

Given bialgebra C and vector space V

then $C \otimes V$ becomes a C -comodule in two ways:

(I) Free C -comodule $C \otimes V$ with action

$$C \otimes V \xrightarrow{\Delta \otimes id} C \otimes C \otimes V$$

(II) Start with C -comodule C , with action

$$\Delta: C \rightarrow C \otimes C$$

Also trivial C -comodule V , with action

$$\begin{array}{ccc} V & \longrightarrow & C \otimes V \\ x & \longrightarrow & 1_C \otimes x \end{array}$$

Gives C -comodule $C \otimes V$ from below LEM 61, with action

$$\begin{array}{ccccccc} C \otimes V & \longrightarrow & C \otimes C \otimes C \otimes V & \longrightarrow & C \otimes C \otimes C \otimes V & \longrightarrow & C \otimes C \otimes V \\ \Delta \otimes (\Delta \otimes id) & & id \otimes T_{C,C} \otimes id & & \mu \otimes id & & \end{array}$$

LEM 64 The above C -comodule structures

I, II on $C \otimes V$ are the same.

pf Compare

$I:$

$$\begin{array}{lcl}
 C \otimes V & \rightarrow & C \otimes C \otimes V \\
 C \otimes x & \rightarrow & \sum_{(c)} c' \otimes c'' \otimes x
 \end{array}$$

$II:$

$$\begin{array}{lcl}
 C \otimes V & \rightarrow & C \otimes C \otimes C \otimes V \rightarrow C \otimes C \otimes C \otimes V \rightarrow C \otimes C \otimes V \\
 C \otimes x & \rightarrow & \left(\sum_{(c)} c' \otimes c'' \right) \otimes (1 \otimes x) \rightarrow \sum_{(c)} c' \otimes 1 \otimes c'' \otimes x \rightarrow \sum_{(c)} (c' \otimes c) \otimes c'' \otimes x \\
 & & = \sum_{(c)} c' \otimes c'' \otimes 1 \otimes x
 \end{array}$$

same

□

LEM 65 Given bialg C and
 C -comodules U, V, W

then the vs is

$$\begin{aligned} (U \otimes V) \otimes W &\longrightarrow U \otimes (V \otimes W) \\ (x \otimes y) \otimes z &\longrightarrow x \otimes (y \otimes z) \end{aligned}$$

is a C -comodule morphism.

pf

Recall actions

$$\begin{aligned} U &\longrightarrow C \otimes U \\ x &\longrightarrow \sum_{\alpha_1} x_{\alpha_1} \otimes x_{\alpha_1} \end{aligned}$$

$$\begin{aligned} V &\longrightarrow C \otimes V \\ y &\longrightarrow \sum_{\alpha_1} y_{\alpha_1} \otimes y_{\alpha_1} \end{aligned}$$

$$\begin{aligned} W &\longrightarrow C \otimes W \\ z &\longrightarrow \sum_{\alpha_1} z_{\alpha_1} \otimes z_{\alpha_1} \end{aligned}$$

$$U \otimes V \longrightarrow C \otimes (U \otimes V)$$

$$x \otimes y \longrightarrow \sum_{\alpha_1} \sum_{\alpha_2} (x_{\alpha_1} y_{\alpha_2}) \otimes x_{\alpha_1} \otimes y_{\alpha_2}$$

etc.

Action for $(u \otimes v) \otimes w$:

$$(x \otimes y) \otimes z \rightarrow \sum_{(x)} \sum_{(y)} \sum_{(z)} (x_c y_c | z_c) \otimes (x_u \otimes y_v) \otimes z_w$$

Action for $u \otimes (v \otimes w)$:

$$x \otimes (y \otimes z) \rightarrow \sum_{(x)} \sum_{(y)} \sum_{(z)} x_c (y_c z_c) \otimes x_u \otimes (y_v \otimes z_w)$$

Result follows since algebra C is associative □

LEM 66 Given bialg C and C -comodule V .

11/6/15
5

then the \cdot vs isomorphisms

$$\begin{aligned} \psi: V &\rightarrow k \otimes V, & \phi: V &\rightarrow V \otimes k \\ v &\rightarrow 1 \otimes v, & v &\rightarrow v \otimes 1 \end{aligned}$$

are comodule morphisms.

pf Consider ψ

Recall C -comodule k

$$\begin{aligned} k &\rightarrow C \otimes k \\ 1 &\rightarrow 1_C \otimes 1 \end{aligned}$$

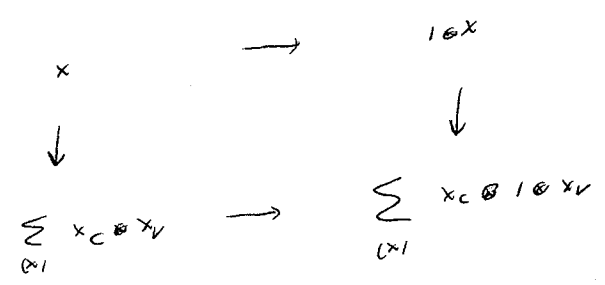
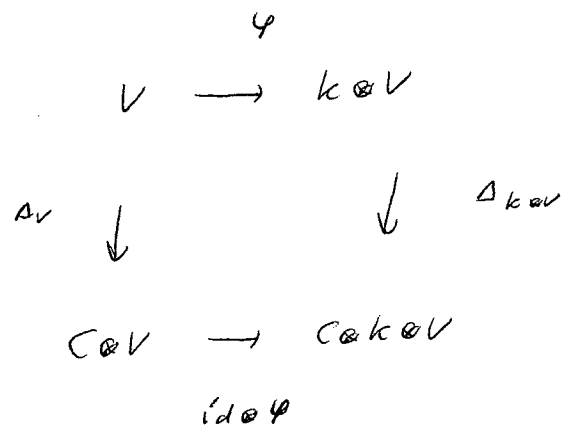
C -comodule V

$$\begin{aligned} V &\rightarrow C \otimes V \\ x &\rightarrow \sum_{\alpha} x_{\alpha} \otimes x_{\alpha} \end{aligned}$$

C -comodule $k \otimes V$:

$$\begin{aligned} k \otimes V &\rightarrow C \otimes (k \otimes V) \\ 1 \otimes x &\rightarrow \sum_{\alpha} (1_C x_{\alpha}) \otimes (1 \otimes x_{\alpha}) \end{aligned}$$

check diagram



OK



the case of ϕ is similar.

