Lec 29 Monday Nov 9 11/9/15

IV the quantum plane

Fix $0 \neq q \in k$

Consider the algebra $k_q[x,y]$

with generators x, y and relation

$$yx = qxy$$

Call $k_q[x,y]$ the <u>quantum plane</u>

$k_q[x,y]$ as an Ore extension

Given alg R, recall Ore extension $R[t, \alpha, \delta]$

where

t = indet

$\alpha : R \to R$ is inj algebra morphism

$\delta : R \to R$ is α-derivation

$$\left(\delta(ab) = \delta(a)b + \alpha(a)\delta(b) \quad \forall a,b \in R \right)$$

$k_q[x,y]$ is Ore ext $R[t, \alpha, \delta]$ where

$R = k[x]$, $t = y$, $\alpha : \begin{matrix} R \to R \\ x \to qx \end{matrix}$, $\delta = 0$

Now from chapter I,

- $K_q[x,y]$ has basis

$$x^i y^j \qquad i,j \in \mathbb{N}$$

- $K_q[x,y]$ has no 0-divisors

- $K_q[x,y]$ is Noetherian

Note that in $K_q[x,y]$,

$$y^i x^j = q^{ij} x^j y^i \qquad i,j \in \mathbb{N}$$

Given alg R and $\mathbb{X}, \mathbb{Y} \in R$

$\exists$ alg morphism

$$k_q[x,y] \longrightarrow R$$

$$x \longrightarrow \mathbb{X}$$

$$y \longrightarrow \mathbb{Y}$$

iff

$$\mathbb{Y}\mathbb{X} = q\,\mathbb{X}\mathbb{Y}$$

In this case call the ordered pair $(\mathbb{X}, \mathbb{Y})$ an

<u>R-point of $k_q[x,y]$</u>

We have bijection

$$\mathrm{Hom}_{alg}\{k_q[x,y], R\} \longleftrightarrow \{R\text{-points of } k_q[x,y]\}$$

Notation For $n \in \mathbb{N}$,

$$[n]_q = 1 + q + q^2 + \cdots + q^{n-1} = \begin{cases} \dfrac{q^n - 1}{q-1} & \text{if } q \neq 1 \\ n & \text{if } q = 1 \end{cases}$$

$$[n]_q! = [n]_q [n-1]_q \cdots [1]_q$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q! \, [n-k]_q!} \qquad 0 \le k \le n$$

(poly in q with integral coefs)

let $\begin{bmatrix} n \\ k \end{bmatrix}_q = 0$ if $k < 0$ or $k > n$

We have

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ n-k \end{bmatrix}_q \qquad 0 \le k \le n$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q + q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q \qquad n \ge 1, \; 0 \le k \le n$$

(q-binom thm) In $k_q[x,y]$,

$$(x+y)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k y^{n-k} \qquad n \in \mathbb{N}$$

— o —

Recall $k[x,y]$ is a comodule algebra for bialg $M(2)$

Next goal: define a bialgebra $M_q(2)$ such that

$k_q[x,y]$ becomes a (left and right) comodule algebra for $M_q(2)$

Until further notice,

fix an algebra M,

fix $a,b,c,d \in M$

LEM 1 TFAE:

(i) $\exists$ an algebra morphism

$$\begin{aligned} k_q[x,y] &\longrightarrow M \otimes k_q[x,y] \\ x &\longrightarrow a\otimes x + b\otimes y \\ y &\longrightarrow c\otimes x + d\otimes y \end{aligned}$$

(ii) In M,

$$ca = qac, \qquad db = qbd,$$

$$qda + cb = qad + q^2 bc$$

pf Write

$$\underline{X} = a \otimes x + b \otimes y$$

$$\underline{Y} = c \otimes x + d \otimes y$$

$$\underline{Y}\,\underline{X} - q\,\underline{X}\,\underline{Y} =$$

$$\begin{array}{c|cc} & a\otimes x & + b\otimes y \\ \hline c\otimes x & ca\otimes x^2 & cb\otimes xy \\ + & & \\ d\otimes y & da\otimes \underset{\substack{\shortparallel \\ qxy}}{yx} & db\otimes y^2 \end{array} \quad - \quad q\left(\begin{array}{c|cc} & c\otimes x & + d\otimes y \\ \hline a\otimes x & ac\otimes x^2 & ad\otimes xy \\ + & & \\ b\otimes y & bc\otimes \underset{\substack{\shortparallel \\ qxy}}{yx} & bd\otimes y^2 \end{array} \right)$$

$=$

coef	$\otimes$	term
$ca \quad - q\,ac$		x^2
$cb \quad + q\,da \quad - q\,ad \quad - q^2 bc$		xy
$db \quad - q\,bd$		y^2

Result follows.

