

LEM 58 Given coalg C and C -comodules

U, V . Given C -comodule morphism $\varphi: U \rightarrow V$.

Then $\varphi^*: V^* \rightarrow U^*$ is a C^* -module morphism.

pf.

Show:

$$\varphi^*(gf) = g \varphi^*(f)$$

$\forall g \in C^* \quad \forall f \in V^*$

Both sides in U^* .

Apply each side to $x \in U$

$$\varphi^*(gf)(x) \stackrel{?}{=} g \varphi^*(f)(x)$$

$$\text{" } gf(\varphi(x))$$

"

$$\sum_{(\varphi(x))} g(\varphi(x)_C) f(\varphi(x)_V)$$

"

$$\text{mk } g \circ f \left(\sum_{(\varphi(x))} \varphi(x)_C \otimes \varphi(x)_V \right)$$

$$\stackrel{?}{=} g \varphi^*(f)(x)$$

"

$$\sum_{(x)} g(x_C) \varphi^*(f)(x_U)$$

"

$$\sum_{(x)} g(x_C) f(\varphi(x_U))$$

"

$$\text{mk } g \circ f \left(\sum_{(x)} x_C \otimes \varphi(x_U) \right)$$

ok

□

Given a coalgebra C and C -comodule V ,

a C -subcomodule of V is a subspace

$U \subseteq V$ such that

$$\Delta_V(U) \subseteq C \otimes U$$

In this case the map

$$\Delta_U : \begin{array}{l} U \longrightarrow C \otimes U \\ x \longrightarrow \Delta_V(x) \end{array}$$

turns U into a C -comodule, and the

inclusion map $U \rightarrow V$ is a C -comodule

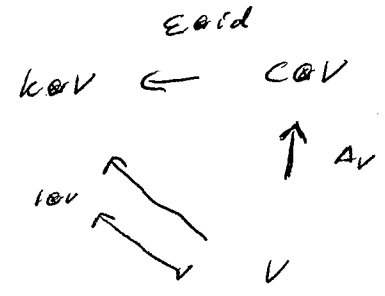
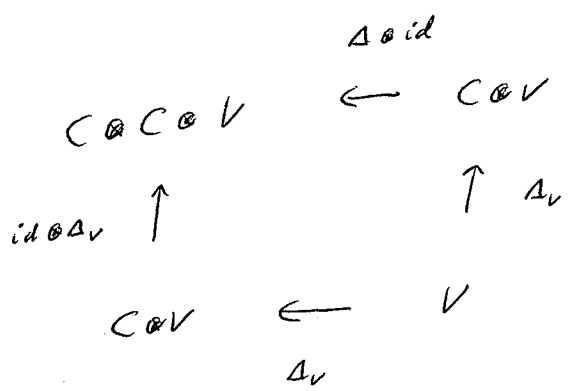
morphism (ex)

LEM 59 Given a coalgebra C ,

then $V = C$ becomes a C -comodule with

$$\Delta_V = \Delta$$

pf Require these diagrams commute:



they commute by the def of Δ, ϵ

□

LEM 60 Given coalgebras C, C'

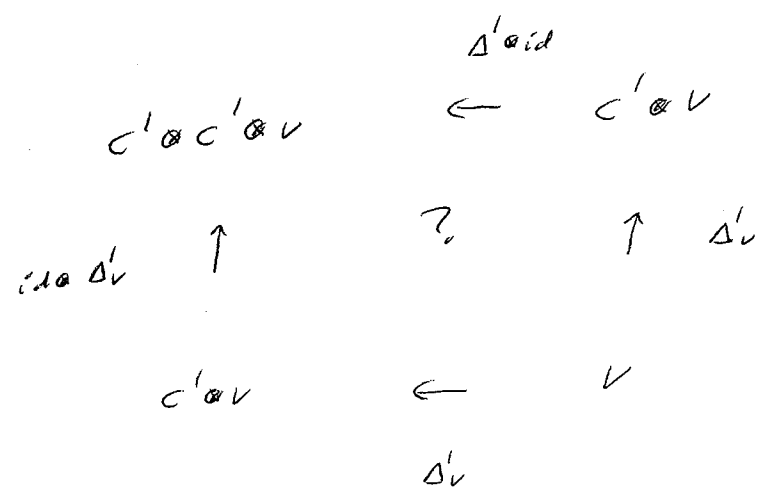
Given coalgebra morphism $\varphi: C \rightarrow C'$