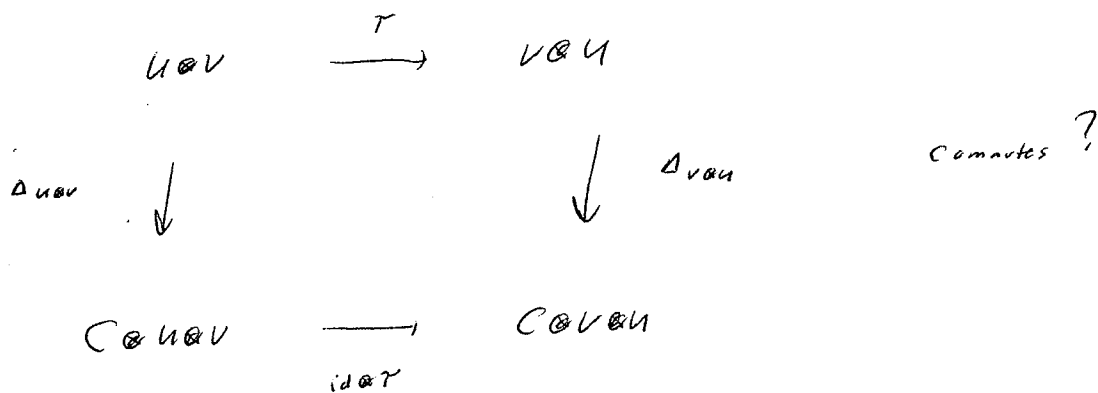
LEMMA Given bialg C and C -comodules U, V

11/6/15
7

the vs iso
 $\tau_{U,V} : U \otimes V \rightarrow V \otimes U$

is a C -comodule morphism provided that
 the algebra C is commutative

pf



$$\begin{array}{ccc}
 x \otimes y & \longrightarrow & y \otimes x \\
 \downarrow & & \downarrow \\
 \sum_{(x)} \sum_{(y)} (x_c y_c) \otimes x_u \otimes y_v & \longrightarrow & \sum_{(x)} \sum_{(y)} y_c x_c \otimes y_v \otimes x_u \quad ??
 \end{array}$$

ok if $x_c y_c = y_c x_c \quad \forall x \in U \quad \forall y \in V$

□

(Aside)

11/6/15
8

Motivation

Given V vs V

Given alg A . Recall the A -module structures on V are in bijection with the algebra morphisms

$$A \rightarrow \text{End}(V)$$

Given coalg C . Expect the C -comodule structures on V are in bijection with the coalgebra

morphisms

$$\text{End}(V) \rightarrow C$$

For convenience assume $\dim V < \infty$

Fix basis $\{v_i\}$ for V

$\text{End}(V)$ has basis $\{e_{ij}\}$

$$e_{ij}(v_r) = \delta_{jr} v_i$$

$\text{End}(V)$ is coalg with

$$\Delta(e_{ij}) = \sum_l e_{il} \otimes e_{lj}$$

$$\varepsilon(e_{ij}) = \delta_{ij}$$

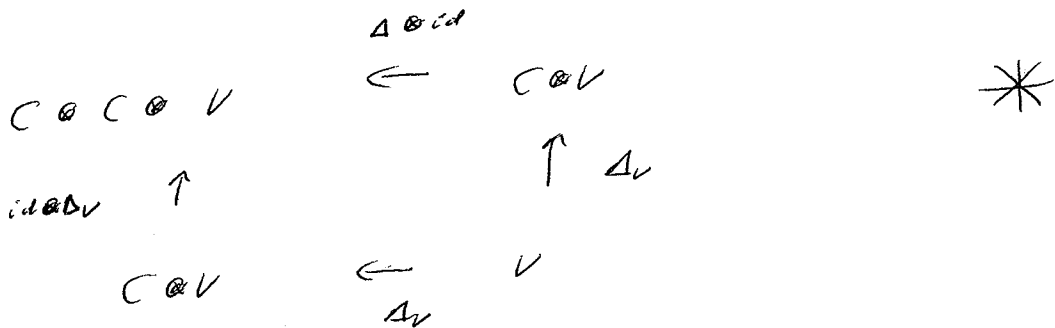
Given any \mathbb{A}^n map

$$\Delta_V : V \rightarrow C \otimes V$$

$$v_i \rightarrow \sum_j c_{ij} \otimes v_j$$

Consider when does Δ_V turn V into C -comodule.

Consider diagrams



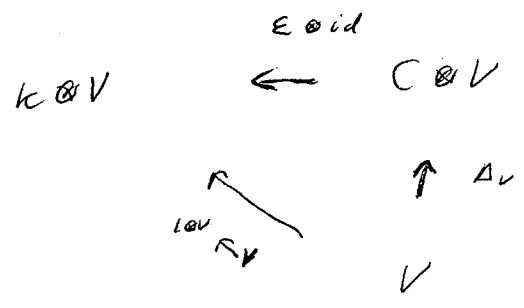
$$\begin{array}{ccc}
 \sum_j \Delta(c_{ij}) \otimes v_j & \xleftarrow{\quad} & \sum_j c_{ij} \otimes v_j \\
 \text{"?"} & & \\
 = \sum_j \left(\sum_l c_{il} \otimes c_{lj} \right) \otimes v_j & & \uparrow \\
 \sum_l c_{il} \otimes \sum_j c_{lj} \otimes v_j & & \\
 \uparrow & & \\
 \sum_l c_{il} \otimes v_l & \xleftarrow{\quad} & v_i
 \end{array}$$

diag * commutes \forall

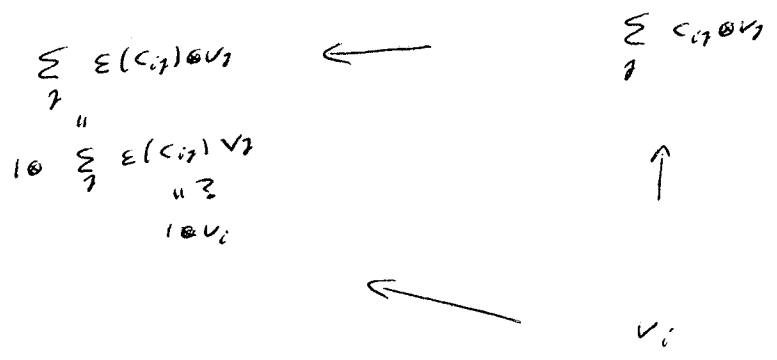
$$\sum_i \Delta(c_{ij}) \otimes v_j = \sum_j \left(\sum_i c_{ij} \otimes c_{ij} \right) \otimes v_j \quad \forall i$$

$$\forall \Delta(c_{ij}) = \sum_l c_{il} \otimes c_{lj} \quad \forall i, j$$

Another diagram



* *



diag * * commutes \forall

$$v_i = \sum_l \epsilon(c_{il}) v_l \quad \forall i$$

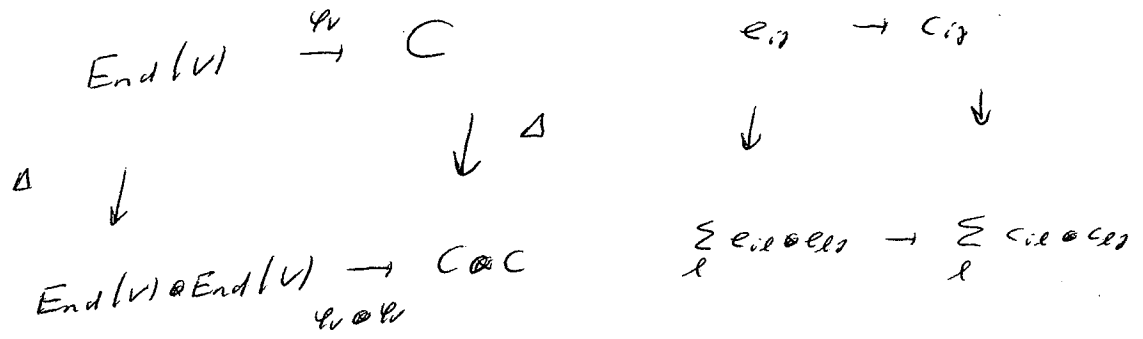
$$\forall \epsilon(c_{ij}) = \delta_{ij} \quad \forall i, j$$

