Lec 27 Wed Nov 4 11/4/15

LEM 58 Given coalg C and C-comodules U, V. Given C-comodule morphism $\varphi: U \to V$.

Then $\varphi^*: V^* \to U^*$ is a C^*-module morphism.

pf. Show:

$$\varphi^*(gf) \overset{?}{=} g\,\varphi^*(f) \qquad \forall g \in C^* \quad \forall f \in V^*$$

Both sides in U^*.

Apply each side to $x \in U$

$$\varphi^*(gf)(x) \overset{?}{=} g\,\varphi^*(f)\,(x)$$

$$\varphi^*(gf)(x) = gf(\varphi(x)) = \sum_{(\varphi(x))} g(\varphi(x)_C)\, f(\varphi(x)_V) = \mu_K\, g \otimes f\left(\sum_{(\varphi(x))} \varphi(x)_C \otimes \varphi(x)_V\right)$$

$$g\,\varphi^*(f)\,(x) = \sum_{(x)} g(x_C)\, \varphi^*(f)(x_U) = \sum_{(x)} g(x_C)\, f(\varphi(x_U)) = \mu_K\, g \otimes f\left(\sum_{(x)} x_C \otimes \varphi(x_U)\right)$$

$$\underbrace{\mu_K\, g \otimes f\left(\sum_{(\varphi(x))} \varphi(x)_C \otimes \varphi(x)_V\right) = \mu_K\, g \otimes f\left(\sum_{(x)} x_C \otimes \varphi(x_U)\right)}_{\text{ok}}$$

□

Given a coalgebra C and C-comodule V,

a <u>C-subcomodule</u> of V is a subspace

$U \subseteq V$ such that

$$\Delta_V(U) \subseteq C \otimes U$$

In this case the map

$$\Delta_U : \begin{array}{ccc} U & \longrightarrow & C \otimes U \\ x & \longmapsto & \Delta_V(x) \end{array}$$

turns U into a C-comodule, and the inclusion map $U \to V$ is a C-comodule morphism (ex)

LEM 59 Given a coalgebra C.

Then $V = C$ becomes a C-comodule with

$$\Delta_V = \Delta$$

pf Require these diags commute:

$$\begin{tikzcd}
C \otimes C \otimes V & C \otimes V \arrow[l, "\Delta \otimes id"'] \\
C \otimes V \arrow[u, "id \otimes \Delta_V"] & V \arrow[l, "\Delta_V"] \arrow[u, "\Delta_V"']
\end{tikzcd}
\qquad
\begin{tikzcd}
k \otimes V \arrow[dr, shift left] & C \otimes V \arrow[l, "\varepsilon \otimes id"'] \\
 & V \arrow[ul, shift left, "1 \otimes v"] \arrow[u, "\Delta_V"']
\end{tikzcd}$$

They commute by the def of Δ, ε $\square$

LEM 60 Given coalgebras C, C'

Given coalgebra morphism $\varphi : C \to C'$

Given C-comodule V.

then V becomes a C'-comodule with action

$$\Delta'_V : V \xrightarrow{\Delta_V} C \otimes V \xrightarrow{\varphi \otimes id} C' \otimes V$$

pf check the diagrams commute

$$\begin{tikzcd}
C' \otimes C' \otimes V & C' \otimes V \arrow[l, "\Delta' \otimes id"'] \\
C' \otimes V \arrow[u, "id \otimes \Delta'_V"] \arrow[ur, phantom, "?"] & V \arrow[l, "\Delta'_V"] \arrow[u, "\Delta'_V"']
\end{tikzcd}$$

detail

$$\begin{tikzcd}
C'\otimes C'\otimes V & & C'\otimes V \arrow[ll, "\Delta'\otimes id"'] \\
C'\otimes C\otimes V \arrow[u, "id\otimes\varphi\otimes id"] \arrow[r, phantom, "\text{by constr}"] & C\otimes C\otimes V \arrow[l, "\varphi\otimes id\otimes id"'] \arrow[ul, "\varphi\otimes\varphi\otimes id"'] \arrow[r, phantom, "\text{since (below) commutes}"] & C\otimes V \arrow[l, "\Delta\otimes id"] \arrow[u, "\varphi\otimes id"'] \\
C'\otimes V \arrow[u, "id\otimes\Delta_V"] \arrow[r, phantom, "\text{since } \varphi\otimes id,\ id\otimes s \text{ commute } \forall r, s"] & C\otimes V \arrow[l, "\varphi\otimes id"] \arrow[u, "id\otimes\Delta_V"] \arrow[r, phantom, "\text{since } V \text{ is } C\text{-comod}"] & V \arrow[l, "\Delta_V"] \arrow[u, "\Delta_V"']
\end{tikzcd}$$

