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Given bialgebra C and vector space V

Then $C \otimes V$ becomes a C-comodule in two ways:

(I) Free C-comodule $C \otimes V$ with action

$$C \otimes V \xrightarrow{\Delta \otimes id} C \otimes C \otimes V$$

(II) Start with C-comodule C, with action

$$\Delta : C \longrightarrow C \otimes C$$

Also trivial C-comodule V, with action

$$V \longrightarrow C \otimes V$$
$$x \longmapsto 1_C \otimes x$$

Given C-comodule $C \otimes V$ from below LEM 61, with action

$$C \otimes V \xrightarrow{\Delta \otimes (\Delta \otimes id)} C \otimes C \otimes C \otimes V \xrightarrow{id \otimes T_{C,C} \otimes id} C \otimes C \otimes C \otimes V \xrightarrow{\mu \otimes id} C \otimes C \otimes V$$

LEM 64 The above C-comodule structures I, II on $C \otimes V$ are the same.

pf Compare

$$\text{I}: \quad C \otimes V \longrightarrow C \otimes C \otimes V$$

$$c \otimes x \longrightarrow \sum_{(c)} c' \otimes c'' \otimes x$$

$$\text{II}: \quad C \otimes V \to C \otimes C \otimes C \otimes V \longrightarrow C \otimes C \otimes C \otimes V \to C \otimes C \otimes V$$

$$c \otimes x \to \Big(\sum_{(c)} c' \otimes c''\Big) \otimes (1_C \otimes x) \to \sum_{(c)} c' \otimes 1_C \otimes c'' \otimes x \to \sum_{(c)} \underbrace{(c' 1_C)}_{=c'} \otimes c'' \otimes x$$

$$= \sum_{(c)} c' \otimes c'' \otimes 1_C \otimes x$$

Same $\square$

LEM 65 Given bialg C and C-comodules U, V, W

then the vs iso

$$(U \otimes V) \otimes W \longrightarrow U \otimes (V \otimes W)$$

$$(x \otimes y) \otimes z \longrightarrow x \otimes (y \otimes z)$$

is a C-comodule morphism.

pf Recall actions

$$U \longrightarrow C \otimes U$$
$$x \longrightarrow \sum_{(x)} x_C \otimes x_U$$

$$V \longrightarrow C \otimes V$$
$$y \longrightarrow \sum_{(y)} y_C \otimes y_V$$

$$W \longrightarrow C \otimes W$$
$$z \longrightarrow \sum_{(z)} z_C \otimes z_W$$

$$U \otimes V \longrightarrow C \otimes (U \otimes V) \qquad \text{etc.}$$
$$x \otimes y \longrightarrow \sum_{(x)} \sum_{(y)} (x_C y_C) \otimes x_U \otimes y_V$$

Action fn $(U \otimes V) \otimes W$:

$$(x \otimes y) \otimes z \longrightarrow \sum_{(x)} \sum_{(y)} \sum_{(z)} (x_c y_c) z_c \otimes (x_u \otimes y_v) \otimes z_w$$

Action fn $U \otimes (V \otimes W)$:

$$x \otimes (y \otimes z) \longrightarrow \sum_{(x)} \sum_{(y)} \sum_{(z)} x_c (y_c z_c) \otimes x_u \otimes (y_v \otimes z_w)$$

Result follows since algebra C is associative $\square$

LEM 66 Given bialg C and C-comodule V.

then the vs isomorphisms

$\varphi: V \to k \otimes V$, $\quad v \to 1 \otimes v$

$\phi: V \to V \otimes k$, $\quad v \to v \otimes 1$

are comodule morphisms.

pf Consider φ

Recall C-comodule k

$$k \to C \otimes k$$
$$1 \to 1_C \otimes 1$$

C-comodule V

$$V \to C \otimes V$$
$$x \to \sum_{(x)} x_C \otimes x_V$$

C-comodule $k \otimes V$:

$$k \otimes V \to C \otimes (k \otimes V)$$
$$1 \otimes x \to \sum_{(x)} \underbrace{(1_C x_C)}_{= x_C} \otimes (1 \otimes x_V)$$

check diagram

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "\Delta_V"'] & k \otimes V \arrow[d, "\Delta_{k\otimes V}"] \\ C \otimes V \arrow[r, "id \otimes \varphi"'] & C \otimes k \otimes V \end{tikzcd}$$

$$\begin{tikzcd} x \arrow[r] \arrow[d] & 1 \otimes x \arrow[d] \\ \sum_{(x)} x_C \otimes x_V \arrow[r] & \sum_{(x)} x_C \otimes 1 \otimes x_V \end{tikzcd}$$

ok

$\square$

The case of ϕ is similar.