Given any linear map

$$\begin{aligned} \varphi_V : \text{End}(V) &\rightarrow C \\ e_{ij} &\rightarrow c_{ij} \end{aligned}$$

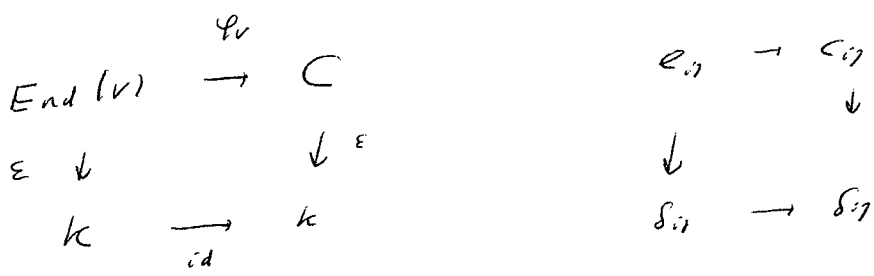
Consider when is φ_V a coalg morphism

Consider diagrams



diag commutes \forall

$$\Delta(c_{ij}) = \sum_l c_{il} \otimes c_{lj} \quad \forall i, j$$



diag commutes \forall

$$\varepsilon(c_{ij}) = \delta_{ij} \quad \forall i, j$$

We conclude:

Prop A68

Given $C, V, \{c_{ij}\}$ as above.

11/6/15
12

TFAE

(i) $\Delta_V : V \rightarrow C \otimes V$ turns V into a C -comodule

(ii) $\Psi_V : \text{End}(V) \rightarrow C$ is coalg morphism

(iii) $\forall i, j$

$$\Delta(c_{ij}) = \sum_l c_{il} \otimes c_{lj}$$

$$\varepsilon(c_{ij}) = \delta_{ij}$$

□

COR 6.9

Given f.d. vectn space V

Given basis $\{v_i\}$ for V

then V is a comodule for the coalg $\text{End}(V)$,

with action

$$\Delta_V: \begin{aligned} V &\longrightarrow \text{End}(V) \otimes V \\ v_i &\longrightarrow \sum_j e_{ij} \otimes v_j \end{aligned}$$

pf recall coalg $\text{End}(V)$ satisfies

$$\Delta(e_{ij}) = \sum_k e_{ik} \otimes e_{kj} \quad \forall i, j$$

$$\varepsilon(e_{ij}) = \delta_{ij}$$

Now apply Prop 6.8 with $C = \text{End}(V)$
and $c_{ij} = e_{ij} \quad \forall i, j$

□

LEM A20 Given Hopf algebra C with antipode S

Given fid. C -comodule V

Pick basis $\{v_i\}$ for V

So $\Delta_V =$

$$V \rightarrow C \otimes V$$

$$v_i \rightarrow \sum_j c_{ij} \otimes v_j$$

Then V^* becomes C -comodule with action

$$\Delta_{V^*}: V^* \rightarrow C \otimes V^*$$

$$v_i \rightarrow \sum_j \underbrace{S(c_{ji})}_{\tilde{c}_{ij}} \otimes v_j^*$$

pf Require by Prop A6B's

$$\Delta(\tilde{c}_{ij}) \stackrel{?}{=} \sum_l \tilde{c}_{il} \otimes \tilde{c}_{lj}$$

Hopf

$$\varepsilon(\tilde{c}_{ij}) \stackrel{?}{=} \delta_{ij}$$

$$\Delta(\tilde{c}_{ij}) \stackrel{?}{=} \sum_l \underbrace{\tilde{c}_{il} \otimes \tilde{c}_{lj}}_{S(c_{ji}) \otimes S(c_{jl})}$$

"

$$\Delta(S(c_{ji})) \stackrel{?}{=} S \otimes S \left(\sum_l \underbrace{c_{ji} \otimes c_{jl}}_{\Delta^{op}(c_{ji})} \right)$$

"

$$(\Delta \circ S)(c_{ji}) \stackrel{?}{=} S \otimes S \circ \Delta^{op}(c_{ji})$$

We saw earlier

$$\Delta \circ S = S \otimes S \circ \Delta^{op} \quad \text{OK}$$

w/6/15
15

$$\mathbb{E}(\tilde{c}_{it}) \stackrel{?}{=} \delta_{it}$$

"

$$\mathbb{E}(S(c_{it}))$$

"

$$\mathbb{E}(c_{it})$$

"

$$\delta_{it}$$

$$\in \mathbb{E}(S(c)) = \mathbb{E}(c) \quad \forall c$$

OK

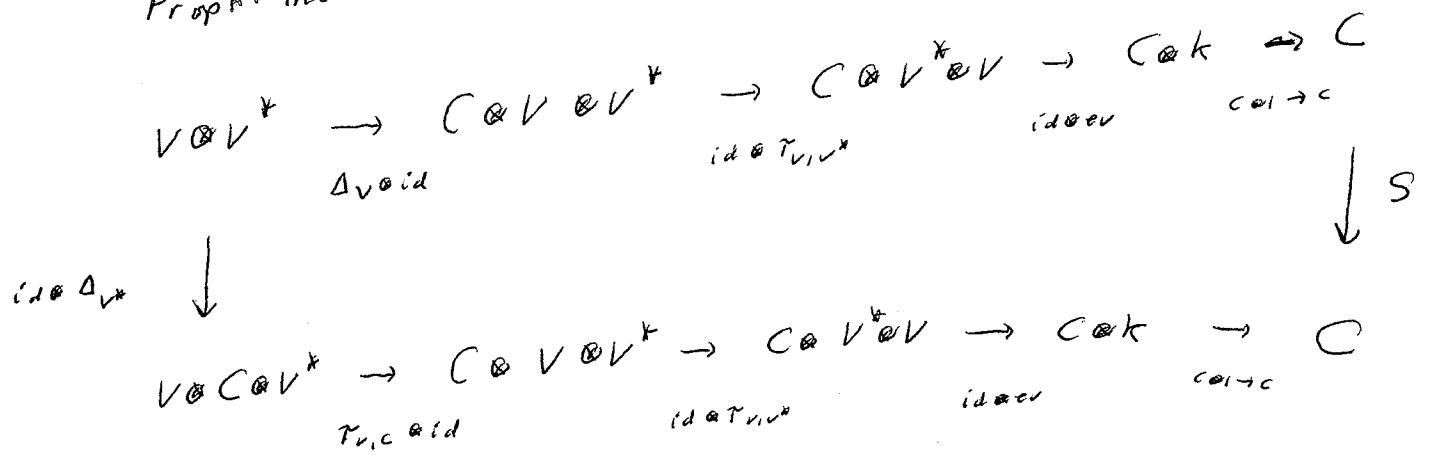
□

Given fid. Hopf alg C with antipode S

Given fid. C -comodule V

Compare C -comodules V, V^*

Prove the following diagram commutes:



pf chase $V_2 \otimes V^*$ around diag

w/6/15
17

$$v_2 \otimes v^r \rightarrow \sum_j c_{2j} \otimes v_j \otimes v^r \rightarrow \sum_j c_{2j} \otimes v^r \otimes v_2 \rightarrow \sum_j c_{2j} \otimes \langle v^r, v_2 \rangle \rightarrow c_{2r}$$



$$v_2 \otimes \sum_t S(c_{tr}) \otimes v^t \rightarrow \sum_t S(c_{tr}) \otimes v_2 \otimes v^t \rightarrow \sum_t S(c_{tr}) \otimes v^t \otimes v_2 \rightarrow \sum_t S(c_{tr}) \otimes \langle v^t, v_2 \rangle \rightarrow S(c_{2r})$$

ok



Given f.d. Hopf alg C with antipode S

11/6/15
18

Given f.d. C -comodule V

Pick basis $\{v_i\}$ for V

Write

$$\Delta_V : \begin{aligned} V &\rightarrow C \otimes V \\ v_i &\rightarrow \sum_j c_{ij} \otimes v_j \end{aligned}$$