Given C -comodule V ,

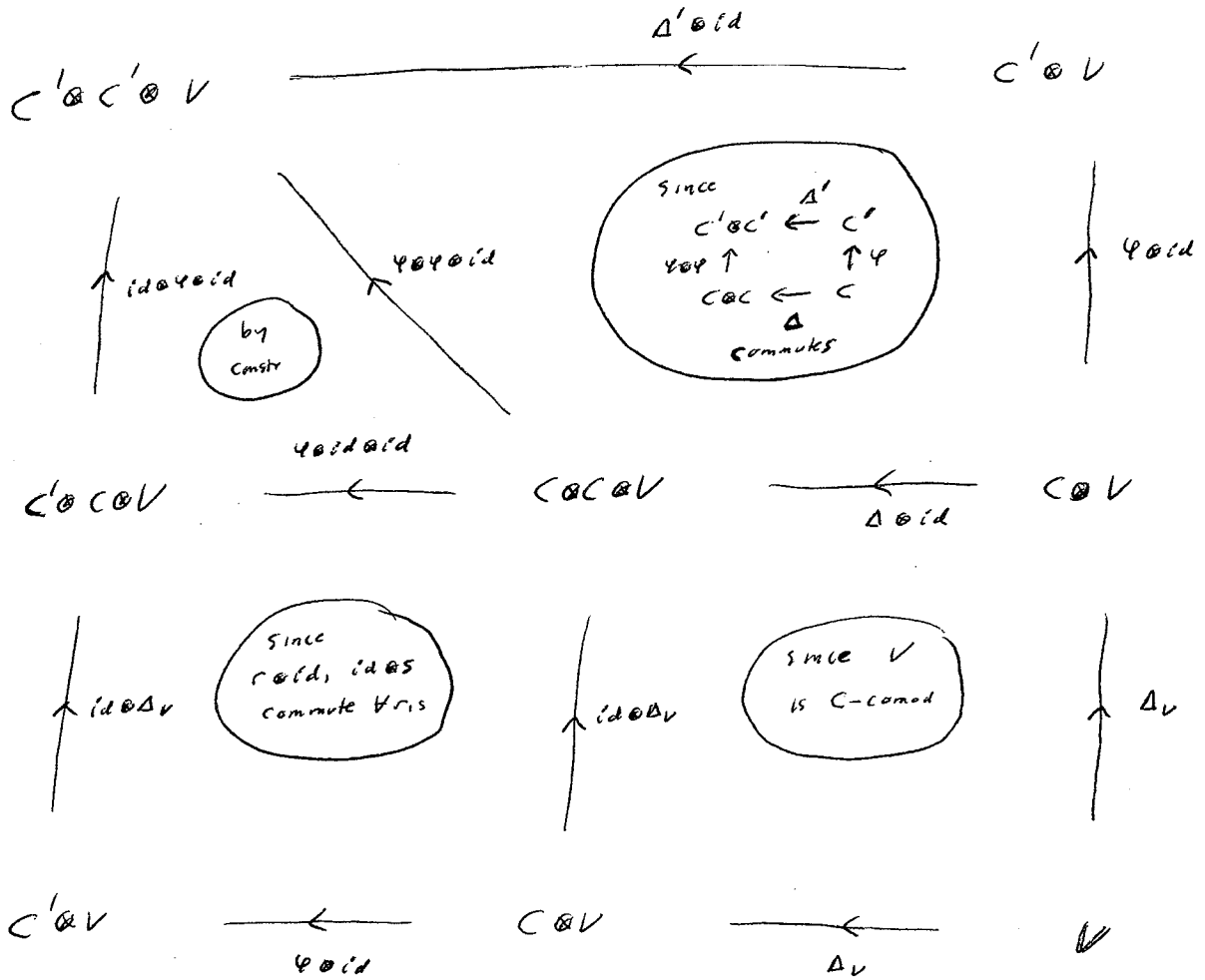
then V becomes a C' -comodule with action

