Lec 3 Wed Sept 9

We saw how the algebra morphisms

$$\Delta : k[x] \to k[x', x'']$$
$$x \to x + x'$$

$$S : k[x] \to k[x]$$
$$x \to -x$$

$$\varepsilon : k[x] \to k$$
$$x \to 0$$

encode the additive abelian group structure of a commutative algebra $A$.

$\Delta, S, \varepsilon$ satisfy some relations that express associativity etc.

this turns $k[x]$ into a Hopf algebra (to be defined shortly)

For an algebra $A$ define

$$A^\times = \{ a \in A \mid a^{-1} \text{ exists} \}$$

Next goal: For $A$ commutative, model the multiplicative group str on $A^\times$

Consider

Mult: $(A^\times)^2 \to A^\times$

$(a, b) \to ab$

Inv: $A^\times \to A^\times$

$a \to a^{-1}$

One: $(A^\times)^0 \to A^\times$

$\emptyset \to 1$

Recall Laurent polynomial algebra $k[x, x^{-1}]$

consists of

$$\sum_{i \in \mathbb{Z}} \alpha_i x^i \qquad \alpha_i \in k$$

fin. many $\alpha_i$ non0

Algebra $k[x, x^{-1}]$ iso to

$$k[x, y]/I \qquad x, y \text{ com indets}$$

$I$ = ideal of $k[x, y]$ gen by $xy - 1$

We have bijections

$$\mathrm{Hom}_{alg}(k, A) \longrightarrow (A^\times)^0$$

$$\mathrm{Hom}_{alg}(k, A) = \{\eta_A\} \qquad (A^\times)^0 = \{\emptyset\}$$

$$\mathrm{Hom}_{alg}(k[x, x^{-1}], A) \longrightarrow A^\times$$

$$F \longrightarrow F(x)$$

$$\mathrm{Hom}_{alg}(k[(x')^{\pm 1}, (x'')^{\pm 1}], A) \longrightarrow (A^\times)^2$$

$$F \longrightarrow (F(x'), F(x''))$$

Via above bijections,

Mult, inv, one

induce maps

$$\overline{Mult} : \quad Hom_{alg}\left(k[(x')^{\pm 1}, (x'')^{\pm 1}], A\right) \to Hom_{alg}\left(k[x, x^{-1}], A\right)$$

$$\overline{Inv} : \quad Hom_{alg}\left(k[x, x^{-1}], A\right) \to Hom_{alg}\left(k[x, x^{-1}], A\right)$$

$$\overline{one} : \quad Hom_{alg}(k, A) \to Hom_{alg}\left(k[x, x^{-1}], A\right)$$

Define algebra homs

$$\Delta : \quad k[x, x^{-1}] \longrightarrow k[(x')^{\pm 1}, (x'')^{\pm 1}]$$

$$x \longmapsto x' x''$$

$$S : \quad k[x, x^{-1}] \longrightarrow k[x, x^{-1}]$$

$$x \longmapsto x^{-1}$$

$$\varepsilon : \quad k[x, x^{-1}] \longrightarrow k$$

$$x \longmapsto 1$$

Prop 15 the following diagrams commute

(i)

$$F \mapsto F \circ \Delta$$

$$\begin{tikzcd} \mathrm{Hom}_{alg}\left(k[(x')^{\pm 1}, (x'')^{\pm 1}], A\right) \arrow[r] \arrow[d] & \mathrm{Hom}_{alg}\left(k[x, x^{-1}], A\right) \arrow[d] \\ (A^\times)^2 \arrow[r, "Mult"'] & A^\times \end{tikzcd}$$

$$F \mapsto (F(x'), F(x'')) \qquad F \mapsto F(x)$$

(ii)

$$F \mapsto F \circ S$$

$$\begin{tikzcd} \mathrm{Hom}_{alg}\left(k[x, x^{-1}], A\right) \arrow[r] \arrow[d] & \mathrm{Hom}_{alg}\left(k[x, x^{-1}], A\right) \arrow[d] \\ A^\times \arrow[r, "Inv"'] & A^\times \end{tikzcd}$$

$$F \mapsto F(x) \qquad F \mapsto F(x)$$

(iii)

$$F \mapsto F \circ \varepsilon$$

$$\begin{tikzcd} \mathrm{Hom}_{alg}\left(k, A\right) \arrow[r] \arrow[d] & \mathrm{Hom}_{alg}\left(k[x, x^{-1}], A\right) \arrow[d] \\ (A^\times)^0 \arrow[r, "one"'] & A^\times \end{tikzcd}$$

$$\eta_A \mapsto \# \qquad F \mapsto F(x)$$

pf chase F around diagrams

pf detail:



(i)

$$\begin{array}{ccc} F & \longmapsto & F \circ \Delta \\ \downarrow & & \downarrow \\ & & F(\Delta(x)) = F(x'x'') \\ & & \parallel \ \checkmark \\ (F(x'); F(x'')) & \longmapsto & F(x')\,F(x'') \end{array}$$

(ii)

$$\begin{array}{ccc} F & \longmapsto & F \circ S \\ \downarrow & & \downarrow \\ & & F(S(x)) = F(x^{-1}) \\ & & \parallel \\ F(x) & \longmapsto & \dfrac{1}{F(x)} \end{array}$$

(iii)

$$\begin{array}{ccc} \eta_A & \longmapsto & \eta_A \circ \varepsilon \\ \downarrow & & \downarrow \\ & & \eta_A(\varepsilon(x)) = \eta_A(1) = 1 \\ & & \checkmark \\ \# & \longmapsto & 1 \end{array}$$

Cor 16 With above notation

(i) $\overline{Mult}$ is

$$\mathrm{Homalg}\left(k[(x')^{\pm 1}, (x'')^{\pm 1}], A\right) \to \mathrm{Homalg}\left(k[x, x^{-1}], A\right)$$
$$F \to F \circ \Delta$$

(ii) $\overline{Inv}$ is

$$\mathrm{Homalg}\left(k[x, x^{-1}], A\right) \to \mathrm{Homalg}\left(k[x, x^{-1}], A\right)$$
$$F \to F \circ S$$

(iii) $\overline{one}$ is

$$\mathrm{Homalg}(k, A) \to \mathrm{Homalg}\left(k[x, x^{-1}], A\right)$$
$$F \to F \circ \varepsilon$$

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Above $\Delta, S, \varepsilon$ turn $k[x, x^{-1}]$ into a Hopf algebra (as we will see)

Until further notice

$A$ is commutative algebra

Consider $2 \times 2$ matrices $M_2(A)$

Next goal: model matrix mult in $M_2(A)$

Mult: $M_2(A) \times M_2(A) \to M_2(A)$

$B \quad C \to BC$

Given commuting indets

$x_{11}, \quad x_{12}, \quad x_{21}, \quad x_{22}$

Consider polynomial algebra

$k[x_{11}, x_{12}, x_{21}, x_{22}] = M(2)$

Given 8 commuting indets

$$x'_{ij}, \quad x''_{ij} \qquad 1 \le i, j \le 2$$

Consider polynomial algebra

$$k[\, x'_{ij},\ x''_{ij},\ 1 \le i, j \le 2\,] = M(2)^{\otimes 2}$$

Bijections

$$\mathrm{Hom}_{alg}(K, A) \longrightarrow M_2(A)^0$$

Eval$_0$

$$\eta_A \longrightarrow \emptyset$$

$$\mathrm{Hom}_{alg}(M(2), A) \longrightarrow M_2(A)$$

Eval$_1$

$$F \longrightarrow \left( F(x_{ij}) \right)_{1 \le i,j \le 2}$$

$$\mathrm{Hom}_{alg}(M(2)^{\otimes 2}, A) \longrightarrow M_2(A)^2$$

Eval$_2$

$$F \longrightarrow \left( \left( F(x'_{ij}) \right)_{1 \le i,j \le 2}, \left( F(x''_{ij}) \right)_{1 \le i,j \le 2} \right)$$

Via above bijections

Mult

induces a map

$$\overline{\text{Mult}} : \text{Hom}_{alg}(M(2)^{\otimes 2}, A) \to \text{Hom}_{alg}(M(2), A)$$