Find $S(c_{ij})$

Recall

$$\Delta : \begin{aligned} C &\rightarrow C \otimes C \\ c_{ij} &\rightarrow \sum_l c_{il} \otimes c_{lj} \end{aligned}$$

$$\varepsilon : \begin{aligned} C &\rightarrow k \\ c_{ij} &\rightarrow \delta_{ij} \end{aligned}$$

Also $\forall x \in C$

$$\varepsilon(x) 1_C = \sum_{(x')} S(x') x'' = \sum_{(x'')} x' S(x'')$$

take $x = c_{ij}$

$$\delta_{ij} 1_C = \sum_l S(c_{il}) c_{lj} = \sum_l c_{il} S(c_{lj})$$

so the matrices

$$\begin{pmatrix} c_{ij} \end{pmatrix} \begin{pmatrix} S(c_{ij}) \end{pmatrix} = \begin{pmatrix} S(c_{ij}) \end{pmatrix} \begin{pmatrix} c_{ij} \end{pmatrix} = \begin{pmatrix} 1_C & & 0 \\ & \ddots & \\ 0 & & 1_C \end{pmatrix}$$

Given f.d. Hopf alg C with antipode S

11/6/15
19

Given f.d. C -comodule V

Consider ϵ_V

$$\epsilon_V: V^* \otimes V \rightarrow k$$

a C -comodule morphism?

Turns out we need to compose with τ

LEMMA The map

$$V \otimes V^* \xrightarrow{\tau_{V, V^*}} V^* \otimes V \xrightarrow{\epsilon_V} k$$

*

is a C -comodule morphism.

pf

Recall actions

Fix basis $\{v_i\}$ for V , dual basis $\{v^i\}$ for V^*

$$V \rightarrow C \otimes V$$

$$v_i \rightarrow \sum_l c_{il} \otimes v_l$$

$$V^* \rightarrow C \otimes V^*$$

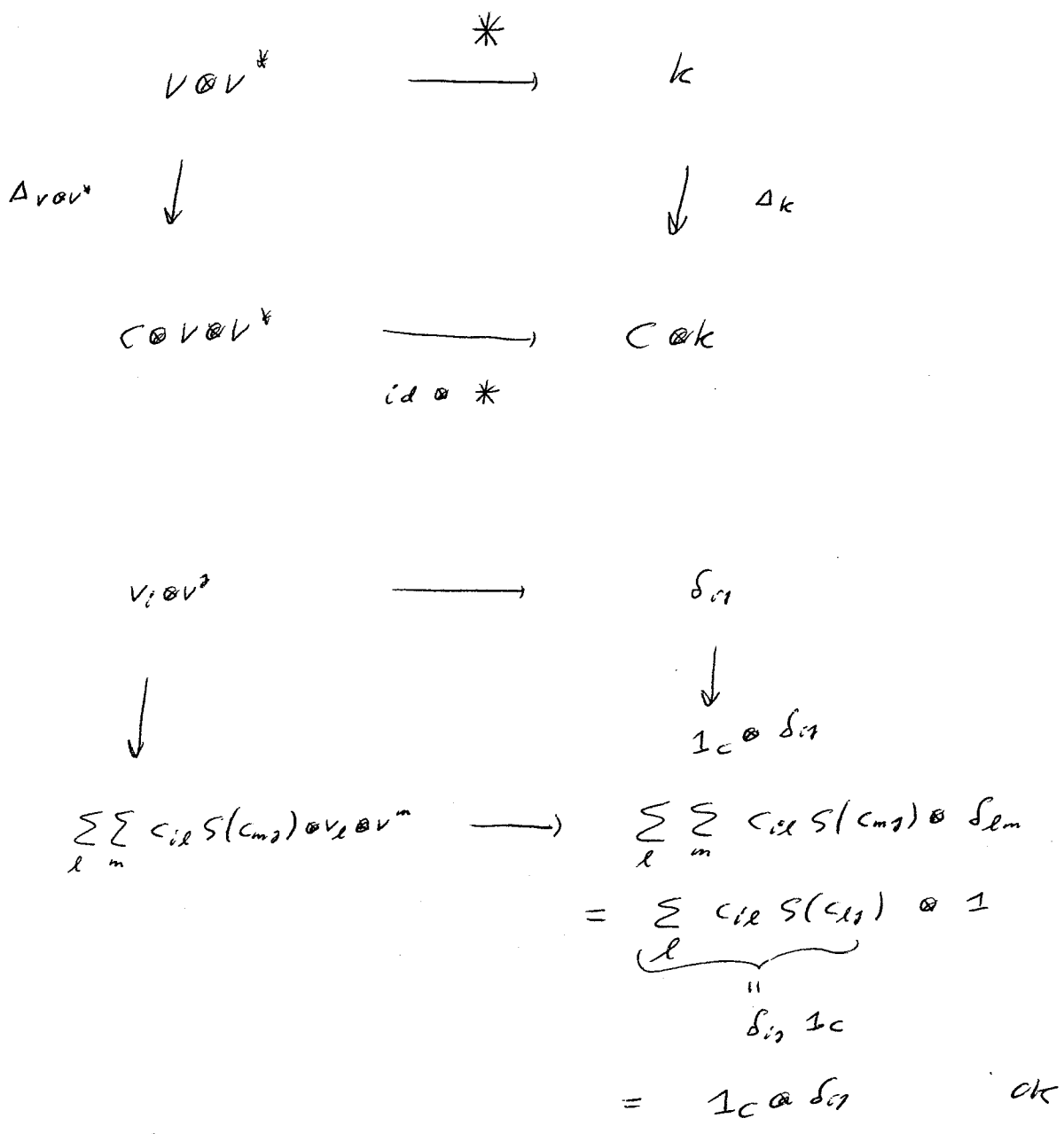
$$v^j \rightarrow \sum_m s(c_{mj}) \otimes v^m$$

$$V \otimes V^* \rightarrow C \otimes V \otimes V^*$$

$\Delta_{V \otimes V^*}:$

$$v_i \otimes v^j \rightarrow \sum_l \sum_m c_{il} s(c_{mj}) \otimes v_l \otimes v^m$$

show this commutes:



□

Natural moves

$$A\text{-module } V \longrightarrow A^*\text{-comodule } V^*$$

$$A\text{-module } V \longrightarrow A\text{-module } V^*$$

$$C\text{-comodule } V \longrightarrow C^*\text{-module } V^*$$

$$C\text{-comodule } V \longrightarrow C\text{-comodule } V^*$$

Combinations

$$A\text{-mod } V \rightarrow A^k\text{-comod } V^* \rightarrow (A^k)^* \text{-mod } (V^k)^*$$

$$A \otimes V \xrightarrow{\text{action}} V$$

$$\begin{array}{ccc}
 \text{not } \otimes \text{ not} & & \text{not} \\
 \downarrow & & \downarrow \\
 (A^k)^* \otimes (V^k)^* & \xrightarrow{\text{action}} & (V^k)^* \\
 & & \text{comodules}
 \end{array}$$

check

A-module V

$$A \otimes V \rightarrow V$$

$$a \otimes v \rightarrow av$$

A*-comodule V*
 ||
 C W

$$V^* \rightarrow A^* \otimes V^*$$

$$f \rightarrow \sum_{(f)} f_{A^*} \otimes f_{V^*}$$

$$f(av) = \sum_{(f)} f_{A^*}(a) f_{V^*}(v)$$

C*-mod W*

$$C^* \otimes W^* \rightarrow W^*$$

$$g \otimes h \rightarrow gh$$

$$(gh)(\rho) = \sum_{(f)} g(f_{A^*}) h(f_{V^*})$$

nat:

$$V \rightarrow (V^*)^* = W^*$$

$$v \rightarrow \hat{v} \quad \hat{v}(f) = f(v)$$

nat:

$$A \rightarrow (A^*)^* = C^*$$

$$a \rightarrow \hat{a} \quad \hat{a}(g) = g(a)$$

$$a \otimes v \rightarrow av$$

$$\downarrow \quad \downarrow$$

$$\hat{a} \otimes \hat{v} \rightarrow \hat{a} \hat{v} \quad \text{"?"}$$

take $g = \hat{a}$
 $h = \hat{v}$

$$\hat{a} \hat{v}(f) = \sum_{(f)} \hat{a}(f_{A^*}) \hat{v}(f_{V^*})$$

$$= \sum_{(f)} f_{A^*}(a) f_{V^*}(v)$$

$$= f(av)$$

$$= \hat{a} \hat{v}(f)$$

so $\hat{a} \hat{v} = a \hat{v}$

ok

C-comodule V

$$V \rightarrow C \otimes V$$

$$x \rightarrow \sum_{(x)} x_c \otimes x_v$$

C*-module V*
 ||
 A ||
 W

$$C^* \otimes V^* \rightarrow V^*$$

$$g \otimes f \rightarrow gf$$

$$gf(x) = \sum_{(x)} g(x_c) f(x_v)$$

A*-comodule W*

$$W^* \rightarrow A^* \otimes U^*$$

$$h \rightarrow \sum_{(h)} h_{A^*} \otimes h_{U^*}$$

$$h(gf) = \sum_{(h)} h_{A^*}(g) h_{U^*}(f)$$

$$x \rightarrow \sum_{(x)} x_c \otimes x_v$$

$$\downarrow$$

$$\hat{x} = h \rightarrow \sum_{(h)} \hat{x}_c \otimes \hat{x}_v$$

" ?

$$\sum_{(h)} h_{A^*} \otimes h_{U^*}$$

$\forall g \in C^* \forall f \in V^*$

$$\sum_{(h)} h_{A^*}(g) h_{U^*}(f) \stackrel{?}{=} \sum_{(x)} \underbrace{\hat{x}_c(g) \hat{x}_v(f)}_{(gf)(x)}$$

" ?

$$h(gf)$$

"

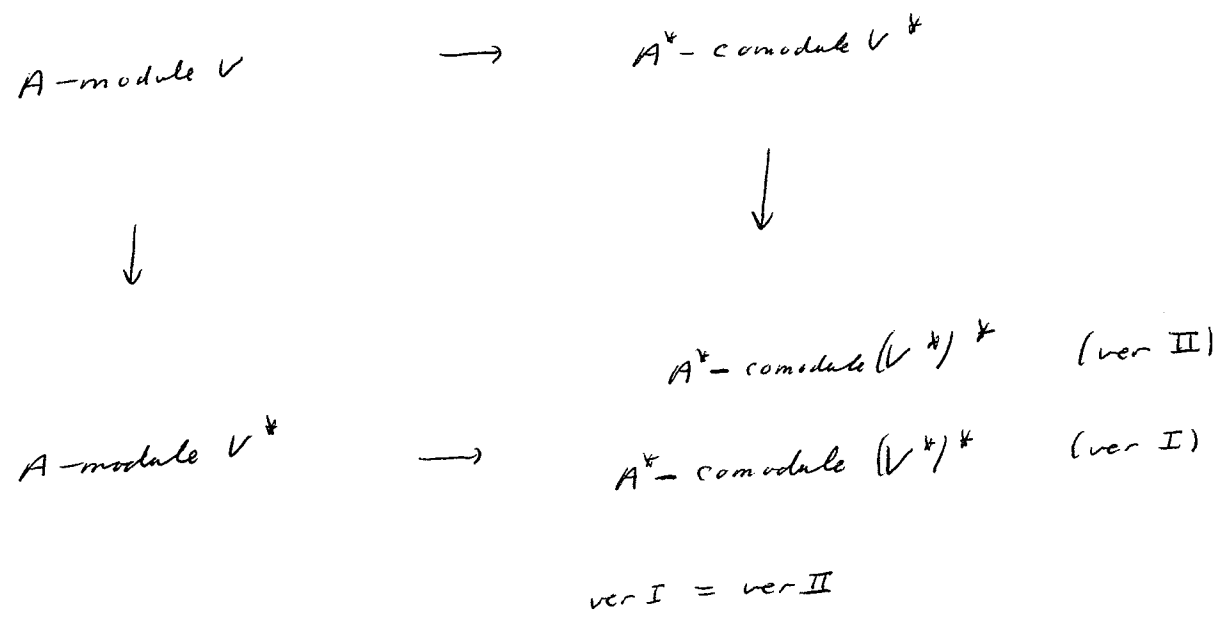
$$\hat{x}(gf)$$

"

$$(gf)(x)$$

ok

Combinations. cont



check

u/6/15
27

A-module V

$$A \otimes V \rightarrow V$$
$$a \otimes v \rightarrow av$$

A^* -module V^*
" " "
C " W

$$V^* \rightarrow A^* \otimes V^*$$
$$f \rightarrow \sum_{(f)} f_A^* \otimes f_{V^*}$$

$$f(av) = \sum_{(f)} f_A^*(a) f_{V^*}(v)$$

$a \in A \quad v \in V \quad f \in V^* = W$

A-module V^*
" " "
C " W

$$A \otimes V^* \rightarrow V^*$$
$$a \otimes f \rightarrow af$$

$$(af)(v) = f(S(a)v)$$

A^* -comod W^* (see I)
" " "
C " W

$$W^* \rightarrow C \otimes W^*$$
$$g \rightarrow \sum_{(g)} g_C \otimes g_{W^*}$$

$$g(af) = \sum_{(g)} g_C(a) g_{W^*}(f)$$



Space	basis
V	v_i
$V^* \cong W$	v^i
W^*	\hat{v}_i

Return to C -comod W

$$W \rightarrow C \otimes W$$

$$v^i \rightarrow \sum_j c_{ij} \otimes v^j$$

So
$$v^i(a v_r) = \sum_j c_{ij}(a) v^j(v_r) = c_{ir}(a)$$

C -comod W^* (view II)