$$\Delta'_V : V \xrightarrow{\Delta_V} C \otimes V \xrightarrow{\varphi \otimes id} C' \otimes V$$

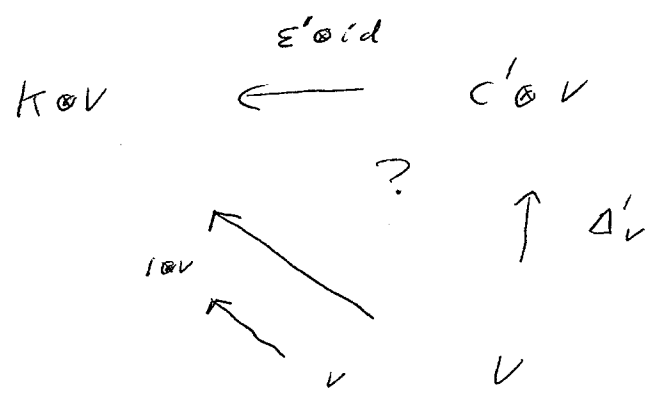
pf check the diagrams commute



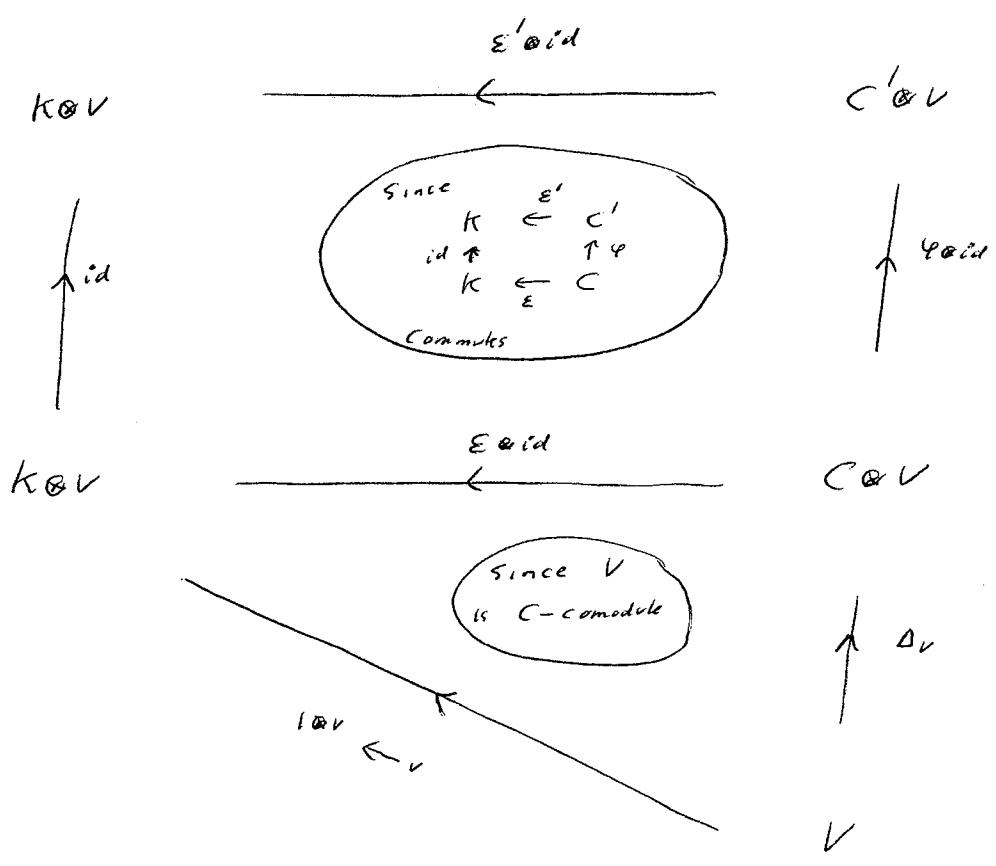
detail



check



detail



□

LEM 6) Given coalgebras C, D

Recall coalg $C \otimes D$

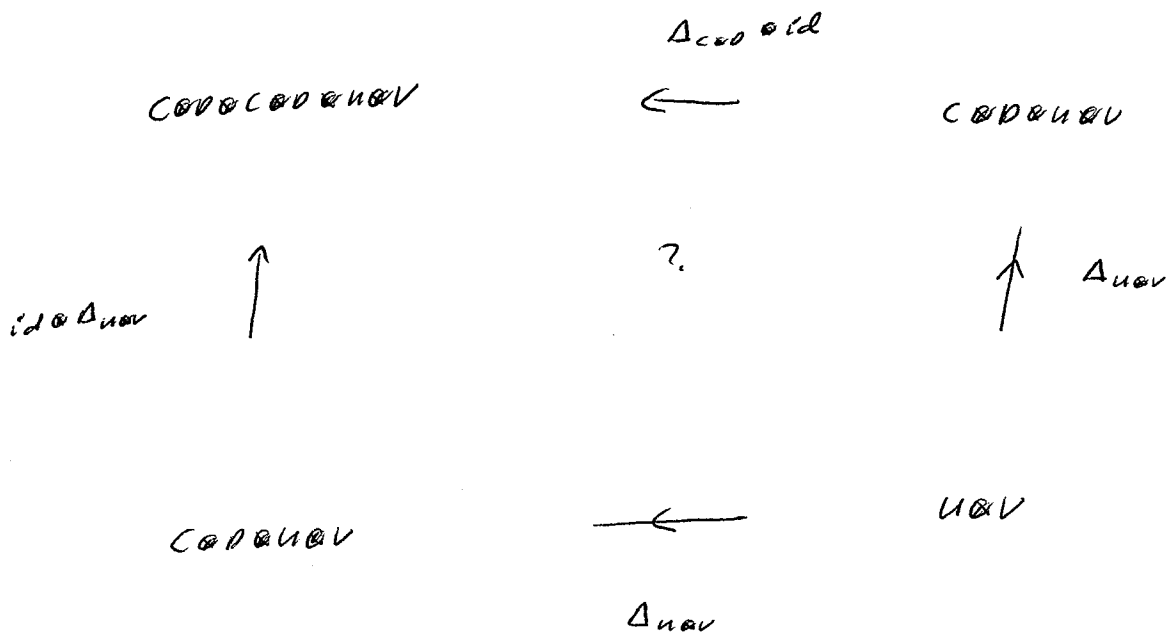
Given C -comodule U and D -comodule V

then $U \otimes V$ becomes a $(C \otimes D)$ -comodule

with action

$$\Delta_{U \otimes V} : U \otimes V \xrightarrow{\Delta_U \otimes \Delta_V} C \otimes U \otimes D \otimes V \xrightarrow{id \otimes \tau_{U,D} \otimes id} C \otimes D \otimes U \otimes V$$

pf check



Recall

$$\Delta_C: \begin{aligned} C &\rightarrow C \otimes C \\ c &\rightarrow \sum_{(c_1)} c'_1 \otimes c''_1 \end{aligned}$$

$$\Delta_D: \begin{aligned} D &\rightarrow D \otimes D \\ d &\rightarrow \sum_{(d_1)} d'_1 \otimes d''_1 \end{aligned}$$

$$\Delta_{C \otimes D}: \begin{aligned} C \otimes D &\rightarrow C \otimes D \otimes C \otimes D \\ c \otimes d &\rightarrow \sum_{(c_1)} \sum_{(d_1)} c'_1 \otimes d'_1 \otimes c''_1 \otimes d''_1 \end{aligned}$$