since

$$\begin{tikzcd}
C'\otimes C' & C' \arrow[l, "\Delta'"'] \\
C\otimes C \arrow[u, "\varphi\otimes\varphi"] & C \arrow[l, "\Delta"] \arrow[u, "\varphi"']
\end{tikzcd}$$

commutes

check

$$
\begin{tikzcd}
K \otimes V & C' \otimes V \arrow[l, "\varepsilon' \otimes id"'] \\
& V \arrow[u, "\Delta'_V"'] \arrow[ul, "1 \otimes v \,\leftarrow\!\shortmid\, v"] \arrow[ul, phantom, "?", near start]
\end{tikzcd}
$$

detail

$$
\begin{tikzcd}
K \otimes V & C' \otimes V \arrow[l, "\varepsilon' \otimes id"'] \\
K \otimes V \arrow[u, "id"] \arrow[ur, phantom, "\text{Since} \; \begin{smallmatrix} K & \xleftarrow{\varepsilon'} & C' \\ {\scriptstyle id}\uparrow & & \uparrow{\scriptstyle \varphi} \\ K & \xleftarrow[\varepsilon]{} & C \end{smallmatrix} \; \text{Commutes}"] & C \otimes V \arrow[l, "\varepsilon \otimes id"'] \arrow[u, "\varphi \otimes id"'] \\
& V \arrow[u, "\Delta_V"'] \arrow[ul, "1 \otimes v \,\leftarrow\!\shortmid\, v"] \arrow[ul, phantom, "\text{Since } V \text{ is } C\text{-comodule}", near start]
\end{tikzcd}
$$

$\square$

LEM 61 Given coalgebras C, D

Recall coalg $C \otimes D$.

Given C-comodule U and D-comodule V

then $U \otimes V$ becomes a $(C \otimes D)$-comodule

with action

$$\Delta_{U \otimes V} : \quad U \otimes V \xrightarrow[\Delta_U \otimes \Delta_V]{} C \otimes U \otimes D \otimes V \xrightarrow[id \otimes \tau_{U,D} \otimes id]{} C \otimes D \otimes U \otimes V$$

pf check

$$\begin{tikzcd}
C\otimes D\otimes C\otimes D\otimes U\otimes V & & C\otimes D\otimes U\otimes V \arrow[ll, "\Delta_{C\otimes D}\otimes id"'] \\
 & ? & \\
C\otimes D\otimes U\otimes V \arrow[uu, "id\otimes\Delta_{U\otimes V}"] & & U\otimes V \arrow[ll, "\Delta_{U\otimes V}"] \arrow[uu, "\Delta_{U\otimes V}"']
\end{tikzcd}$$

Recall

$$\Delta_C : \quad C \to C\otimes C$$
$$c \mapsto \sum_{(c)} c'\otimes c''$$

$$\Delta_D : \quad D \to D\otimes D$$
$$d \mapsto \sum_{(d)} d'\otimes d''$$

$$\Delta_{C\otimes D} : \quad C\otimes D \to C\otimes D\otimes C\otimes D$$
$$c\otimes d \mapsto \sum_{(c)}\sum_{(d)} c'\otimes d'\otimes c''\otimes d''$$

$\forall x \in U \quad \forall y \in V$

$$
\begin{tikzcd}
\displaystyle\sum_{(x)}\sum_{(y)} x_c' \otimes y_D' \otimes x_c'' \otimes y_D'' \otimes x_u \otimes y_v \arrow[d, phantom, "\ast \quad \|?"] & \displaystyle\sum_{(x)}\sum_{(y)} x_c \otimes y_D \otimes x_u \otimes y_v \arrow[l] \\
\displaystyle\sum_{(x)}\sum_{(y)} x_c \otimes y_D \otimes (x_u)_c \otimes (y_v)_D \otimes (x_u)_u \otimes (y_v)_v & \\
\displaystyle\sum_{(x)}\sum_{(y)} x_c \otimes y_D \otimes x_u \otimes x_v \arrow[u] & x \otimes y \arrow[l] \arrow[uu]
\end{tikzcd}
$$

Since U is C-comodule,

$$\sum_{(x)} x_c' \otimes x_c'' \otimes x_u = \sum_{(x)} x_c \otimes (x_u)_c \otimes (x_u)_u$$