LEM 67 Given bialg C and C-comodules U, V

the vs iso $\tau_{U,V} : U \otimes V \to V \otimes U$

is a C-comodule morphism provided that the algebra C is commutative

pf

$$\begin{tikzcd} U \otimes V \arrow[r, "\tau"] \arrow[d, "\Delta_{U\otimes V}"'] & V \otimes U \arrow[d, "\Delta_{V \otimes U}"] \\ C \otimes U \otimes V \arrow[r, "id \otimes \tau"'] & C \otimes V \otimes U \end{tikzcd}$$

commutes ?

$$\begin{tikzcd} x \otimes y \arrow[r] \arrow[dd] & y \otimes x \arrow[d] \\ & \sum_{(x)} \sum_{(y)} y_C x_C \otimes y_V \otimes x_U \\ \sum_{(x)} \sum_{(y)} (x_C y_C) \otimes x_U \otimes y_V \arrow[r] & \sum_{(x)} \sum_{(y)} (x_C y_C) \otimes y_V \otimes x_U \end{tikzcd}$$

$=$?

ok if $x_C y_C = y_C x_C \quad \forall x \in U \ \forall y \in V$

$\square$

(Aside)

Motivation

Given vs V

Given alg A. Recall the A-module structures on V are in bijection with the algebra morphisms

$$A \to \mathrm{End}(V)$$

Given coalg C. Expect the C-comodule structures on V are in bijection with the coalgebra morphisms $\mathrm{End}(V) \to C$

For convenience assume $\dim V < \infty$

Fix basis $\{v_i\}$ for V

$\mathrm{End}(V)$ has basis $\{e_{ij}\}$

$$e_{ij}(v_r) = \delta_{jr} v_i$$

$\mathrm{End}(V)$ is coalg with

$$\Delta(e_{ij}) = \sum_{l} e_{il} \otimes e_{lj}$$

$$\varepsilon(e_{ij}) = \delta_{ij}$$

Given any lin map

$$\Delta_V : \quad V \longrightarrow C \otimes V$$
$$\qquad v_i \longrightarrow \sum_j c_{ij} \otimes v_j$$

Consider when does Δ_V turn V into C-comodule.

Consider diagrams

$$\begin{tikzcd}
C \otimes C \otimes V & C \otimes V \arrow[l, "\Delta \otimes id"'] \\
C \otimes V \arrow[u, "id \otimes \Delta_V"] & V \arrow[l, "\Delta_V"] \arrow[u, "\Delta_V"']
\end{tikzcd} \qquad *$$

$$\begin{tikzcd}
\sum_j \Delta(c_{ij}) \otimes v_j \overset{?}{=} \sum_j \left(\sum_l c_{il} \otimes c_{lj} \right) \otimes v_j & \sum_j c_{ij} \otimes v_j \arrow[l] \\
\sum_l c_{il} \otimes \sum_j c_{lj} \otimes v_j & \\
\sum_l c_{il} \otimes v_l \arrow[u] & v_i \arrow[l] \arrow[uu]
\end{tikzcd}$$

diag $*$ commutes iff

$$\sum_j \Delta(c_{ij}) \otimes v_j = \sum_j \left(\sum_\ell c_{i\ell} \otimes c_{\ell j} \right) \otimes v_j \qquad \forall i$$

iff

$$\Delta(c_{ij}) = \sum_\ell c_{i\ell} \otimes c_{\ell j} \qquad \forall i, j$$

Another diagram

$$\begin{tikzcd}
k \otimes V & C \otimes V \arrow[l, "\varepsilon \otimes id"'] \\
 & V \arrow[u, "\Delta_V"'] \arrow[ul, "1 \otimes v \mapsfrom v"]
\end{tikzcd} \qquad **$$

$$\begin{tikzcd}
\sum_j \varepsilon(c_{ij}) \otimes v_j & \sum_j c_{ij} \otimes v_j \arrow[l] \\
 & v_i \arrow[u] \arrow[ul]
\end{tikzcd}$$

$$\sum_j \varepsilon(c_{ij}) \otimes v_j = 1 \otimes \sum_j \varepsilon(c_{ij}) v_j \overset{?}{=} 1 \otimes v_i$$

Diag $**$ commutes iff

$$v_i = \sum_j \varepsilon(c_{ij})\, v_j \qquad \forall i$$

iff

$$\varepsilon(c_{ij}) = \delta_{ij} \qquad \forall i, j$$

Given any linear map

$$\varphi_V : \mathrm{End}(V) \to C$$
$$e_{ij} \to c_{ij}$$

Consider when is φ_V a coalg morphism

Consider diagrams

$$\begin{array}{ccc} \mathrm{End}(V) & \xrightarrow{\varphi_V} & C \\ \Delta \downarrow & & \downarrow \Delta \\ \mathrm{End}(V) \otimes \mathrm{End}(V) & \xrightarrow[\varphi_V \otimes \varphi_V]{} & C \otimes C \end{array} \qquad \begin{array}{ccc} e_{ij} & \to & c_{ij} \\ \downarrow & & \downarrow \\ \sum_\ell e_{i\ell} \otimes e_{\ell j} & \to & \sum_\ell c_{i\ell} \otimes c_{\ell j} \end{array}$$

diag commutes iff

$$\Delta(c_{ij}) = \sum_\ell c_{i\ell} \otimes c_{\ell j} \qquad \forall i,j$$

$$\begin{array}{ccc} \mathrm{End}(V) & \xrightarrow{\varphi_V} & C \\ \varepsilon \downarrow & & \downarrow \varepsilon \\ k & \xrightarrow[id]{} & k \end{array} \qquad \begin{array}{ccc} e_{ij} & \to & c_{ij} \\ \downarrow & & \downarrow \\ \delta_{ij} & \to & \delta_{ij} \end{array}$$

diag commutes iff

$$\varepsilon(c_{ij}) = \delta_{ij} \qquad \forall i,j.$$

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We conclude:

Prop A68 Given C, V, $\{c_{ij}\}$ as above.