$$W^* \rightarrow C \otimes W^*$$

$$\hat{v}_r \rightarrow \sum_j S^*(c_{jr}) \otimes \hat{v}_j$$

**

Compare *, **, Equal iff

$$\hat{v}_r(a f) = \sum_j S^*(c_{jr})(a) \hat{v}_j(f)$$

$\forall a \in A, \forall f \in V^*$

take $f = v^i$

$$\hat{v}_r(a v^i) = \sum_j S^*(c_{jr})(a) \underbrace{\hat{v}_j^*(v^i)}_{\delta_{ij}}$$

$$(a v^i)(v_r)$$

$$v^i(S(a)v_r)$$

$$c_{ir}(S(a))$$

$$S^*(c_{ir})(a)$$

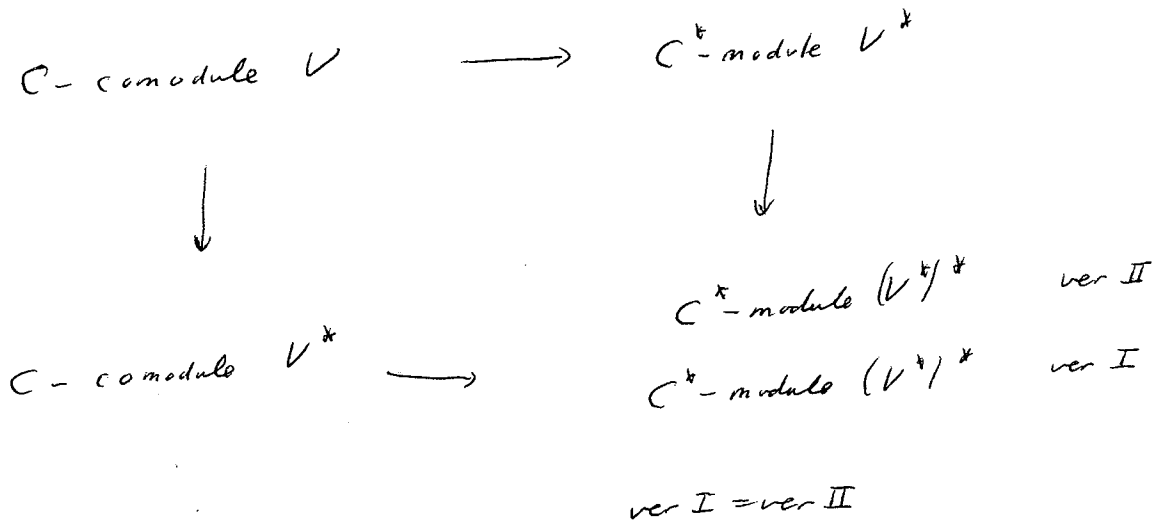
$$\langle S^*(c_{ir}), a \rangle$$

$$\langle c_{ir}, S(a) \rangle$$

$$c_{ir}(S(a))$$

OK.

Combinations, cont



check

Space	basis
V	v_i) dual
$V^* = W$	v^i) dual
W^*	\hat{v}_i

C-comodule V

$$V \rightarrow C \otimes V$$

$$v_r \rightarrow \sum_j c_{rj} \otimes v_j$$

C*-module V*
" " W
A

$$C^* \otimes V^* \rightarrow V^*$$

$$g \otimes f \rightarrow gf$$

$$(gf)(x) = \sum_{x_i} g(x_i) f(x_i)$$

f = v^i, x = v_r

$$g v^i(v_r) = \sum_j g(c_{rj}) v^i(v_j)$$

" δ_{ij}

$$= g(c_{ri})$$

A-module W* (over II)

$$A \otimes W^* \rightarrow W^*$$

$$a \otimes h \rightarrow ah$$

$$ah(f) = h(S^*(a)f)$$

h = \hat{v}_r, f = v^i

$$a \hat{v}_r(v^i) = \hat{v}_r(S^*(a)v^i)$$

$$= S^*(a)v^i(v_r)$$

$$= \langle S^*(a)v^i, v_r \rangle$$



C-comodule V*

$$V^* \rightarrow C \otimes V^*$$

$$v^i \rightarrow \sum_j S(c_{ji}) \otimes v^j$$

A-module W^* (ver I)

11/6/15
32

$$A \otimes W^* \rightarrow W^*$$

$$a \otimes h \rightarrow ah$$

$$(ah)(v^i) = \sum_j a(S(c_{ij})) h(v^j)$$

$$h = \hat{v}_r$$

$$a \hat{v}_r(v^i) = \sum_j a(S(c_{ij})) \hat{v}_r(v^j)$$

$\hat{v}_r(v^j) \underset{\text{def}}{=} \delta_{jr}$

$$= a(S(c_{ir}))$$

**

Compare *, **

Equal !!

$$\langle \underbrace{S^*(a)}_?, v^i, v_r \rangle \stackrel{?}{=} a(S(c_{ir}))$$

$\forall i, r$

$$S^*(a)(c_{ir})$$

$$\langle S^*(a), c_{ir} \rangle$$

$$\langle a, S(c_{ir}) \rangle$$

$$a(S(c_{ir}))$$

ok

□

LEM A73

w/0/15
33

Given Hopf alg C and

f.d. comodules U, V

recall vs iso

$$\begin{array}{ccc} V^* \otimes U^* & \longrightarrow & (U \otimes V)^* \\ \varphi & & \\ g \otimes f & \longrightarrow & H \quad H(u \otimes v) = f(u)g(v) \end{array}$$

Then φ is a C -comodule morphism

pf Fix basis $\{u_i\}$ for U ,
 $\{v_j\}$ for V

Acts

$$\begin{aligned} U &\longrightarrow C \otimes U \\ u_i &\longrightarrow \sum_r c_{ir} \otimes u_r \end{aligned}$$

$$\begin{aligned} V &\longrightarrow C \otimes V \\ v_j &\longrightarrow \sum_\lambda \tilde{c}_{j\lambda} \otimes v_\lambda \end{aligned}$$

$$\begin{aligned} U^* &\longrightarrow C \otimes U^* \\ u_i^* &\longrightarrow \sum_r S(c_{ri}) \otimes u_r^* \end{aligned}$$

$$\begin{aligned} V^* &\longrightarrow C \otimes V^* \\ v_j^* &\longrightarrow \sum_\lambda S(\tilde{c}_{\lambda j}) \otimes v_\lambda^* \end{aligned}$$

$$U \otimes V \longrightarrow C \otimes (U \otimes V)$$

$$u_i \otimes v_j \longrightarrow \sum_{r,s} c_{ir} \tilde{e}_{rs} \otimes (u_r \otimes v_s)$$

$$V^* \otimes U^* \longrightarrow C \otimes (V^* \otimes U^*)$$

$\Delta_{V^* \otimes U^*} :$

$$v^a \otimes u^i \longrightarrow \sum_{r,s} S(\tilde{e}_{rs}) S(c_{ri}) \otimes (v^a \otimes u^r)$$

Let $\{\overline{u_i \otimes v_j}\}$ denote the basis for $(U \otimes V)^*$
dual to $\{u_i \otimes v_j\}$ basis for $U \otimes V$

$$\text{So } \overline{u_i \otimes v_j} (u_r \otimes v_s) = \delta_{ir} \delta_{js}$$