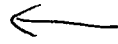
$\forall x \in U \forall y \in V$

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$$\sum_{(x)} \sum_{(y)} x_c' \otimes y_0' \otimes x_c'' \otimes y_0'' \otimes x_u \otimes y_v$$

* 11?



$$\sum_{(x)} \sum_{(y)} x_c \otimes y_0 \otimes x_u \otimes y_v$$

$$\sum_{(x)} \sum_{(y)} x_c \otimes y_0 \otimes (x_u)_c \otimes (y_v)_0 \otimes (x_u)_u \otimes (y_v)_v$$



$$\sum_{(x)} \sum_{(y)} x_c \otimes y_0 \otimes x_u \otimes x_v$$



$$x \otimes y$$

Since U is C -comodule,

$$\sum_{(x)} x_c' \otimes x_c'' \otimes x_u = \sum_{(x)} x_c \otimes (x_u)_c \otimes (x_u)_u$$

Since V is D -comodule,

$$\sum_{(y)} y_0' \otimes y_0'' \otimes y_v = \sum_{(y)} y_0 \otimes (y_v)_0 \otimes (y_v)_v$$

So

$$\left(\sum_{(x)} x_c' \otimes x_c'' \otimes x_u \right) \otimes \left(\sum_{(y)} y_o' \otimes y_o'' \otimes y_v \right)$$

$$= \left(\sum_{(x)} x_c \otimes (x_u)_c \otimes (x_u)_u \right) \otimes \left(\sum_{(y)} y_o \otimes (y_v)_o \otimes (y_v)_v \right)$$

Expanding each side etc

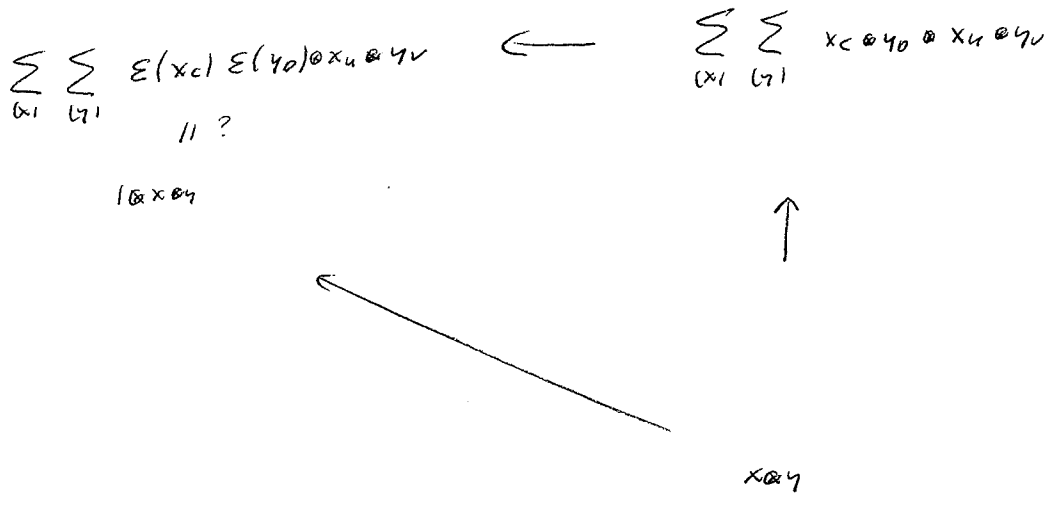
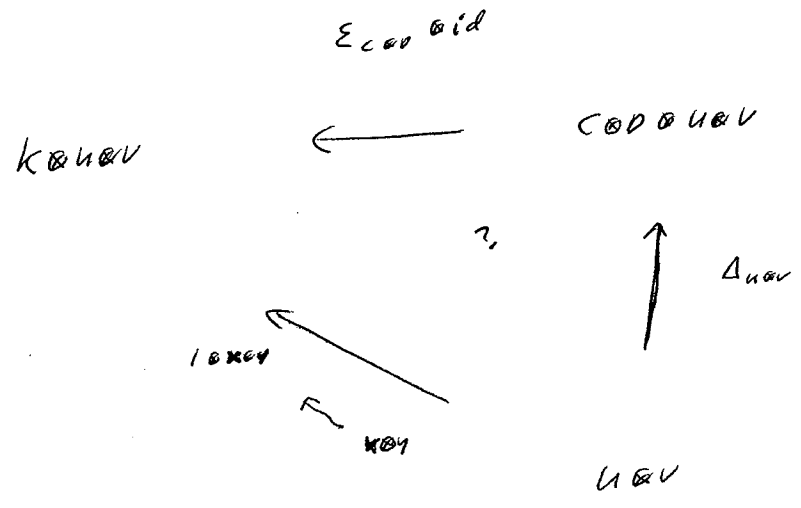
$$\sum_{(x)} \sum_{(y)} x_c' \otimes x_c'' \otimes x_u \otimes y_o' \otimes y_o'' \otimes y_v$$