TFAE

(i) $\Delta_V : V \to C \otimes V$ turns V into a C-comodule

(ii) $\varphi_V : \mathrm{End}(V) \to C$ is coalg morphism

(iii) $\forall\, i, j$

$$\Delta(c_{ij}) = \sum_{l} c_{il} \otimes c_{lj}$$

$$\varepsilon(c_{ij}) = \delta_{ij}$$

$\square$

COR A69 Given f.d. vector space V

Given basis $\{v_i\}$ for V

then V is a comodule for the coalg $End(V)$,

with action

$$\Delta_V : \begin{array}{ccc} V & \longrightarrow & End(V) \otimes V \\ v_i & \longrightarrow & \sum_j e_{ij} \otimes v_j \end{array}$$

pf Recall coalg $End(V)$ satisfies

$$\Delta(e_{ij}) = \sum_l e_{il} \otimes e_{lj} \qquad \forall i,j$$

$$\varepsilon(e_{ij}) = \delta_{ij}$$

Now apply Prop 68 with $C = End(V)$

and $c_{ij} = e_{ij}$ $\forall i,j$

$\square$

LEM A70 Given Hopf algebra C with antipode S

Given f.d. C-comodule V

Pick basis $\{v_i\}$ for V

So
$$\Delta_V : \begin{array}{ccc} V & \longrightarrow & C \otimes V \\ v_i & \longrightarrow & \sum_j c_{ij} \otimes v_j \end{array}$$

Then V^* becomes C-comodule with action

$$\Delta_{V^*} : \begin{array}{ccc} V^* & \rightarrow & C \otimes V^* \\ v^i & \rightarrow & \sum_j \underbrace{S(c_{ji})}_{\tilde{c}_{ij}} \otimes v^j \end{array}$$

pf Require by Prop A68:

$$\Delta(\tilde{c}_{ij}) \stackrel{?}{=} \sum_l \tilde{c}_{il} \otimes \tilde{c}_{lj} \qquad \forall i j$$

$$\varepsilon(\tilde{c}_{ij}) \stackrel{?}{=} \delta_{ij}$$

$$\Delta(\tilde{c}_{ij}) \stackrel{?}{=} \underbrace{\sum_l \underset{S(c_{li})}{\tilde{c}_{il}} \otimes \underset{S(c_{jl})}{\tilde{c}_{lj}}}$$

$$= \Delta(S(c_{ji})) = S \otimes S \Big(\underbrace{\sum_l c_{li} \otimes c_{jl}}_{= \Delta^{op}(c_{ji})} \Big)$$

$$= (\Delta \circ S)(c_{ji})$$

$$\Delta \circ S \; (c_{ji}) \stackrel{?}{=} S \otimes S \circ \Delta^{op} \; (c_{ji})$$

We saw earlier
$$\Delta \circ S = S \otimes S \circ \Delta^{op} \qquad \text{OK}$$

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$$\varepsilon(\tilde{c}_{ij}) \overset{?}{=} \delta_{ij}$$

$\|$

$$\varepsilon(S(c_{ji})) \quad \Leftarrow \quad \varepsilon(S(c)) \underset{\forall c}{=} \varepsilon(c)$$

$\|$

$$\varepsilon(c_{ji})$$

$\|$ OK

$$\delta_{ji}$$

$\square$

Given f.d. Hopf alg C with antipode S

Given f.d. C-comodule V

Compare C-comodules V, V^*

Prop. The following diagram commutes:

$$
\begin{tikzcd}
V \otimes V^* \arrow[r, "\Delta_V \otimes id"'] \arrow[d, "id \otimes \Delta_{V^*}"'] & C \otimes V \otimes V^* \arrow[r, "id \otimes \tau_{V,V^*}"'] & C \otimes V^* \otimes V \arrow[r, "id \otimes ev"'] & C \otimes k \arrow[r, "c \otimes 1 \mapsto c"'] & C \arrow[d, "S"] \\
V \otimes C \otimes V^* \arrow[r, "\tau_{V,C} \otimes id"'] & C \otimes V \otimes V^* \arrow[r, "id \otimes \tau_{V,V^*}"'] & C \otimes V^* \otimes V \arrow[r, "id \otimes ev"'] & C \otimes k \arrow[r, "c \otimes 1 \mapsto c"'] & C
\end{tikzcd}
$$

pf chase $v_s \otimes v^r$ around diag.