Get action

$$(U \otimes V)^* \longrightarrow C \otimes (U \otimes V)^*$$

$\Delta_{(U \otimes V)^*} :$

$$\overline{u_i \otimes v_j} \longrightarrow \sum_{r,s} S(c_{ri} \tilde{e}_{rs}) \overline{u_r \otimes v_s}$$

check diag commutes:

w/6/15
35

$$V^* \otimes U^* \xrightarrow{\varphi} (U \otimes V)^*$$

$$\Delta_{V^* \otimes U^*} \downarrow \quad \quad \quad \downarrow \quad \quad \quad \Delta_{(U \otimes V)^*}$$

$$C \otimes V^* \otimes U^* \xrightarrow{id \otimes \varphi} C \otimes (U \otimes V)^*$$

$$v^a \otimes u^i \xrightarrow{\quad} \overline{u_r \otimes v_s}$$

$$\downarrow \quad \quad \quad \downarrow$$

$$\sum_{r,s} S(c_{ri} \tilde{c}_{sj}) \overline{u_r \otimes v_s}$$

|| since $S(xy) = S(y)S(x)$

$$\sum_{r,s} S(\tilde{c}_{sj}) S(c_{ri}) \otimes v^a \otimes u^i \rightarrow \sum_{r,s} S(\tilde{c}_{sj}) S(c_{ri}) \otimes \overline{u_r \otimes v_s}$$

□

Recall bialg $M(z)$

Algebra $M(z)$: poly alg $k[x_{ij} \mid 1 \leq i, j \leq 2]$

Coalg $M(z)$:

$$\Delta: \begin{array}{l} M(z) \rightarrow M(z) \otimes M(z) \\ x_{ij} \rightarrow \sum_l x_{il} \otimes x_{lj} \end{array}$$

$$\varepsilon: \begin{array}{l} M(z) \rightarrow k \\ x_{ij} \rightarrow \delta_{ij} \end{array}$$

Consider poly alg $A = k[x_1, x_2]$

Consider the alg

$$M(z) \otimes A \cong k[x_{11}, x_{12}, x_{21}, x_{22}, x_1, x_2]$$

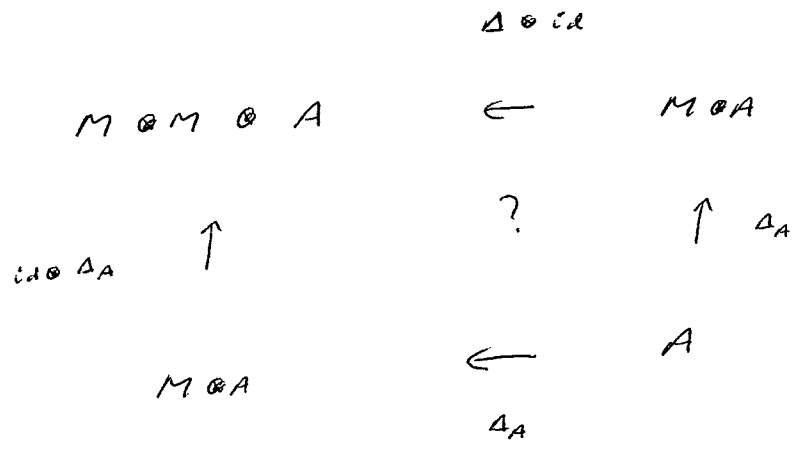
Since $M(z) \otimes A$ is commutative, \exists alg morphism

$$\Delta_A: \begin{array}{l} A \longrightarrow M(z) \otimes A \\ x_1 \longrightarrow x_{11} \otimes x_1 + x_{12} \otimes x_2 \\ x_2 \longrightarrow x_{21} \otimes x_1 + x_{22} \otimes x_2 \end{array}$$

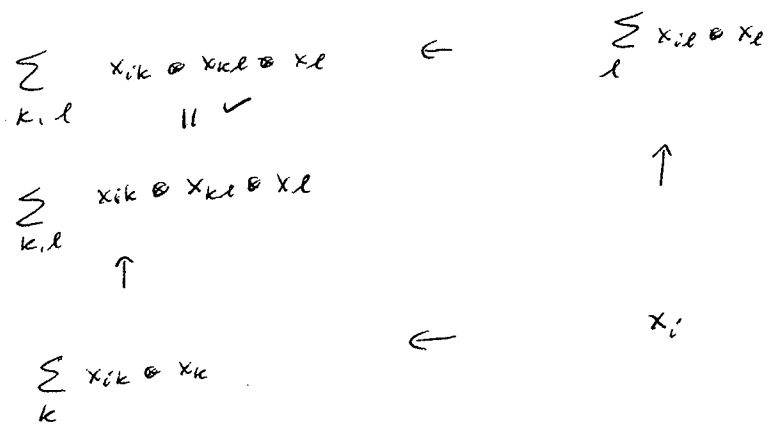
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37

claim Δ_A turns A into an $M(2)$ -comodule.
" M

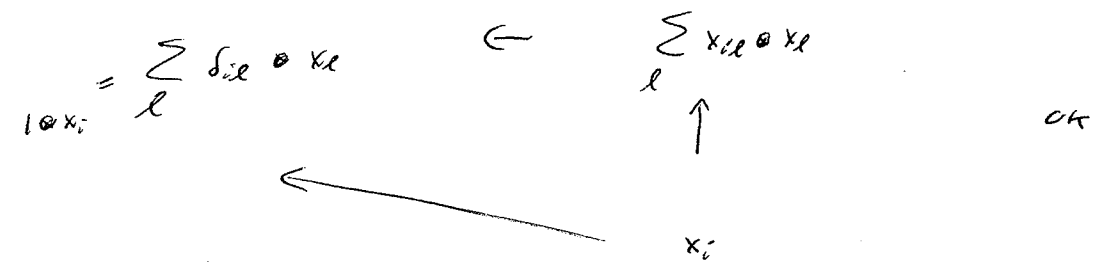
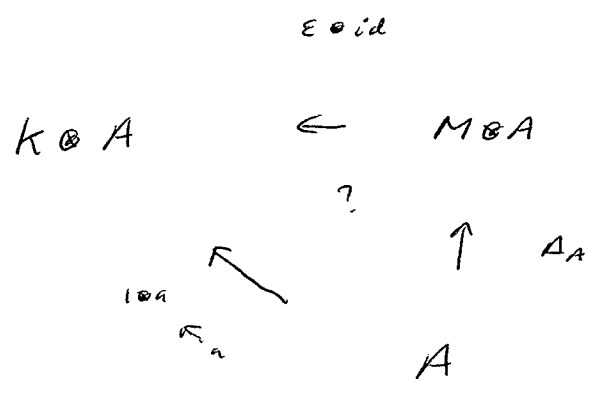
check diag



All maps are alg morphisms. suf to chase gens x_i around diag



ok



DEF 68 Given bialg H .

A_n H -comodule algebra is an algebra A

together with an alg morphism

$$\Delta_A : A \rightarrow H \otimes A$$

that turns A into an H -comodule

LEM 69 Given bialg H and
 H -comodule alg A .

