

IV the quantum planeFix $0 \neq q \in k$ Consider the algebra $k_q[x, y]$
with generators x, y and relation

$$yx = qxy$$

Call $k_q[x, y]$ the quantum plane $k_q[x, y]$ as an Ore extension.Given alg R , recall Ore extension $R[t, \alpha, \delta]$

where

$$t = \text{indot}$$

 $\alpha: R \rightarrow R$ is ring algebra morphism $\delta: R \rightarrow R$ is α -derivation

$$\left(\delta(ab) = \delta(a)b + \alpha(a)\delta(b) \quad \forall a, b \in R \right)$$

 $k_q[x, y]$ is Ore ext $R[t, \alpha, \delta]$ where

$$R = k[x], \quad t = y,$$

$$\alpha: R \rightarrow R, \quad x \mapsto qx$$

$$\delta = 0$$

Now from chapter I,

- $K_2[x, y]$ has basis

$$x^i y^j \quad i, j \in \mathbb{N}$$

- $K_2[x, y]$ has no 0-divisors

- $K_2[x, y]$ is Noetherian

Note that in $K_2[x, y]$,

$$y^i x^j = x^j y^i \quad i, j \in \mathbb{N}$$

Given alg R and $X, Y \in R$

\exists alg morphism

$$k_q[x, y] \longrightarrow R$$

$$x \longrightarrow X$$

$$y \longrightarrow Y$$

\forall

$$YX = qXY$$

In this case call the ordered pair (X, Y) an

R -point of $k_q[x, y]$

We have bijection

$$\text{Hom}_{\text{alg}} \{ k_q[x, y], R \} \longleftrightarrow \{ R\text{-points of } k_q[x, y] \}$$

Notation For $n \in \mathbb{N}$,

$$[n]_q = 1 + q + q^2 + \dots + q^{n-1} = \begin{cases} \frac{q^n - 1}{q - 1} & \text{if } q \neq 1 \\ n & \text{if } q = 1 \end{cases}$$

$$[n]_q! = [n]_q [n-1]_q \dots [1]_q$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q! [n-k]_q!} \quad 0 \leq k \leq n$$

(poly in q with integral coeffs)

let $\begin{bmatrix} n \\ k \end{bmatrix}_q = 0$ if $k < 0$ or $k > n$

We have

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ n-k \end{bmatrix}_q \quad 0 \leq k \leq n$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q + q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q$$

$n \geq 1$,
 $0 \leq k \leq n$

(q -binom thm) In $K_q[x, y]$,

$$(x+y)^n = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q x^k y^{n-k} \quad n \in \mathbb{N}$$

— 0 —

Recall $k[x, y]$ is a comodule algebra for bialg $M(2)$

Next goal: define a bialgebra $M_q(2)$ such that

$k_q[x, y]$ becomes a (left and right) comodule algebra for $M_q(2)$

Until further notice,

fix an algebra M ,

fix $a, b, c, d \in M$

LEM 1 TFAE:

(i) \exists an algebra morphism

$$\begin{array}{rcl}
 k_q[x, y] & \longrightarrow & M \otimes k_q[x, y] \\
 x & \longrightarrow & a \otimes x + b \otimes y \\
 y & \longrightarrow & c \otimes x + d \otimes y
 \end{array}$$

(ii) In M ,

$$ca = qac, \quad db = qbd,$$

$$qda + cb = qad + q^2bc$$

pf write

$$\bar{X} = ax + by$$

$$\bar{Y} = cx + dy$$

$$\bar{Y}\bar{X} - q\bar{X}\bar{Y} =$$

	$ax + by$	
cx	$ca \otimes x^2$	$cb \otimes xy$
$+ dy$	$da \otimes xy$	$db \otimes y^2$

$$-$$

	$cx + dy$	
ax	$ac \otimes x^2$	$ad \otimes xy$
$+ by$	$bc \otimes xy$	$bd \otimes y^2$

coef	\otimes	term
ca	$-qac$	x^2
$cb + qda$	$-qad - q^2bc$	xy
db	$-qbd$	y^2

Result follows.