$\square$

LEM 2 TFAE

(i) $\exists$ alg morphism

$$K_q[x,y] \longrightarrow K_q[x,y] \otimes M$$

$$x \longrightarrow x \otimes a + y \otimes c$$

$$y \longrightarrow x \otimes b + y \otimes d$$

(ii) In M,

$$ba = q\,ab, \qquad dc = q\,cd$$

$$q\,da + bc = q\,ad + q^2 cb$$

pf (sim to pf of LEM 1)

Write $\underline{X} = x\otimes a + y\otimes c$

$\underline{Y} = x\otimes b + y\otimes d$

$\underline{Y}\,\underline{X} - q\,\underline{X}\,\underline{Y} =$

$$\begin{array}{c|cc} & \multicolumn{2}{c}{x\otimes a + y\otimes c} \\ \hline x\otimes b & x^2\otimes ba & xy\otimes bc \\ + & & \\ y\otimes d & \underset{qxy}{\underset{\shortparallel}{yx}}\otimes da & y^2\otimes dc \end{array} \quad - q \left(\begin{array}{c|cc} & \multicolumn{2}{c}{x\otimes b + y\otimes d} \\ \hline x\otimes a & x^2\otimes ab & xy\otimes ad \\ + & & \\ y\otimes c & \underset{qxy}{\underset{\shortparallel}{yx}}\otimes cb & y^2\otimes cd \end{array} \right)$$

$$= \begin{array}{c|l} \text{term } \otimes & \text{coef} \\ \hline x^2 & ba - qab \\ xy & bc + qda - qad - q^2cb \\ y^2 & dc - qcd \end{array}$$

Result follows. $\square$

LEM 3. Assume $q^4 \neq 1$. Then

(i), (ii) hold in both LEM 1, 2 iff

$$\left.\begin{array}{ll} ba = q\,ab, & dc = q\,cd \\ ca = q\,ac, & db = q\,bd, \\ bc = cb, & ad - da = bc(q^{-1} - q) \end{array}\right\} *$$

pf Use LEM 1, 2 $\square$

DEF 4. The algebra $M_q(2)$ has gens a, b, c, d and relations $*$.

Define

$$x_{11} = a \qquad x_{12} = b$$

$$x_{21} = c \qquad x_{22} = d$$

LEM 5 $\exists$ alg morphisms

$$\Delta: \quad M_q(2) \longrightarrow M_q(2) \otimes M_q(2)$$

$$x_{ij} \longrightarrow \sum_{\ell} x_{i\ell} \otimes x_{\ell j} \qquad 1 \le i,j \le 2$$

$$\varepsilon: \quad M_q(2) \longrightarrow k$$

$$x_{ij} \longrightarrow \delta_{ij} \qquad 1 \le i,j \le 2$$

pf (Δ) write

$A = a\otimes a + b\otimes c$ $\qquad B = a\otimes b + b\otimes d$

$C = c\otimes a + d\otimes c$ $\qquad D = c\otimes b + d\otimes d$

$BA - qAB =$

$$\begin{array}{c|cc} & a\otimes a & + \; b\otimes c \\ \hline a\otimes b & a^2\otimes ba & ab\otimes bc \\ + & & \\ b\otimes d & \underset{\underset{qab}{\parallel}}{ba}\otimes da & b^2\otimes dc \end{array} \quad - \; q\left(\begin{array}{c|cc} & a\otimes b & + \; b\otimes d \\ \hline a\otimes a & a^2\otimes ab & ab\otimes ad \\ + & & \\ b\otimes c & \underset{\underset{qab}{\parallel}}{ba}\otimes cb & b^2\otimes cd \end{array} \right)$$

term ⊗	coef		
a^2	ba	$-q\,ab$	$(=0)$
ab	$bc \quad +q\,da$	$-q\,ad - q^2 cb$	$(=0)$
b^2	dc	$-q\,cd$	$(=0)$

so

$$BA = qAC$$

We sim obtain

$$DC = qCD, \qquad CA = qAC, \qquad DB = qBD$$

$$BC = CB, \qquad AD - DA = BC(q^{-1} - q).$$

So alg morphism Δ exists.