$$
\begin{tikzcd}
v_s \otimes v^r \arrow[r] \arrow[d] & \sum_j c_{sj} \otimes v_j \otimes v^r \arrow[r] & \sum_j c_{sj} \otimes v^r \otimes v_s \arrow[r] & \sum_j c_{sj} \otimes \overset{\delta_{rs}}{\overset{\parallel}{\langle v^r, v_s \rangle}} \arrow[r] & c_{sr} \arrow[d] \\
v_s \otimes \sum_t S(c_{tr}) \otimes v^t \arrow[r] & \sum_t S(c_{tr}) \otimes v_s \otimes v^t \arrow[r] & \sum_t S(c_{tr}) \otimes v^t \otimes v_s \arrow[r] & \sum_t S(c_{tr}) \otimes \underset{\delta_{ts}}{\underset{\parallel}{\langle v^t, v_s \rangle}} \arrow[r] & S(c_{sr})
\end{tikzcd}
$$

ok

$\square$

Given f.d. Hopf alg C with antipode S

Given f.d. C-comodule V

Pick basis $\{v_i\}$ for V

Write

$$\Delta_V : \begin{array}{ccc} V & \to & C \otimes V \\ v_i & \to & \sum_j c_{ij} \otimes v_j \end{array}$$

Find $S(c_{ij})$

Recall

$$\Delta : \begin{array}{ccc} C & \to & C \otimes C \\ c_{ij} & \to & \sum_l c_{il} \otimes c_{lj}. \end{array} \qquad \varepsilon : \begin{array}{ccc} C & \to & k \\ c_{ij} & \to & \delta_{ij} \end{array}$$

Also $\forall x \in C$

$$\varepsilon(x)\, 1_C = \sum_{(x)} S(x')\, x'' = \sum_{(x)} x'\, S(x'')$$

take $x = c_{ij}$

$$\delta_{ij}\, 1_C = \sum_l S(c_{il})\, c_{lj} = \sum_l c_{il}\, S(c_{lj})$$

so the matrices

$$(c_{ij})\,(S(c_{ij})) = (S(c_{ij}))\,(c_{ij}) = \begin{pmatrix} 1_C & & 0 \\ & 1_C & \\ 0 & & \ddots \; 1_C \end{pmatrix}$$

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Given f.d. Hopf alg C with antipode S

Given f.d. C-comodule V

Consider is

$$ev: V^* \otimes V \longrightarrow k$$

a C-comodule morphism?

Turns out we need to compose with τ

LEMMA 72 The map

$$V \otimes V^* \xrightarrow{\tau_{V,V^*}} V^* \otimes V \xrightarrow{ev} k$$

is a C-comodule morphism.

pf Recall actions

Fix basis $\{v_i\}$ for V, dual basis $\{v^i\}$ for V^*

$$V \to C \otimes V \qquad V^* \to C \otimes V^*$$

$$v_i \to \sum_l c_{il} \otimes v_l \qquad v^j \to \sum_m S(c_{mj}) \otimes v^m$$

$$V \otimes V^* \to C \otimes V \otimes V^*$$

$$\Delta_{V \otimes V^*}: v_i \otimes v^j \to \sum_l \sum_m c_{il} S(c_{mj}) \otimes v_l \otimes v^m$$

show this commutes:

$$\begin{tikzcd} V\otimes V^* \arrow[r, "*"] \arrow[d, "\Delta_{V\otimes V^*}"'] & k \arrow[d, "\Delta_k"] \\ C\otimes V\otimes V^* \arrow[r, "id\otimes *"'] & C\otimes k \end{tikzcd}$$

$$\begin{tikzcd} v_i\otimes v^j \arrow[r] \arrow[d] & \delta_{ij} \arrow[d] \\ & 1_C\otimes\delta_{ij} \\ \sum_l\sum_m c_{il}S(c_{mj})\otimes v_l\otimes v^m \arrow[r] & \sum_l\sum_m c_{il}S(c_{mj})\otimes\delta_{lm} \end{tikzcd}$$

$$= \underbrace{\sum_l c_{il}S(c_{lj})}_{=\,\delta_{ij}1_C}\otimes 1$$

$$= 1_C\otimes\delta_{ij} \qquad \text{ok}$$

$\square$

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Natural moves

A - module V $\longrightarrow$ A^*-comodule V^*

A - module V $\longrightarrow$ A-module V^*

C - comodule V $\longrightarrow$ C^*-module V^*

C - comodule V $\longrightarrow$ C - comodule V^*

Combinations

$$A\text{-mod } V \longrightarrow A^*\text{-comod } V^* \longrightarrow (A^*)^*\text{-mod } (V^*)^*$$

$$\begin{tikzcd}
A \otimes V \arrow[r, "\text{action}"] \arrow[d, "\text{nat} \otimes \text{nat}"'] & V \arrow[d, "\text{nat}"] \\
(A^*)^* \otimes (V^*)^* \arrow[r, "\text{action}"'] & (V^*)^*
\end{tikzcd}$$

