Lec 26 Monday Nov 2 11/2/15

Describe Δ_{V^*}

$$\Delta_{V^*}: \quad \begin{array}{ccccccc} V^* & \longrightarrow & (A \otimes V)^* & \longrightarrow & V^* \otimes A^* & \longrightarrow & A^* \otimes V^* \\ f & \longrightarrow & F & \longrightarrow & \sum_{(f)} f_{V^*} \otimes f_{A^*} & \longrightarrow & \sum_{(f)} f_{A^*} \otimes f_{V^*} \end{array}$$

$\forall a \in A \quad \forall v \in V$

$$\begin{aligned} F(a \otimes v) &= \langle F, a \otimes v \rangle \\ &= \langle (\mu_V)^* f, a \otimes v \rangle \\ &= \langle f, \mu_V(a \otimes v) \rangle \\ &= \langle f, av \rangle \\ &= f(av) \end{aligned}$$

Also

$$F(a \otimes v) = \sum_{(f)} \underset{\uparrow k}{f_{V^*}(v)} \, \underset{\uparrow k}{f_{A^*}(a)} = \sum_{(f)} f_{A^*}(a) f_{V^*}(v)$$

So

$$f(av) = \sum_{(f)} f_{A^*}(a) f_{V^*}(v) \qquad \forall a \in A, \ \forall v \in V$$

check diagrams

$$\begin{tikzcd} A^* \otimes A^* \otimes V^* & A^* \otimes V^* \arrow[l, "\Delta \otimes id"'] \\ A^* \otimes V^* \arrow[u, "id \otimes \Delta_{V^*}"] & V^* \arrow[l, "\Delta_{V^*}"] \arrow[u, "\Delta_{V^*}"'] \end{tikzcd}$$

$\forall f \in V^*$

$$\begin{tikzcd} \begin{array}{c} \sum_{(f)} \Delta(f_{A^*}) \otimes f_{V^*} \\ ?\,\| \\ \sum_{(f)} f_{A^*} \otimes \Delta_{V^*}(f_{V^*}) \end{array} & \sum_{(f)} f_{A^*} \otimes f_{V^*} \arrow[l] \\ \sum_{(f)} f_{A^*} \otimes f_{V^*} \arrow[u] & f \arrow[l] \arrow[u] \end{tikzcd}$$

write

$$\sum_{(f)} \Delta(f_{A^*}) \otimes f_{V^*} = \sum_{(f)} (f_{A^*})' \otimes (f_{A^*})'' \otimes f_{V^*} \quad *$$

$$?\,\|$$

$$\sum_{(f)} f_{A^*} \otimes \Delta_{V^*}(f_{V^*}) = \sum_{(f)} f_{A^*} \otimes (f_{V^*})_{A^*} \otimes (f_{V^*})_{V^*} \quad **$$

$\forall\ a, b \in A \quad \forall\ v \in V$

$$(ab)v = a(bv)$$

So

$$f((ab)v) = f(a(bv))$$

$$= \sum_{(f)} f_{A^*}(ab) f_{V^*}(v) = \sum_{(f)} (f_{A^*})'(a) (f_{A^*})''(b) f_{V^*}(v)$$

$$f(a(bv)) = \sum_{(f)} f_{A^*}(a) f_{V^*}(bv) = \sum_{(f)} f_{A^*}(a) (f_{V^*})_{A^*}(b) (f_{V^*})_{V^*}(v)$$

Since A is f.d. we have vs iso

$$\theta \qquad A^* \otimes A^* \otimes V^* \simeq (V \otimes A \otimes A)^*$$

Under θ, images of $*$, $**$ are equal.

So $\quad * = **$

check diagrams, cnt.

$$\begin{tikzcd}
k \otimes V^* & & A^* \otimes V^* \arrow[ll, "\varepsilon \otimes id"'] \\
& & V^* \arrow[u, "\Delta_{V^*}"'] \arrow[ull]
\end{tikzcd}$$

$$1 \otimes f \longleftarrow f$$

$$\begin{tikzcd}
\sum_{(f)} \varepsilon(f_{A^*}) \otimes f_{V^*} & \sum_{(f)} f_{A^*} \otimes f_{V^*} \arrow[l] \\
& f \arrow[u]
\end{tikzcd}$$

$$\sum_{(f)} \varepsilon(f_{A^*}) \otimes f_{V^*} = 1 \otimes \sum_{(f)} \varepsilon(f_{A^*}) f_{V^*} \overset{?}{=} 1 \otimes f \longleftarrow f$$

Require

$$f \overset{?}{=} \sum_{(f)} \varepsilon(f_{A^*}) f_{V^*}$$

$\forall v \in V$

$$f(v) \overset{?}{=} \underbrace{\sum_{(f)} \underset{f_{A^*}(1_A)}{\underset{\shortparallel}{\varepsilon(f_{A^*})}} f_{V^*}(v)}_{\substack{\shortparallel \\ f(1_A v)}} = f(v) \quad \text{ok.}$$

We have shown that Δ_{V^*} turns V^* into an A^*-comodule.

$\square$

Given Coalg C Get algebra C^*.