(E) Define

$$\bar{a} = 1, \qquad \bar{b} = 0,$$

$$\bar{c} = 0, \qquad \bar{d} = 1.$$

then $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ satisfy $*$.

therefore alg morphism ε exists. □

LEM 6 the maps Δ, ε in LEM 5 turn $M_q(2)$ into a bialgebra.

pf Abbr $C = M_q(2)$ One checks these diags commute:

$$\begin{tikzcd}
C \otimes C \otimes C & C \otimes C \arrow[l, "\Delta \otimes id"'] \\
C \otimes C \arrow[u, "id \otimes \Delta"] & C \arrow[l, "\Delta"] \arrow[u, "\Delta"']
\end{tikzcd}$$

$$\begin{tikzcd}
k \otimes C & C \otimes C \arrow[l, "\varepsilon \otimes id"'] \arrow[r, "id \otimes \varepsilon"] & C \otimes k \\
& C \arrow[ul, "1 \otimes c \mapsfrom c"] \arrow[u, "\Delta"] \arrow[ur, "c \mapsto c \otimes 1"'] &
\end{tikzcd}$$

$\square$

LEM 7 For

$$A = k_q[x,y], \qquad C = M_q(2)$$

(i) The map $\Delta_A : A \to C \otimes A$ from LEM 1 (i) turns A into a left C-comodule algebra

(ii) The map $\Delta'_A : A \to A \otimes C$ from LEM 2 (i) turns A into a right C-comodule algebra.

pf (i) One checks these diagrams commute:

$$\begin{tikzcd}
C \otimes C \otimes A & C \otimes A \arrow[l, "\Delta \otimes id"'] \\
C \otimes A \arrow[u, "id \otimes \Delta_A"] & A \arrow[l, "\Delta_A"] \arrow[u, "\Delta_A"']
\end{tikzcd}$$

$$\begin{tikzcd}
k \otimes A & C \otimes A \arrow[l, "\varepsilon \otimes id"'] \\
 & A \arrow[u, "\Delta_A"'] \arrow[ul, "1 \otimes a \,\mapsfrom\, a"]
\end{tikzcd}$$

(ii) Similar. $\square$

~~A~~ basis for $M_q(2)$

Recall alg $M_q(2)$

Generators: $\begin{matrix} a & b \\ c & d \end{matrix}$

Relations:

$$\left.\begin{matrix} ba = qab & dc = qcd \\ ca = qac & db = qbd \\ bc = cb & ad - da = (q^{-1} - q)bc \end{matrix}\right\} \quad *$$

Next goal: show that $M_q(2)$ has a basis

$$a^i b^j c^k d^l \qquad i, j, k, l \in \mathbb{N}$$

We define a sequence of algebras

algebra	M_1	M_2	M_3	M_4
gens	a	a, b	a, b, c	a, b, c, d
rels	ϕ	$ba = qab$	$ba = qab$ $ca = qac$ $bc = cb$	$*$

So

$M_1 = k[a]$, $\quad M_2 = k_q[a, b]$, $\quad M_4 = M_q(2)$

For $i=2,3,4$ show M_i is an Ore extension of M_{i-1}

We saw M_2 is Ore ext of M_1

Show M_3 is an Ore ext of M_2

Since $ba=qab$ is the defining relation for M_2,

$\exists$ alg morphism $\alpha: M_2 \to M_2$ that sends

$$a \to qa, \qquad b \to b$$

(Indeed for $\bar{a}=qa$, $\bar{b}=b$ we have $\bar{b}\bar{a}=q\bar{a}\bar{b}$)

Obs α is a bijection.

Also, $\delta=0$ is an α-derivation of M_2

For the Ore ext $M_2[t,\alpha,\delta]$,

$$ta = \delta(a)+\alpha(a)t = qat$$
$$tb = \delta(b)+\alpha(b)t = bt$$

So $\exists$ alg morphism

$$\theta: M_3 \to M_2[t,\alpha,\delta]$$

that sends

$$a \to a, \quad b \to b, \quad c \to t$$