commutes

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Check

A-module V $\qquad$ $A \otimes V \to V$

$a \otimes v \to av$

A^*-comodule V^* $\qquad$ $V^* \to A^* \otimes V^*$

$\shortparallel \qquad \shortparallel$

$C \qquad W$

$$f \to \sum_{(f)} f_{A^*} \otimes f_{V^*}$$

$$f(av) = \sum_{(f)} f_{A^*}(a)\, f_{V^*}(v)$$

C^*-mod W^* $\qquad$ $C^* \otimes W^* \to W^*$

$g \otimes h \to gh$

$$(gh)(f) = \sum_{(f)} g(f_{A^*})\, h(f_{V^*})$$

nat: $V \to (V^*)^* = W^*$

$v \to \hat{v} \qquad \hat{v}(f) = f(v)$

nat: $A \to (A^*)^* = C^*$

$a \to \hat{a} \qquad \hat{a}(g) = g(a)$

$$\begin{array}{ccc} a \otimes v & \longrightarrow & av \\ \downarrow & & \downarrow \\ & & \widehat{av} \\ & & \shortparallel ? \\ \hat{a} \otimes \hat{v} & \longrightarrow & \hat{a}\hat{v} \end{array}$$

take $g = \hat{a}$

$h = \hat{v}$

$$\hat{a}\hat{v}(f) = \sum_{(f)} \hat{a}(f_{A^*})\, \hat{v}(f_{V^*})$$

$$= \sum_{(f)} f_{A^*}(a)\, f_{V^*}(v)$$

$$= f(av)$$

$$= \widehat{av}(f)$$

so $\hat{a}\hat{v} = \widehat{av}$

ok

Combinations, cont

$$C\text{-comod } V \to C^*\text{-mod } V^* \to (C^*)^*\text{-comod } (V^*)^*$$

$$\begin{array}{ccc} & \text{coaction} & \\ V & \longrightarrow & C \otimes V \\ \text{nat} \downarrow & & \downarrow \text{nat} \otimes \text{nat} \\ (V^*)^* & \longrightarrow & (C^*)^* \otimes (V^*)^* \\ & \text{coaction} & \end{array}$$

Commutes

C- comodule V

$$V \to C \otimes V$$
$$x \to \sum_{(x)} x_C \otimes x_V$$

C^*-module V^*
$\parallel \quad \parallel$
$A \quad W$

$$C^* \otimes V^* \to V^*$$
$$g \otimes f \to gf$$
$$gf(x) = \sum_{(x)} g(x_C)\, f(x_V)$$

A^* - comodule W^*

$$W^* \to A^* \otimes W^*$$
$$h \to \sum_{(h)} h_{A^*} \otimes h_{W^*}$$

$$h(gf) = \sum_{(h)} h_{A^*}(g)\, h_{W^*}(f)$$

$$\begin{array}{ccc} x & \longrightarrow & \sum_{(x)} x_C \otimes x_V \\ \downarrow & & \downarrow \\ & & \sum_{(x)} \hat{x}_C \otimes \hat{x}_V \\ & & \parallel ? \\ \hat{x} = h & \longrightarrow & \sum_{(h)} h_{A^*} \otimes h_{W^*} \end{array}$$

$\forall\ g \in C^* \quad \forall f \in V^*$

$$\sum_{(h)} h_{A^*}(g)\, h_{W^*}(f) \overset{?}{=} \underbrace{\sum_{(x)} \underset{g(x_C)}{\overset{\parallel}{\hat{x}_C(g)}}\ \underset{f(x_V)}{\overset{\parallel}{\hat{x}_V(f)}}}_{\overset{\parallel}{(gf)(x)}}$$

$\parallel$
$h(gf)$
$\parallel$
$\hat{x}(gf)$
$\parallel$
$(gf)(x)$

ok

Combinations. cont

$$\begin{tikzcd}
A\text{-module } V \arrow[r] \arrow[d] & A^*\text{-comodule } V^* \arrow[d] \\
& A^*\text{-comodule } (V^*)^* \quad (\text{ver II}) \\
A\text{-module } V^* \arrow[r] & A^*\text{-comodule } (V^*)^* \quad (\text{ver I})
\end{tikzcd}$$

ver I = ver II

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check

A-module V $\quad A\otimes V \to V$

$a\otimes v \to av$

A^*-comodule V^*
$\overset{\shortparallel}{C}$ $\quad \overset{\shortparallel}{W}$

$$V^* \to A^*\otimes V^*$$
$$f \to \sum_{(f)} f_{A^*}\otimes f_{V^*}$$

$$f(av) = \sum_{(f)} f_{A^*}(a)\, f_{V^*}(v)$$

$a\in A \quad v\in V \qquad f\in V^* = W$

A-module V^*
$\quad\quad\quad\overset{\shortparallel}{W}$

$$A\otimes V^* \to V^*$$
$$a\otimes f \to af$$

$$(af)(v) = f(S(a)v)$$

A^*-comod W^* (ver I)
$\overset{\shortparallel}{C}$

$$W^* \to C\otimes W^*$$
$$g \to \sum_{(g)} g_C\otimes g_{W^*}$$

$$g(af) = \sum_{(g)} g_C(a)\, g_{W^*}(f)$$

*

Space	basis
V	v_i
$V^* = W$	v^i
W^*	$\hat{v}_i$

Return to C-comod W

$$W \to C \otimes W$$
$$v^i \to \sum_j c_{ij} \otimes v^j$$

So
$$v^i(a v_r) = \sum_j c_{ij}(a) \underbrace{v^j(v_r)}_{\delta_{jr}} = c_{ir}(a)$$

