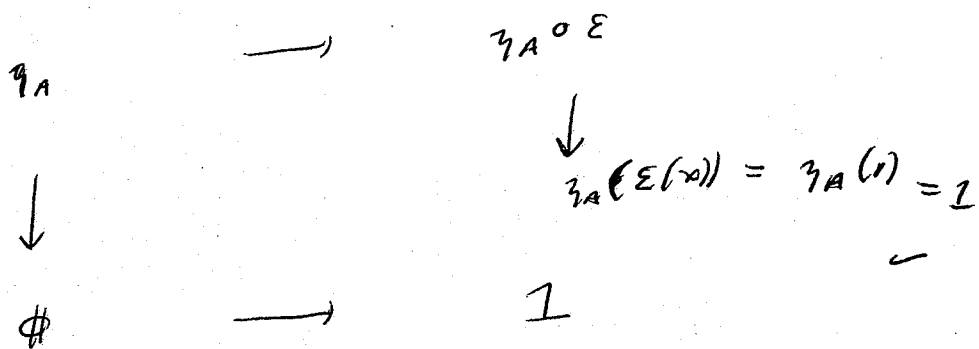
(i)



(ii)



(iii)



Cor 16 With above notation

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(i) $\overline{\text{Mult}}$ is

$$\text{Homalg}(k[(x')^{\pm 1}, (x'')^{\pm 1}], A) \rightarrow \text{Homalg}(k[x, x^{-1}], A)$$
$$F \rightarrow F \circ \Delta$$

(ii) $\overline{\text{Inv}}$ is

$$\text{Homalg}(k[x, x^{-1}], A) \rightarrow \text{Homalg}(k[x, x^{-1}], A)$$
$$F \rightarrow F \circ S$$

(iii) $\overline{\text{one}}$ is

$$\text{Homalg}(k, A) \rightarrow \text{Homalg}(k[x, x^{-1}], A)$$
$$F \rightarrow F \circ E$$

————— 0 —————
Above Δ, S, E turn $k[x, x^{-1}]$ into a Hopf algebra (as we will see)

Until further notice

A is commutative algebra

Consider 2×2 matrices $M_2(A)$

Next goal: model matrix mult in $M_2(A)$

$$\begin{array}{l}
 \text{Mult:} \\
 M_2(A) \times M_2(A) \rightarrow M_2(A) \\
 B \quad C \quad \rightarrow BC
 \end{array}$$

Given commuting indets

$$x_{11}, x_{12}, x_{21}, x_{22}$$

Consider polynomial algebra

$$k[x_{11}, x_{12}, x_{21}, x_{22}] = M(2)$$

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Given 8 commuting indets

$$x'_{ij}, \quad x''_{ij}, \quad 1 \leq i, j \leq 2$$

Consider polynomial algebra

$$k[x'_{ij}, x''_{ij}, 1 \leq i, j \leq 2] = M(2)^{\otimes 2}$$

Bijections

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$$\begin{array}{l} \text{Hom}_{\text{alg}}(K, A) \longrightarrow M_2(A)^0 \\ \text{Eval}_0 \quad \eta_A \longrightarrow \emptyset \end{array}$$

$$\begin{array}{l} \text{Hom}_{\text{alg}}(M(2), A) \longrightarrow M_2(A) \\ \text{Eval}_1 \quad F \longrightarrow \left(F(x_{ij}) \right)_{1 \leq i, j \leq 2} \end{array}$$

$$\begin{array}{l} \text{Hom}_{\text{alg}}(M(2)^{\otimes 2}, A) \longrightarrow M_2(A)^2 \\ \text{Eval}_2 \quad F \longrightarrow \left(F(x_{ij}^i) \right)_{1 \leq i, j \leq 2}, \left(F(x_{ij}^{ii}) \right)_{1 \leq i, j \leq 2} \end{array}$$

Via above bijections

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Mult

Induces a map

$$\overline{\text{Mult}} : \text{Homalg}(M(2)^{\otimes 2}, A) \rightarrow \text{Homalg}(M(2), A)$$

Describe $\overline{\text{Mult}}$

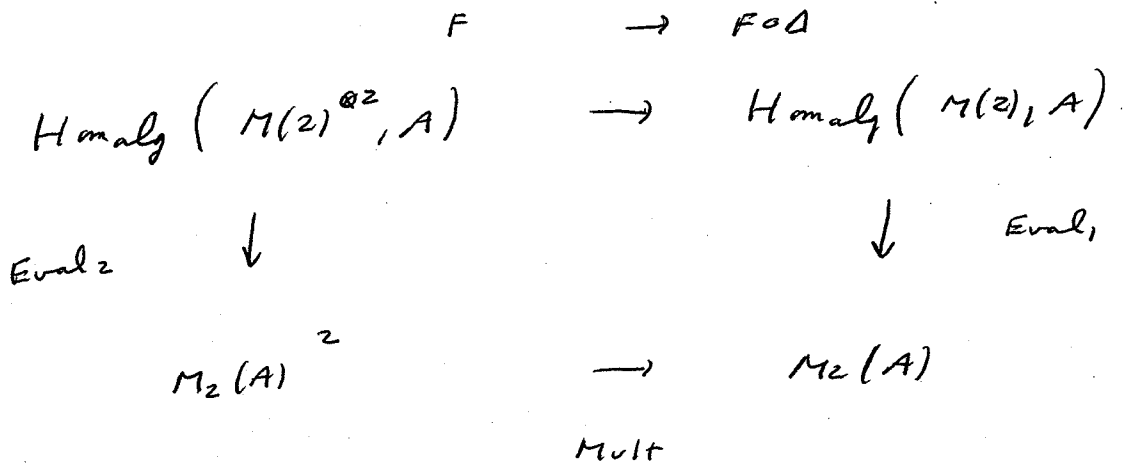
Define algebra hom

$$\Delta : \begin{array}{l} M(2) \rightarrow M(2)^{\otimes 2} \\ x_{ij} \rightarrow \sum_{k=1}^2 (x^k)_{ie} (x^k)_{kj} \end{array}$$