□

LEM 2

TFAE

(i) \exists alg morphism

$$K_q[x, y] \rightarrow K_q[x, y] \otimes M$$

$$x \rightarrow x \otimes a + y \otimes c$$

$$y \rightarrow x \otimes b + y \otimes d$$

(ii) In M ,

$$ba = q ab, \quad dc = q cd$$

$$q da + bc = q ad + q^2 cb$$

pf (sim to pf + LEM 1)

Write $X = xa + y \circ c$

$Y = x \circ b + y \circ d$

$YX - \gamma XY =$

	$xa + y \circ c$
$x \circ b$	$x^2 \circ ba \quad x \gamma \circ bc$
+	
$y \circ d$	$y \gamma \circ da \quad y^2 \circ dc$
	" " $\gamma \times \gamma$

- γ

	$x \circ b + y \circ d$
xa	$x^2 \circ ab \quad x \gamma \circ ad$
+	
$y \circ c$	$y \gamma \circ cb \quad y^2 \circ cd$
	" " $\gamma \times \gamma$

=

term	coef
x^2	$ba - \gamma ab$
$x \gamma$	$bc + \gamma da - \gamma ad - \gamma^2 cb$
y^2	$dc - \gamma cd$

Result follows.



LEM 3. Assume $q^4 \neq 1$. Then

(i), (ii) hold in both LEM 1, 2 \mathcal{M}

$$ba = q ab, \quad dc = q cd$$

$$ca = q ac, \quad db = q bd$$

$$bc = cb, \quad ad - da = bc(q^2 - q)$$

} *

pf use LEM 1, 2 □

DEF 4. The algebra $M_q(2)$ has gens a, b, c, d and relations *

define

$$x_{11} = a$$

$$x_{12} = b$$

$$x_{21} = c$$

$$x_{22} = d$$

LEM 5 \exists alg morphisms

$\Delta: M_q(\mathbb{Z}) \rightarrow M_q(\mathbb{Z}) \otimes M_q(\mathbb{Z})$

$x_{ij} \rightarrow \sum_l x_{il} \otimes x_{lj} \quad 1 \leq i, j \leq 2$

$\varepsilon: M_q(\mathbb{Z}) \rightarrow k$

$x_{ij} \rightarrow \delta_{ij} \quad 1 \leq i, j \leq 2$

pf (Δ) write

$A = a \otimes a + b \otimes c$

$B = a \otimes b + b \otimes d$

$C = c \otimes a + d \otimes c$

$D = c \otimes b + d \otimes d$

$BA - \varepsilon AB =$

	$a \otimes a + b \otimes c$
$a \otimes b$	$a^2 \otimes ba + ab \otimes bc$
$+ b \otimes d$	$ba \otimes da + b^2 \otimes dc$
	gab

$- \varepsilon \left(\begin{array}{c|c} a \otimes b + b \otimes d & \\ \hline a \otimes a & a^2 \otimes ab + ab \otimes ad \\ + & \\ b \otimes c & ba \otimes cb + b^2 \otimes cd \\ & \\ & gab \end{array} \right)$

term	coef		
a^2	ba	$-q ab$	$(=0)$
ab	$bc + q da$	$-q ad - q^2 cb$	$(=0)$
b^2	dc	$-q cd$	$(=0)$

so

$$BA = qAC$$

We sim obtain

$$DC = qCD, \quad CA = qAC, \quad DB = qBD$$

$$BC = CB, \quad AD - DA = BC(q^2 - q).$$

So alg morphism Δ exists.

(E)

Define

$$\bar{a} = 1, \quad \bar{b} = 0,$$

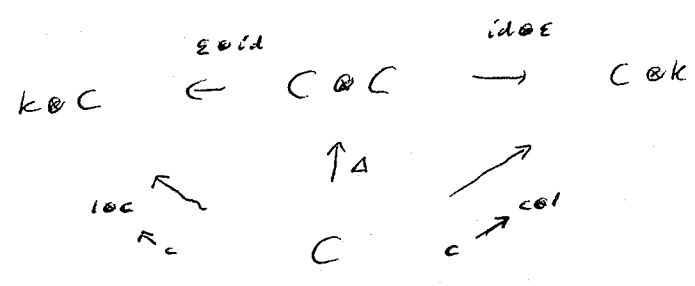
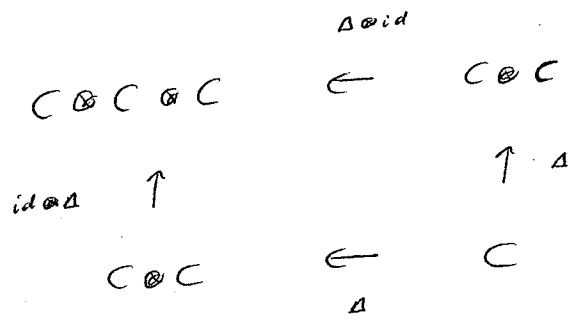
$$\bar{c} = 0, \quad \bar{d} = 1.$$