=

$$\sum_{(x)} \sum_{(y)} x_c \otimes (x_u)_c \otimes (x_u)_u \otimes y_o \otimes (y_v)_o \otimes (y_v)_v$$

In this eqn permute the tensor factors to get *

check



Have

$$x = \sum_{(x)} E(x_C) x_U$$

$$y = \sum_{(y)} E(y_D) y_V$$

So

$$x \otimes y = \sum_{(x)} \sum_{(y)} E(x_C) E(y_D) x_U \otimes y_V$$

OK

□

Given bialgebra C

C is coalgebra

$C \otimes C$ is coalgebra

$\mu: C \otimes C \rightarrow C$ is coalgebra morphism

Given C -comodules U, V

By LEM 61

$U \otimes V$ is $(C \otimes C)$ -comodule

with action

* $U \otimes V \xrightarrow{\Delta_{U \otimes V}} C \otimes U \otimes C \otimes V \xrightarrow{id \otimes \tau_{U,C} \otimes id} C \otimes C \otimes U \otimes V$

By LEM 60,

$U \otimes V$ is C -comodule

with action

$\Delta_{U \otimes V}: U \otimes V \xrightarrow{*} C \otimes C \otimes U \otimes V \xrightarrow{\mu \otimes id} C \otimes U \otimes V$

Describe $\Delta_{U \otimes V}$:

Write

$$\Delta_U : \begin{aligned} U &\rightarrow C \otimes U \\ x &\rightarrow \sum_{(x)} x_c \otimes x_u \end{aligned}$$

$$\Delta_V : \begin{aligned} V &\rightarrow C \otimes V \\ y &\rightarrow \sum_{(y)} y_c \otimes y_v \end{aligned}$$

$$\Delta_{U \otimes V} : \begin{aligned} U \otimes V &\rightarrow C \otimes U \otimes V \\ x \otimes y &\rightarrow \sum_{(x)} \sum_{(y)} x_c y_c \otimes x_u \otimes y_v \end{aligned}$$