Since V is D-comodule,

$$\sum_{(y)} y_D' \otimes y_D'' \otimes y_v = \sum_{(y)} y_D \otimes (y_v)_D \otimes (y_v)_v$$

So

$$\left(\sum_{(x)} x_c' \otimes x_c'' \otimes x_u\right) \otimes \left(\sum_{(y)} y_o' \otimes y_o'' \otimes y_v\right)$$

$$=$$

$$\left(\sum_{(x)} x_c \otimes (x_u)_c \otimes (x_u)_u\right) \otimes \left(\sum_{(x)} y_o \otimes (y_v)_o \otimes (y_v)_v\right)$$

Expanding each side get

$$\sum_{(x)} \sum_{(y)} x_c' \otimes x_c'' \otimes x_u \otimes y_o' \otimes y_o'' \otimes y_v$$

$$=$$

$$\sum_{(x)} \sum_{(y)} x_c \otimes (x_u)_c \otimes (x_u)_u \otimes y_o \otimes (y_v)_o \otimes (y_v)_v$$

In this eqn permute the tensor factors to get $*$

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Check

$$\begin{tikzcd}
k \otimes U \otimes V & & C \otimes D \otimes U \otimes V \arrow[ll, "\varepsilon_{C\otimes D} \otimes id"'] \\
& & \\
& & U \otimes V \arrow[uu, "\Delta_{U\otimes V}"'] \arrow[uull, "1 \otimes x \otimes y"] \\
& ? &
\end{tikzcd}$$

$$x \otimes y$$

$$\begin{tikzcd}
\sum_{(x)} \sum_{(y)} \varepsilon(x_C)\,\varepsilon(y_D) \otimes x_U \otimes y_V & & \sum_{(x)} \sum_{(y)} x_C \otimes y_D \otimes x_U \otimes y_V \arrow[ll] \\
& & \\
& & x \otimes y \arrow[uu] \arrow[uull]
\end{tikzcd}$$

$\overset{?}{=}$ $1 \otimes x \otimes y$

Have

$$x = \sum_{(x)} \varepsilon(x_C)\, x_U$$

$$y = \sum_{(y)} \varepsilon(y_D)\, y_V$$

So

$$x \otimes y = \sum_{(x)} \sum_{(y)} \varepsilon(x_C)\,\varepsilon(y_D)\, x_U \otimes y_V$$

ok

$\square$

Given bialgebra C

C is coalgebra

$C \otimes C$ is coalgebra

$\mu : C \otimes C \to C$ is coalgebra morphism

Given C-comodules U, V

By LEM 61

$U \otimes V$ is $(C \otimes C)$-comodule

with action

$$* \quad U \otimes V \xrightarrow{\Delta_U \otimes \Delta_V} C \otimes U \otimes C \otimes V \xrightarrow{id \otimes T_{U,C} \otimes id} C \otimes C \otimes U \otimes V$$

By LEM 60,

$U \otimes V$ is C-comodule

with action

$$\Delta_{U \otimes V} : U \otimes V \xrightarrow{*} C \otimes C \otimes U \otimes V \xrightarrow{\mu \otimes id} C \otimes U \otimes V$$

Describe $\Delta_{U \otimes V}$:

Write

$$\Delta_U : \quad \begin{array}{ccc} U & \longrightarrow & C \otimes U \\ x & \longmapsto & \sum_{(x)} x_C \otimes x_U \end{array}$$

$$\Delta_V : \quad \begin{array}{ccc} V & \longrightarrow & C \otimes V \\ y & \longmapsto & \sum_{(y)} y_C \otimes y_V \end{array}$$

$$\Delta_{U \otimes V} : \quad \begin{array}{ccc} U \otimes V & \longrightarrow & C \otimes U \otimes V \\ x \otimes y & \longmapsto & \sum_{(x)} \sum_{(y)} x_C y_C \otimes x_U \otimes y_V \end{array}$$

LEM 62 Given bialgebra C

Given vector space V

then V becomes a C-comodule with action

$$\Delta_V : \begin{array}{ccc} V & \longrightarrow & C \otimes V \\ x & \longrightarrow & 1_C \otimes x \end{array}$$