Given C-comodule V, expect V^* is C^*-module.

with action

$$\mu_{C^*}: \quad C^* \otimes V^* \xrightarrow[\tau_{C^*, V^*}]{} V^* \otimes C^* \to (C \otimes V)^* \xrightarrow[\Delta_V^*]{} V^*$$

Describe μ_{C^*}. $\quad \forall g \in C^* \ \forall f \in V^*$

$$g \otimes f \longrightarrow f \otimes g \longrightarrow H \longrightarrow \Delta_V^* H$$

$$H(c \otimes v) = g(c) f(v)$$

$\forall x \in V$

$$\begin{aligned} \Delta_V^* H(x) &= \langle \Delta_V^* H, x \rangle \\ &= \langle H, \Delta_V x \rangle \\ &= \langle H, \sum_{(x)} x_C \otimes x_V \rangle \\ &= \sum_{(x)} g(x_C) f(x_V) \end{aligned}$$

So $gf \in V^*$ satisfies

$$(gf)(x) = \sum_{(x)} g(x_C) f(x_V) \qquad \forall x \in V$$

Check diagrams

$\forall\ g, h \in C^*\quad \forall f \in V^*$

$$(gh)f \overset{?}{=} g(hf)$$

Both sides in V^*. Apply each side to $x \in V$:

$$\big((gh)f\big)(x) = \sum_{(x)} (gh)(x_C)\ f(x_V)$$

$$= \sum_{(x)} g(x_C')\ h(x_C'')\ f(x_V)$$

$= \mu_K\ g \otimes h \otimes f$ at $\sum_{(x)} x_C' \otimes x_C'' \otimes x_V$

$$\big(g(hf)\big)(x) = \sum_{(x)} g(x_C)\ (hf)(x_V)$$

$$= \sum_{(x)} g(x_C)\ h\big((x_V)_C\big)\ f\big((x_V)_V\big)$$

$= \mu_K\ g \otimes h \otimes f$ at $\sum_{(x)} x_C \otimes (x_V)_C \otimes (x_V)_V$

ok

check $\quad 1_{C^*} f \overset{?}{=} f \qquad \forall f \in V^*$

$\forall x \in V,$

$$(1_{C^*} f)(x) = \sum_{(x)} \underset{\varepsilon}{\underset{\shortparallel}{1_{C^*}}}(x_C)\ f(x_V)$$

$$= f\Big(\sum_{(x)} \varepsilon(x_C)\ x_V\Big)$$

$$= f(x)$$

ok

We have shown that $\mathcal{M}_{C^*}$ turns V^* into a C^*-module.

11/2/15
7

Given coalg C, and C-comodules U, V

A linear map $\varphi : U \to V$ is a <u>comodule morphism</u>

whenever

$$\begin{tikzcd} C \otimes U \arrow[r, "id \otimes \varphi"] & C \otimes V \\ U \arrow[u, "\Delta_U"] \arrow[r, "\varphi"'] & V \arrow[u, "\Delta_V"'] \end{tikzcd}$$

commutes

So $\forall x \in U$

$$\begin{tikzcd} \sum_{(x)} x_C \otimes x_U \arrow[r] & \sum_{(x)} x_C \otimes \varphi(x_U) \overset{?}{=} \sum_{(\varphi(x))} \varphi(x)_C \otimes (\varphi(x))_V \\ x \arrow[u] \arrow[r] & \varphi(x) \arrow[u] \end{tikzcd}$$

Require

$$\sum_{(x)} x_C \otimes \varphi(x_U) = \sum_{(\varphi(x))} \varphi(x)_C \otimes \varphi(x)_V \qquad \forall x \in U$$

LEM 57 Given f.d. algebra A and A-modules U, V.

Given A-module morphism $\varphi : U \to V$

then $\varphi^* : V^* \to U^*$ is an A^*-comodule morphism.

pf Show this diag commutes:

$$\begin{tikzcd} A^* \otimes V^* \arrow[r, "id \otimes \varphi^*"] & A^* \otimes U^* \\ V^* \arrow[u, "\Delta_{V^*}"] \arrow[r, "\varphi^*"'] & U^* \arrow[u, "\Delta_{U^*}"'] \end{tikzcd}$$

$\forall f \in V^*$ require

$$\sum_{(f)} f_{A^*} \otimes \varphi^*(f_{V^*}) \qquad *$$

$\stackrel{?}{=}$

$$\sum_{(\varphi^*(f))} (\varphi^*(f))_{A^*} \otimes (\varphi^*(f))_{U^*} \qquad **$$

Have: $\forall a \in A \quad \forall u \in \mathcal{U}$

$$a\varphi(u) = \varphi(au)$$

So $\forall f \in V^*$

$$f(a\varphi(u)) = f(\varphi(au))$$

$$f(a\varphi(u)) = \sum_{(f)} f_{A^*}(a)\, f_{V^*}(\varphi(u)) = \sum_{(f)} f_{A^*}(a)\, \varphi^*(f_{V^*})(u)$$

$$f(\varphi(au)) = \varphi^*(f)(au) = \sum_{(\varphi^*(f))} (\varphi^*(f))_{A^*}(a)\, \varphi^*(f)_{\mathcal{U}^*}(u)$$

Recall VS 150

$$\theta : A^* \otimes \mathcal{U}^* \longrightarrow (\mathcal{U} \otimes A)^*$$

Under θ, images of $*$, $**$ are equal.

So $* = **$ $\square$