C-comod W^* (view II)

$$W^* \to C \otimes W^* \qquad **$$
$$\hat{v}_r \longrightarrow \sum_j S^*(c_{jr}) \otimes \hat{v}_j$$

Compare $*$, $**$. Equal iff

$$\hat{v}_r(af) \overset{?}{=} \sum_j S^*(c_{jr})(a) \ \hat{v}_j(f) \qquad \forall a \in A, \forall f \in V^*$$

take $f = v^i$

$$\hat{v}_r(av^i) \overset{?}{=} \underbrace{\sum_j S^*(c_{jr})(a)\ \underbrace{\hat{v}_j(v^i)}_{=\delta_{ij}}}_{}$$

$$\hat{v}_r(av^i) = (av^i)(v_r) = v^i(S(a)v_r) = c_{ir}(S(a))$$

$$S^*(c_{ir})(a) = \langle S^*(c_{ir}), a\rangle = \langle c_{ir}, S(a)\rangle = c_{ir}(S(a))$$

ok.

Combinations, cont

$$\begin{array}{ccc}
C\text{-comodule } V & \longrightarrow & C^*\text{-module } V^* \\
\downarrow & & \downarrow \\
 & & C^*\text{-module } (V^*)^* \quad \text{ver II} \\
C\text{-comodule } V^* & \longrightarrow & C^*\text{-module } (V^*)^* \quad \text{ver I}
\end{array}$$

ver I = ver II

check

Space	basis
V	v_i
	) dual
$V^* = W$	v^i
	) dual
W^*	$\hat{v}_i$

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C-comodule V

$$V \to C \otimes V$$
$$v_r \to \sum_j c_{rj} \otimes v_j$$

C^*-module V^*
$\overset{\shortparallel}{A}$ $\quad\quad$ $\overset{\shortparallel}{W}$

$$C^* \otimes V^* \to V^*$$
$$g \otimes f \to gf$$

$$(gf)(x) = \sum_{(x)} g(x_c)\, f(x_v)$$

$f = v^i, \quad x = v_r$

$$g v^i (v_r) = \sum_j g(c_{rj}) \underbrace{v^i(v_j)}_{\delta_{ij}}$$
$$= g(c_{ri})$$

A-module W^* (ver II)

$$A \otimes W^* \to W^*$$
$$a \otimes h \to ah$$

$$ah(f) = h(S^*(a) f)$$

$h = \hat{v}_r \qquad f = v^i$

$$a \hat{v}_r (v^i) = \hat{v}_r (S^*(a) v^i)$$
$$= S^*(a) v^i (v_r)$$
$$= \langle S^*(a) v^i, v_r \rangle$$

$*$

C-comodule V^*

$$V^* \to C \otimes V^*$$
$$v^i \to \sum_j S(c_{ji}) \otimes v^j$$

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A-module W^* (ver I)

$$A \otimes W^* \to W^*$$
$$a \otimes h \to ah$$

$$(ah)(v^i) = \sum_j a(S(c_{ji}))\ h(v^j)$$

$h = \hat{v}_r$

$$a\hat{v}_r\ (v^i) = \sum_j a(S(c_{ji}))\ \underset{\underset{\underset{\delta_{jr}}{\shortparallel}}{v^j(v_r)}}{\underset{\shortparallel}{\hat{v}_r(v^j)}}$$

$$= a(S(c_{ri}))$$ **

Compare *, **

Equal iff

$$\langle \underbrace{S^*(a)}_{g}\ v^i, v_r \rangle \overset{?}{=} a(S(c_{ri})) \quad \forall i, r$$

$$\shortparallel$$

$$S^*(a)\ (c_{ri})$$

$$\shortparallel$$

$$\langle S^*(a), c_{ri} \rangle$$

$$\shortparallel$$

$$\langle a, S(c_{ri}) \rangle \qquad \text{ok}$$

$$\shortparallel$$

$$a(S(c_{ri}))$$

$\square$

LEM A73

Given Hopf alg C and

f.d. comodules U, V

Recall vs iso

$$V^* \otimes U^* \longrightarrow (U \otimes V)^*$$

$$\varphi \qquad g \otimes f \longrightarrow H \qquad H(u \otimes v) = f(u)\, g(v)$$