Then the mult map

$$\mu: A \otimes A \rightarrow A$$

is an H -comodule morphism

pf

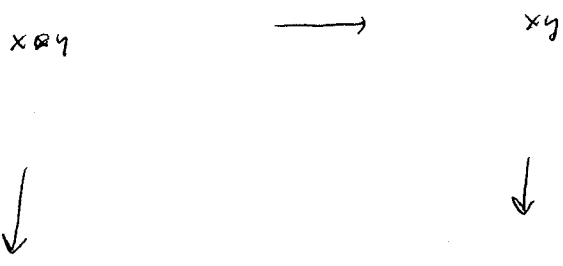
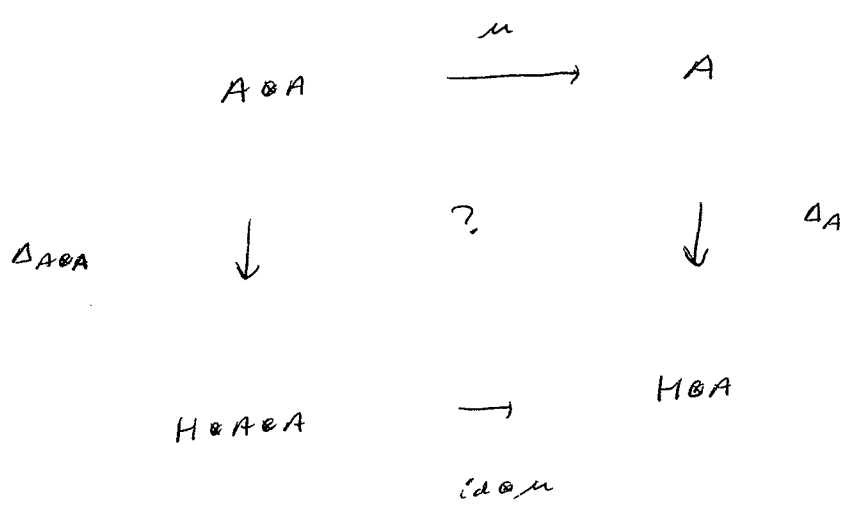
H -comodule A :

$$\begin{aligned} \Delta_A: A &\rightarrow H \otimes A \\ x &\rightarrow \sum_{(x)} x_H \otimes x_A \end{aligned}$$

H -comodule $A \otimes A$:

$$\begin{aligned} \Delta_{A \otimes A}: A \otimes A &\rightarrow H \otimes A \otimes A \\ x \otimes y &\rightarrow \sum_{(x)} \sum_{(y)} (x_H \otimes y_H) \otimes x_A \otimes y_A \end{aligned}$$

check diag



$$\begin{aligned}
 \sum_{(x)} \sum_{(y)} x_H y_H \otimes x_A \otimes y_A & \longrightarrow \sum_{(x)} \sum_{(y)} x_H y_H \otimes x_A y_A \\
 & = \left(\sum_{(x)} x_H \otimes x_A \right) \left(\sum_{(y)} y_H \otimes y_A \right) \\
 & = \Delta_A(x) \Delta_A(y)
 \end{aligned}$$

ok

LEM 20 Given bialg H and

H -comodule alg A

then the unit map

$$\eta: k \rightarrow A$$

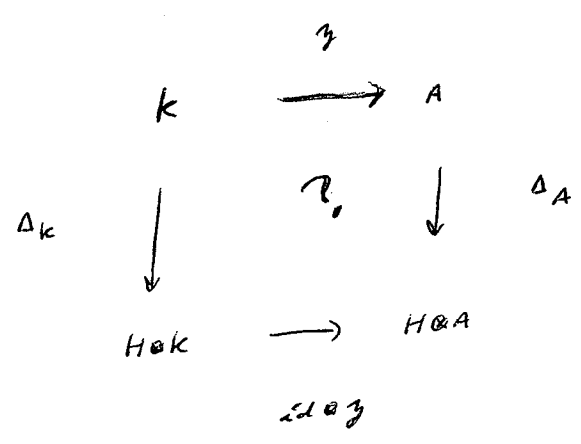
is an H -comodule morphism

pf.

Recall H -comodule k :

$$\Delta_k: \begin{array}{ccc} k & \longrightarrow & H \otimes k \\ 1 & \longrightarrow & 1_H \otimes 1 \end{array}$$

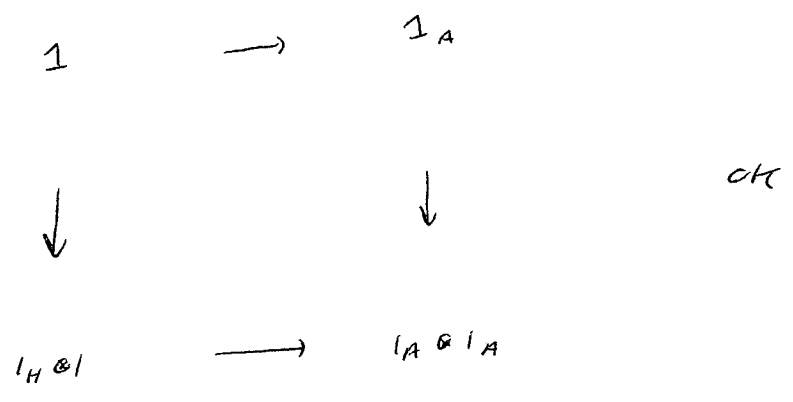
check diag:



Δ_A is alg morphism so it sends

$$1_A \longrightarrow 1_{H \otimes A} = 1_H \otimes 1_A$$

u/6/15
43



We saw that $k[x_1, x_2]$ becomes a comodule algebra for $M(2)$
 Similarly $k[x_1, x_2]$ \dots $SL(2)$

Abbrev

$$\begin{aligned} x &= x_{11} & y &= x_{12} \\ a &= x_{11} & b &= x_{12} \\ c &= x_{21} & d &= x_{22} \end{aligned}$$

So Δ_A sends

$$\begin{aligned} x &\rightarrow a \otimes x + b \otimes y \\ y &\rightarrow c \otimes x + d \otimes y \end{aligned}$$

For $i, j \in \mathbb{N}$,

$$\begin{aligned} \Delta_A(x^i y^j) &= (\Delta_A(x))^i (\Delta_A(y))^j \\ &= (a \otimes x + b \otimes y)^i (c \otimes x + d \otimes y)^j \\ &\quad \text{(binom)} \\ &= \sum_{r=0}^i \sum_{s=0}^j \binom{i}{r} \binom{j}{s} a^r b^{i-r} c^s d^{j-s} \otimes x^{r+s} y^{i+j-r-s} \quad (*) \end{aligned}$$

$A = k[x, y]$ has a grading

$$A = \sum_{n \in \mathbb{N}} A_n \quad (ds)$$

A_n consists of homog polynomials with total deg n .

LEM 71 For $H = \mathfrak{sl}(2)$ or $SL(2)$ consider the H -comodule algebra $A = k[x, y]$.

Then for $n \in \mathbb{N}$,

A_n is an H -cosubmodule of A

pf show

$$\Delta_A(A_n) \subseteq H \otimes A_n$$

A_n has basis

$$x^i y^j \quad i+j = n$$

B_j *

$$\Delta_A(x^i y^j) \in H \otimes A_n \quad \text{if } i+j = n$$

Result follows.