Describe $\overline{\text{Mult}}$

Define algebra hom

$$\Delta : \quad M(2) \longrightarrow M(2)^{\otimes 2}$$

$$x_{ij} \longrightarrow \sum_{\ell=1}^{2} (x')_{i\ell} (x'')_{\ell j}$$

Prop 17 the following diagram commutes:

$$\begin{array}{ccc} F & \longrightarrow & F \circ \Delta \\ \mathrm{Homalg}(M(2)^{\otimes 2}, A) & \longrightarrow & \mathrm{Homalg}(M(2), A) \\ \mathrm{Eval}_2 \downarrow & & \downarrow \mathrm{Eval}_1 \\ M_2(A)^2 & \xrightarrow[\mathrm{Mult}]{} & M_2(A) \end{array}$$

pf Chase $F$ around diag $\square$

Detail

$$\begin{tikzcd} F \arrow[r] \arrow[d] & F \circ \Delta \arrow[d] \\ {} & {} \end{tikzcd}$$

$$\begin{pmatrix} F(\Delta(x_{11})) & F(\Delta(x_{12})) \\ F(\Delta(x_{21})) & F(\Delta(x_{22})) \end{pmatrix}$$

$$\| *$$

$$\begin{pmatrix} F(x'_{11}) & F(x'_{12}) \\ F(x'_{21}) & F(x'_{22}) \end{pmatrix}, \begin{pmatrix} F(x''_{11}) & F(x''_{12}) \\ F(x''_{21}) & F(x''_{22}) \end{pmatrix} \to \left(\begin{array}{c|c} F(x'_{11})F(x''_{11}) + F(x'_{12})F(x''_{21}) & F(x'_{11})F(x''_{12}) + F(x'_{12})F(x''_{22}) \\ \hline F(x'_{21})F(x''_{11}) + F(x'_{22})F(x''_{21}) & F(x'_{21})F(x''_{12}) + F(x'_{22})F(x''_{22}) \end{array}\right)$$

$x_{ij}$: For $1 \le i, j \le 2$

$$F(\Delta(x_{ij})) = F\left(\sum_l x'_{il} x''_{lj}\right)$$

$$= \sum_l F(x'_{il}) F(x''_{lj})$$

9/9/15

COR 18 $\overline{\text{Mult}}$ 15

16

$$\mathrm{Hom}_{alg}(M(2)^{\otimes 2}, A) \longrightarrow \mathrm{Hom}_{alg}(M(2), A)$$

$$F \longrightarrow F \circ \Delta$$

Given commutative algebra $A$

define

$$GL_2(A) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(A) \;\middle|\; ad-bc \in A^\times \right\}$$

Next goal: describe the mult group structure on $GL_2(A)$

Mult: $GL_2(A) \times GL_2(A) \to GL_2(A)$

$B, C \mapsto BC$

Inv: $GL_2(A) \to GL_2(A)$

$B \mapsto B^{-1}$

ident: $GL_2(A)^0 \to GL_2(A)$

$\emptyset \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Recall poly algebra

$$M(2) = k[x_{11}, x_{12}, x_{21}, x_{22}]$$

Consider commuting indets

$$x_{11}, x_{12}, x_{21}, x_{22}, t \qquad (*)$$

Define

$GL(2)$ = comm algebra with gens $(*)$

and relations

$$(x_{11}x_{22} - x_{12}x_{21})\,t = 1$$

So $GL(2)$ iso $M(2)[t]/I$

ideal $I$ gen by

$$(x_{11}x_{22} - x_{12}x_{21})\,t - 1$$

Comm indets

$x'_{ij}, \quad x''_{ij}, \qquad t', \; t''$ $\qquad$ **

Def $\quad GL(2)^{\otimes 2}$ is comm alg with gens **

and relations

$(x'_{11} x'_{22} - x'_{12} x'_{21})\, t' = 1,$

$(x''_{11} x''_{22} - x''_{12} x''_{21})\, t'' = 1.$

Bijections

$$\mathrm{Hom}_{alg}(k, A) \longleftrightarrow GL_2(A)^0$$

Eval 0 $\qquad \| \qquad\qquad \|$

$\qquad\qquad \{A\} \qquad\qquad \emptyset$

$$\mathrm{Hom}_{alg}(GL(2), A) \longleftrightarrow GL_2(A)$$

Eval$_1$ $\qquad F \longleftrightarrow (F(x_{ij}))_{1 \le i,j \le 2}$

$$\mathrm{Hom}_{alg}(GL(2)^{\otimes 2}, A) \longleftrightarrow GL_2(A)^2$$

Eval$_2$ $\qquad F \longleftrightarrow \left(F(x'_{ij})\right)_{1\le i,j\le 2}\left(F(x''_{ij})\right)_{1\le i,j\le 2}$