Then φ is a C-comodule morphism

pf Fix basis $\{u_i\}$ for U,
$\{v_j\}$ for V

Actions

$$U \longrightarrow C \otimes U \qquad\qquad V \longrightarrow C \otimes V$$

$$u_i \longrightarrow \sum_r c_{ir} \otimes u_r \qquad\qquad v_j \longrightarrow \sum_s \tilde{c}_{js} \otimes v_s$$

$$U^* \longrightarrow C \otimes U^* \qquad\qquad V^* \longrightarrow C \otimes V^*$$

$$u^i \longrightarrow \sum_r S(c_{ri}) \otimes u^r \qquad\qquad v^j \longrightarrow \sum_s S(\tilde{c}_{sj}) \otimes v^s$$

$$U \otimes V \longrightarrow C \otimes (U \otimes V)$$

$$u_i \otimes v_j \longrightarrow \sum_{r,s} c_{ir} \quad \tilde{c}_{js} \otimes (u_r \otimes v_s)$$

$$V^* \otimes U^* \longrightarrow C \otimes (V^* \otimes U^*)$$

$\Delta_{V^* \otimes U^*}$:

$$v^j \otimes u^i \longrightarrow \sum_{r,s} S(\tilde{c}_{sj})\, S(c_{ri}) \otimes (v^s \otimes u^r)$$

Let $\{\overline{u_i \otimes v_j}\}$ denote the basis for $(U \otimes V)^*$ dual to $\{u_i \otimes v_j\}$ basis for $U \otimes V$

So $\quad \overline{u_i \otimes v_j} \quad (u_r \otimes v_s) \quad = \quad \delta_{ir}\, \delta_{js}$

Get action

$$(U \otimes V)^* \longrightarrow C \otimes (U \otimes V)^*$$

$\Delta_{(U \otimes V)^*}$

$$\overline{u_i \otimes v_j} \longrightarrow \sum_{r,s} S(c_{ri}\, \tilde{c}_{sj}) \quad \overline{u_r \otimes v_s}$$

check diag commutes:

$$\begin{tikzcd}
V^* \otimes U^* \arrow[r, "\varphi"] \arrow[d, "\Delta_{V^*\otimes U^*}"'] \arrow[dr, phantom, "?"] & (U \otimes V)^* \arrow[d, "\Delta_{(U\otimes V)^*}"] \\
C \otimes V^* \otimes U^* \arrow[r, "id \otimes \varphi"'] & C \otimes (U \otimes V)^*
\end{tikzcd}$$

$$\begin{tikzcd}
v^j \otimes u^i \arrow[r] \arrow[dd] & \overline{u_i \otimes v_j} \arrow[d] \\
 & \sum_{r,a} S(c_{ri} \tilde{c}_{aj}) \, \overline{u_r \otimes v_a} \arrow[d, equal, "\text{since } S(xy) = S(y)S(x)"] \\
\sum_{r,a} S(\tilde{c}_{aj}) S(c_{ri}) \otimes v^a \otimes u^r \arrow[r] & \sum_{r,a} S(\tilde{c}_{aj}) S(c_{ri}) \otimes \overline{u_r \otimes v_a}
\end{tikzcd}$$

$\square$

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Recall bialg $M(2)$

Algebra $M(2)$: poly alg $k[x_{ij} \mid 1 \le i,j \le 2]$

Coalg $M(2)$:

$$\Delta: \quad \begin{aligned} M(2) &\to M(2) \otimes M(2) \\ x_{ij} &\to \sum_{l} x_{il} \otimes x_{lj} \end{aligned}$$

$$\varepsilon: \quad \begin{aligned} M(2) &\to k \\ x_{ij} &\to \delta_{ij} \end{aligned}$$

Consider poly alg $A = k[x_1, x_2]$

Consider the alg

$$M(2) \otimes A \cong k[x_{11}, x_{12}, x_{21}, x_{22}, x_1, x_2]$$

Since $M(2) \otimes A$ is commutative, $\exists$ alg morphism

$$\Delta_A: \quad \begin{aligned} A &\longrightarrow M(2) \otimes A \\ x_1 &\longrightarrow x_{11} \otimes x_1 + x_{12} \otimes x_2 \\ x_2 &\longrightarrow x_{21} \otimes x_1 + x_{22} \otimes x_2 \end{aligned}$$

<u>claim</u> Δ_A turns A into an $\underset{\overset{\parallel}{M}}{M(2)}$-comodule.

check diags

$$\begin{tikzcd} M \otimes M \otimes A & M \otimes A \arrow[l, "\Delta \otimes id"'] \\ M \otimes A \arrow[u, "id \otimes \Delta_A"] \arrow[ur, phantom, "?"] & A \arrow[l, "\Delta_A"] \arrow[u, "\Delta_A"'] \end{tikzcd}$$

All maps are alg morphisms. suf to chase gens x_i around diag

$$\begin{tikzcd} \sum_{k,l} x_{ik} \otimes x_{kl} \otimes x_l \arrow[d, phantom, "\parallel \checkmark"] & \sum_l x_{il} \otimes x_l \arrow[l] \\ \sum_{k,l} x_{ik} \otimes x_{kl} \otimes x_l & \\ \sum_k x_{ik} \otimes x_k \arrow[u] & x_i \arrow[l] \arrow[uu] \end{tikzcd}$$

ok

$$\begin{tikzcd}
K \otimes A & & M \otimes A \arrow[ll, "\varepsilon \otimes id"'] \arrow[dll, phantom, "?"] \\
 & & A \arrow[u, "\Delta_A"'] \arrow[ull, "1\otimes a \,\mapsfrom\, a"]
\end{tikzcd}$$

$$\begin{tikzcd}
1 \otimes x_i = \sum_\ell \delta_{i\ell} \otimes x_\ell & & \sum_\ell x_{i\ell} \otimes x_\ell \arrow[ll] \\
 & & x_i \arrow[u] \arrow[ull]
\end{tikzcd}$$

ok

DEF 68 Given bialg H.