LEM 62 Given bialgebra C

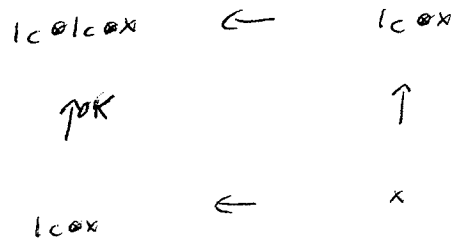
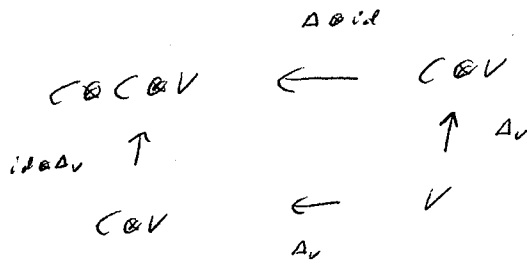
Given vector space V

then V becomes a C -comodule with action

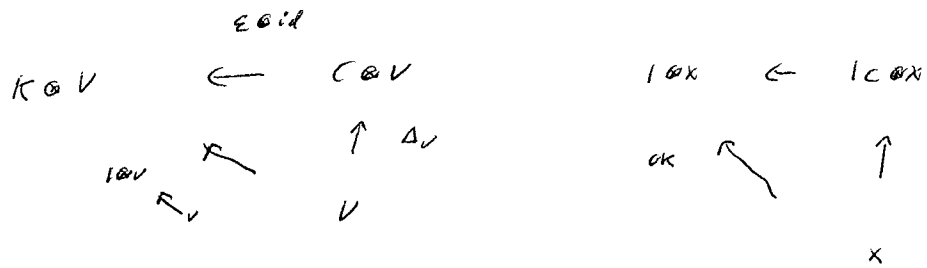
$$\Delta_V : \begin{aligned} V &\longrightarrow C \otimes V \\ x &\longrightarrow 1_C \otimes x \end{aligned}$$

"trivial comodule"

pf check diagrams



check



□

Given f.d. bialg A

Given vector space V

Recall the trivial A -module V :

$$av = \varepsilon(a)v \quad \forall a \in A \quad \forall v \in V.$$

Obs A^* is bialg and

V^* is A^* -comodule.

LEM 63 The above A^* -comodule V^* is trivial in the sense of LEM 62.

pf Recall

$$\begin{aligned} \Delta_{V^*} : V^* &\rightarrow A^* \otimes V^* \\ f &\rightarrow \sum_{(f)} f_{A^*} \otimes f_{V^*} \end{aligned}$$

satisfies

$$f(av) = \sum_{(f)} f_{A^*}(a) f_{V^*}(v) \quad \forall a \in A \quad \forall v \in V.$$

show Δ_{V^*} is

$$\begin{aligned} \Delta_{V^*} : V^* &\rightarrow A^* \otimes V^* \\ f &\rightarrow \sum_{\varepsilon} f_{A^*} \otimes f \end{aligned}$$

check:

$$f(av) \stackrel{?}{=} \underbrace{\varepsilon(av)}_{\varepsilon(a)\varepsilon(v)} = \varepsilon(a)f(v) \quad \forall a \in A \quad \forall v \in V$$

ok

□

Free comodules

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Motivation:

Given algebra A and vector space V
recall the free A -module $A \otimes V$ has action

$$a(b \otimes v) = (ab) \otimes v \quad \forall a, b \in A \quad \forall v \in V$$