Then $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ satisfy *therefore alg morphism ε exists. \square

LEM 6 the maps Δ, ε in LEM 5

turn $M_q(2)$ into a bialgebra.

pf Abb $C = M_q(2)$ One checks these diag commute:



□

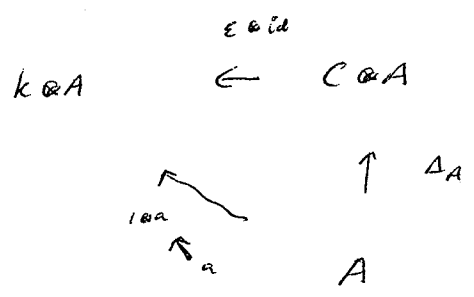
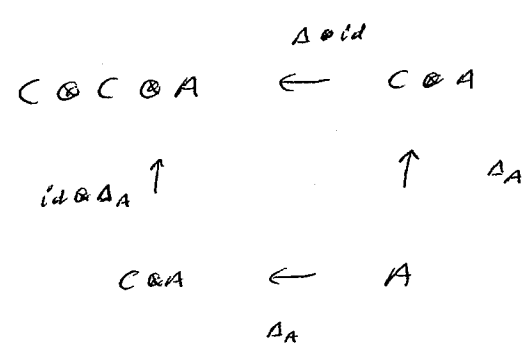
LEM 7 For

$$A = k_q[x, y], \quad C = M_q(2)$$

(i) the map $\Delta_A : A \rightarrow C \otimes A$ from LEM 2 (i) turns A into a left C -comodule algebra

(ii) the map $\Delta'_A : A \rightarrow A \otimes C$ from LEM 2 (i) turns A into a right C -comodule algebra.

pt (i) One checks these diagrams commute:



(ii) Similar.



A basis for $M_q(2)$

Recall alg $M_q(2)$

Generators: $\begin{matrix} a & b \\ c & d \end{matrix}$

Relations:

$ba = qab$

$dc = qcd$

$ca = qac$

$db = qbd$

$bc = cb$

$ad - da = (q-1)bc$

} *

Next goal: show that $M_q(2)$ has a basis

$a^i b^j c^k d^l$

$i, j, k, l \in \mathbb{N}$

We define a sequence of algebras

algebra	M_1	M_2	M_3	M_4
gens	a	a, b	a, b, c	a, b, c, d
rels	\emptyset	$ba = qab$	$ba = qab$ $ca = qac$ $bc = cb$	*

so

$M_1 = k[a]$,

$M_2 = k_q[a, b]$,

$M_4 = M_q(2)$

For $i=2,3,4$ show M_i is an Ore extension of M_{i-1}

11/9/15
15

We saw M_2 is Ore ext of M_1

Show M_3 is an Ore ext of M_2

Since $ba = qab$ is the defining relation for M_2 ,

\exists alg morphism $\alpha: M_2 \rightarrow M_2$ that sends

$$a \rightarrow qa, \quad b \rightarrow b$$

(Indeed for $\bar{a} = qa, \bar{b} = b$ we have $\bar{b}\bar{a} = q\bar{a}\bar{b}$)

Obs α is a bijection.

Also, $\delta = 0$ is an α -derivation of M_2

For the Ore ext $M_2[t, \alpha, \delta]$,

$$ta = \delta(a) + \alpha(a)t = qat$$

$$tb = \delta(b) + \alpha(b)t = bt$$

So \exists alg morphism

$$\theta: M_3 \rightarrow M_2[t, \alpha, \delta]$$

that sends

$$a \rightarrow a, \quad b \rightarrow b, \quad c \rightarrow t$$