"trivial comodule"

pf check diagrams

$$\begin{tikzcd} C \otimes C \otimes V & C \otimes V \arrow[l, "\Delta \otimes id"'] \\ C \otimes V \arrow[u, "id \otimes \Delta_V"] & V \arrow[l, "\Delta_V"] \arrow[u, "\Delta_V"'] \end{tikzcd}$$

$$\begin{tikzcd} 1_C \otimes 1_C \otimes x & 1_C \otimes x \arrow[l] \\ 1_C \otimes x \arrow[u, "ok"] & x \arrow[l] \arrow[u] \end{tikzcd}$$

check

$$\begin{tikzcd} K \otimes V & C \otimes V \arrow[l, "\varepsilon \otimes id"'] \\ & V \arrow[ul, "1 \otimes v"] \arrow[u, "\Delta_V"'] \end{tikzcd} \qquad \begin{tikzcd} 1 \otimes x & 1_C \otimes x \arrow[l] \\ & x \arrow[ul, "ok"] \arrow[u] \end{tikzcd}$$

$\square$

Given f.d. bialg A

Given vector space V

Recall the trivial A-module V:

$$av = \varepsilon(a)v \qquad \forall a \in A \quad \forall v \in V.$$

Obs A^* is bialg and

V^* is A^*-comodule.

LEM 63 The above A^*-comodule V^* is trivial in the sense of LEM 62.

pf Recall

$$\Delta_{V^*} : \quad V^* \longrightarrow A^* \otimes V^*$$
$$f \quad \rightarrow \sum_{(f)} f_{A^*} \otimes f_{V^*}$$

satisfies

$$f(av) = \sum_{(f)} f_{A^*}(a)\, f_{V^*}(v) \qquad \forall a \in A \quad \forall v \in V$$

show Δ_{V^*} is

$$\Delta_{V^*} : \quad V^* \longrightarrow A^* \otimes V^*$$
$$f \quad \rightarrow \underset{\overset{\shortparallel}{\varepsilon}}{1_{A^*}} \otimes f$$

check:

$$\underbrace{\underset{\underset{\varepsilon(a)v}{\shortparallel}}{f(av)}}_{\varepsilon(a) f(v)} \overset{?}{=} \varepsilon(a) f(v) \qquad \forall a \in A \ \forall v \in V$$

ok

$\square$

Free comodules

Motivation:

Given algebra A and vector space V

Recall the free A-module $A \otimes V$ has action

$$a(b \otimes v) = (ab) \otimes v \qquad \forall a, b \in A \quad \forall v \in V$$

Action is

$$A \otimes (A \otimes V) \xrightarrow[\mu \otimes id]{} A \otimes V$$

Given coalg C and vector space V

expect $C \otimes V$ becomes a C-comodule with action

$$\Delta_{C \otimes V} : \quad C \otimes V \xrightarrow[\Delta \otimes id]{} C \otimes (C \otimes V)$$

It works.
Check diagrams commute

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Write $W = C \otimes V$

$$
\begin{tikzcd}
C \otimes C \otimes W & C \otimes W \arrow[l, "\Delta \otimes id"'] \\
C \otimes W \arrow[u, "id \otimes \Delta_W"] \arrow[ur, phantom, "?"] & W \arrow[l, "\Delta_W"] \arrow[u, "\Delta_W"']
\end{tikzcd}
$$

expand

$$
\begin{tikzcd}
C \otimes C \otimes C \otimes V & C \otimes C \otimes V \arrow[l, "\Delta \otimes id \otimes id"'] \\
C \otimes C \otimes V \arrow[u, "id \otimes \Delta \otimes id"] \arrow[ur, phantom, "?"] & C \otimes V \arrow[l, "\Delta \otimes id"] \arrow[u, "\Delta \otimes id"']
\end{tikzcd}
$$

diag commutes since Δ is coassoc

$$
\begin{tikzcd}
K \otimes W & C \otimes W \arrow[l, "\varepsilon \otimes id"'] \\
 & W \arrow[u, "\Delta_W"'] \arrow[ul, "1 \otimes w \,\leftarrow\!\shortmid\, w"] \arrow[ul, phantom, "?", shift left=4]
\end{tikzcd}
$$

Detail

$$
\begin{tikzcd}
k \otimes C \otimes V & C \otimes C \otimes V \arrow[l, "\varepsilon \otimes id \otimes id."'] \\
 & C \otimes V \arrow[u, "\Delta \otimes id"'] \arrow[ul, "1 \otimes c \otimes v \,\leftarrow\!\shortmid\, c \otimes v"] \arrow[ul, phantom, "?", shift left=4]
\end{tikzcd}
$$

diag commutes since ε is counit for C

ok

Above C-comodule $C \otimes V$ is called <u>free</u>