Action is

$$A \otimes (A \otimes V) \xrightarrow{\text{mod}} A \otimes V$$

Given coalg C and vector space V

expect $C \otimes V$ becomes a C -comodule with action

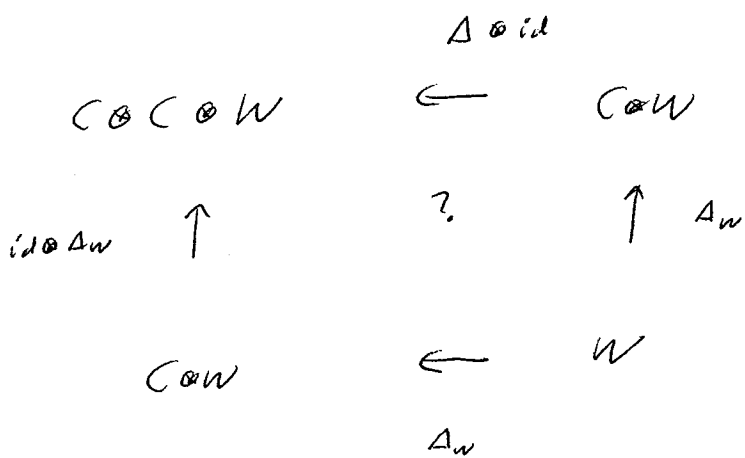
$$\Delta_{C \otimes V} : C \otimes V \xrightarrow{\Delta_{\text{id}}} C \otimes (C \otimes V)$$

It works.

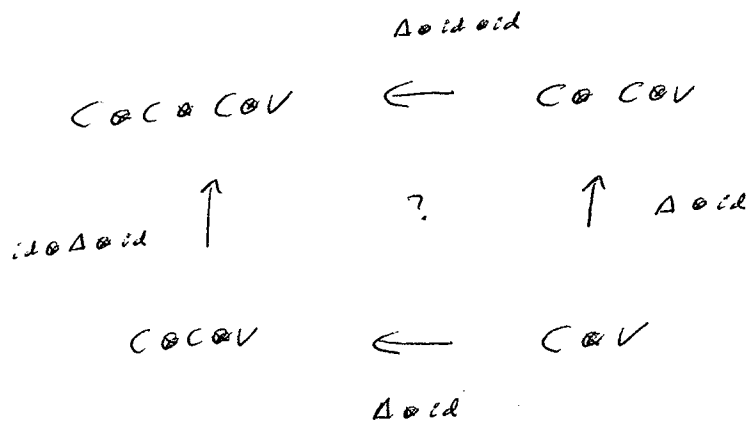
Check diagrams commute

Write $w = C \otimes V$

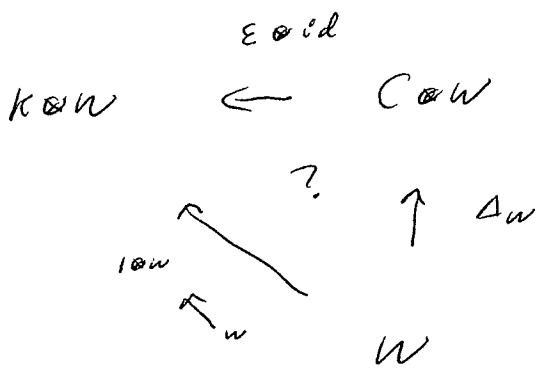
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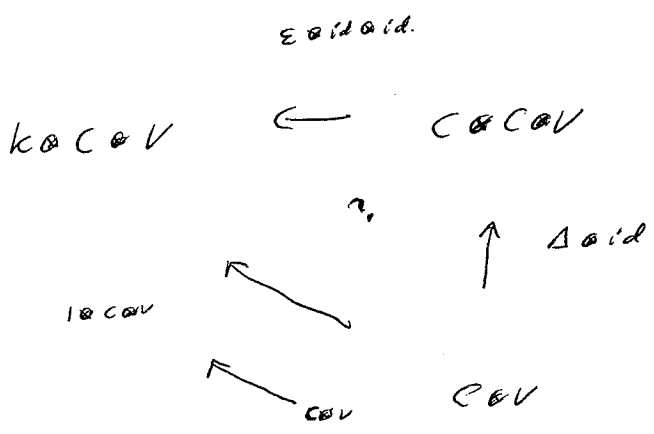
expand



diag commutes
since Δ is
coassoc



Detail



diag commutes
since ε is
counit for C

OK

Above C -comodule $C \otimes V$ is called free