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Prop 18 the following diagram commutes:



pf Chase F around diag

□

Detail

$$\begin{array}{ccc}
 F & \longrightarrow & F \circ \Delta \\
 \downarrow & & \downarrow \begin{pmatrix} F(\Delta(x_{11})) & F(\Delta(x_{12})) \\ F(\Delta(x_{21})) & F(\Delta(x_{22})) \end{pmatrix} \\
 & & \parallel *
 \end{array}$$

$$\begin{pmatrix} F(x'_{11}) & F(x'_{12}) \\ F(x'_{21}) & F(x'_{22}) \end{pmatrix} \cdot \begin{pmatrix} F(x''_{11}) & F(x''_{12}) \\ F(x''_{21}) & F(x''_{22}) \end{pmatrix} \rightarrow \begin{pmatrix} F(x'_{11})F(x''_{11}) + & F(x'_{11})F(x''_{12}) + \\ F(x'_{12})F(x''_{11}) + & F(x'_{12})F(x''_{12}) \\ \hline F(x'_{21})F(x''_{11}) + & F(x'_{21})F(x''_{12}) + \\ F(x'_{22})F(x''_{11}) + & F(x'_{22})F(x''_{12}) \end{pmatrix}$$

*: F_n ($1 \leq i, j \leq 2$)

$$\begin{aligned}
 F(\Delta(x_{ij})) &= F\left(\sum_l x'_{il} x''_{lj}\right) \\
 &= \sum_l F(x'_{il}) F(x''_{lj})
 \end{aligned}$$

Cor is $\overline{\text{Mult}}$ is

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$$\begin{array}{ccc} \text{Hom}_{\text{alg}}(M(2)^{\otimes 2}, A) & \longrightarrow & \text{Hom}_{\text{alg}}(M(2), A) \\ F & \longrightarrow & F \circ A \end{array}$$

Given commutative algebra A

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define

$$GL_2(A) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(A) \mid ad - bc \in A^\times \right\}$$

Next goal: describe the mult group structure on $GL_2(A)$

$$\begin{array}{l} \text{Mult:} \\ GL_2(A) \times GL_2(A) \rightarrow GL_2(A) \\ B, C \rightarrow BC \end{array}$$

$$\begin{array}{l} \text{Inv:} \\ GL_2(A) \rightarrow GL_2(A) \\ B \rightarrow B^{-1} \end{array}$$

$$\begin{array}{l} \text{Ident:} \\ GL_2(A)^0 \rightarrow GL_2(A) \\ \emptyset \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array}$$

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Recall poly algebra

$$M(2) = k[x_{11}, x_{12}, x_{21}, x_{22}]$$

Consider commuting indets

$$x_{11}, x_{12}, x_{21}, x_{22}, t$$

(*)

Define

$$GL(2) = \text{comm algebra with gens } (*)$$

and relations

$$(x_{11}x_{22} - x_{12}x_{21})t = 1$$

So

$$GL(2) \cong$$

$$M(2)[t]/I$$

ideal I gen by

$$(x_{11}x_{22} - x_{12}x_{21})t - 1$$

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(9)

Comm inlets

$$x_{12}^{\prime}, x_{12}^{\prime\prime}, t^{\prime}, t^{\prime\prime}$$

**

Def $GL(2)^{\otimes 2}$ is comm alg with gens **

and relations

$$(x_{11}^{\prime} x_{22}^{\prime} - x_{12}^{\prime} x_{21}^{\prime}) t^{\prime} = 1,$$

$$(x_{11}^{\prime\prime} x_{22}^{\prime\prime} - x_{12}^{\prime\prime} x_{21}^{\prime\prime}) t^{\prime\prime} = 1.$$

Bijections

$\text{Hom}_{\mathbb{Z}}(\mathbb{Z}, A) \xleftrightarrow{\cong} \text{GL}_2(A)^0$
 $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}, A)$

$\text{Hom}_{\mathbb{Z}}(\text{GL}(2), A) \xleftrightarrow{\cong} \text{GL}_2(A)$
 $\text{Hom}_{\mathbb{Z}}(F, A) \xleftrightarrow{\cong} (F(x_{ij}))_{1 \leq i, j \leq 2}$

$\text{Hom}_{\mathbb{Z}}(\text{GL}(2)^{\otimes 2}, A) \xleftrightarrow{\cong} \text{GL}_2(A)^2$
 $\text{Hom}_{\mathbb{Z}}(F, A) \xleftrightarrow{\cong} (F(x'_{ij}))_{1 \leq i, j \leq 2} (F(x''_{ij}))_{1 \leq i, j \leq 2}$