An <u>H-comodule algebra</u> is an algebra A

together with an alg morphism

$$\Delta_A : A \to H \otimes A$$

that turns A into an H-comodule

LEM 69 Given bialg H and

H-comodule alg A.

Then the mult map

$$\mu : A \otimes A \to A$$

is an H-comodule morphism

pf

H-comodule A:

$$\Delta_A : \begin{array}{ccc} A & \to & H \otimes A \\ x & \to & \sum_{(x)} x_H \otimes x_A \end{array}$$

H-comodule $A \otimes A$:

$$\Delta_{A \otimes A} : \begin{array}{ccc} A \otimes A & \to & H \otimes A \otimes A \\ x \otimes y & \to & \sum_{(x)} \sum_{(y)} (x_H y_H) \otimes x_A \otimes y_A \end{array}$$

check diag

$$\begin{tikzcd}
A \otimes A \arrow[r, "\mu"] \arrow[d, "\Delta_{A \otimes A}"'] \arrow[dr, phantom, "?"] & A \arrow[d, "\Delta_A"] \\
H \otimes A \otimes A \arrow[r, "id \otimes \mu"'] & H \otimes A
\end{tikzcd}$$

$$\begin{tikzcd}
x \otimes y \arrow[r] \arrow[d] & xy \arrow[d] \\
\sum\limits_{(x)} \sum\limits_{(y)} x_H y_H \otimes x_A \otimes y_A \arrow[r] & \Delta_A(xy) = \Delta_A(x)\, \Delta_A(y)
\end{tikzcd}$$

$$\sum_{(x)} \sum_{(y)} x_H y_H \otimes x_A y_A$$

$$= \left(\sum_{(x)} x_H \otimes x_A \right) \left(\sum_{(y)} y_H \otimes y_A \right) \qquad \text{ok}$$

$$= \Delta_A(x)\ \Delta_A(y)$$

LEM 70 Given bialg H and

H-comodule alg A

then the unit map

$$\eta : k \to A$$

is an H-comodule morphism

pf.

Recall H-comodule k:

$$\Delta_k : \begin{array}{ccc} k & \longrightarrow & H \otimes k \\ 1 & \longmapsto & 1_H \otimes 1 \end{array}$$

check diag:

$$\begin{tikzcd} k \arrow[r, "\eta"] \arrow[d, "\Delta_k"'] \arrow[dr, phantom, "?"] & A \arrow[d, "\Delta_A"] \\ H \otimes k \arrow[r, "id \otimes \eta"'] & H \otimes A \end{tikzcd}$$

Δ_A is alg morphism so it sends

$$1_A \longmapsto 1_{H \otimes A} = 1_H \otimes 1_A$$

$$\begin{tikzcd} 1 \arrow[r] \arrow[d] & 1_A \arrow[d] \\ 1_H \otimes 1 \arrow[r] & 1_A \otimes 1_A \end{tikzcd}$$

ok

$\square$

We saw that $\overset{A}{\overset{\shortparallel}{k[x_1, x_2]}}$ becomes a comodule algebra for $M(2)$

Similarly $k[x_1, x_2]$ ---- $SL(2)$

Abbrev

$$x = x_1 \qquad y = x_2$$

$$a = X_{11} \qquad b = X_{12}$$

$$c = X_{21} \qquad d = X_{22}$$

So Δ_A sends

$$x \longrightarrow a \otimes x + b \otimes y$$

$$y \mapsto c \otimes x + d \otimes y$$

For $i, j \in \mathbb{N}$,

$$\Delta_A(x^i y^j) = (\Delta_A(x))^i (\Delta_A(y))^j$$

$$= (a \otimes x + b \otimes y)^i (c \otimes x + d \otimes y)^j$$

(binom)

$$= \sum_{r=0}^{i} \sum_{s=0}^{j} \binom{i}{r}\binom{j}{s} a^r b^{i-r} c^s d^{j-s} \otimes x^{r+s} y^{i+j-r-s} \qquad (*)$$

$A = k[x,y]$ has a grading

$$A = \sum_{n \in \mathbb{N}} A_n \qquad (ds)$$

A_n consists of homog polynomials with total deg n.

LEM 71 For $H = M(2)$ or $SL(2)$ consider the H-comodule algebra $A = k[x,y]$.

Then for $n \in \mathbb{N}$,

A_n is an H-cosubmodule of A

pf show

$$\Delta_A(A_n) \subseteq H \otimes A_n$$

A_n has basis

$$x^i y^j \qquad i+j=n$$

By $*$

$$\Delta_A(x^i y^j) \in H \otimes A_n \qquad \text{if } i+j=n$$

Result follows. $\square$