

Describe Δ_{V^*}

$$\begin{array}{l} V^* \longrightarrow (A \otimes V)^* \longrightarrow V^* \otimes A^* \longrightarrow A^* \otimes V^* \\ \Delta_{V^*}: f \longrightarrow F \longrightarrow \sum_{(f)} f_{V^*} \otimes f_{A^*} \longrightarrow \sum_{(f)} f_{A^*} \otimes f_{V^*} \end{array}$$

 $\forall a \in A \quad \forall v \in V$

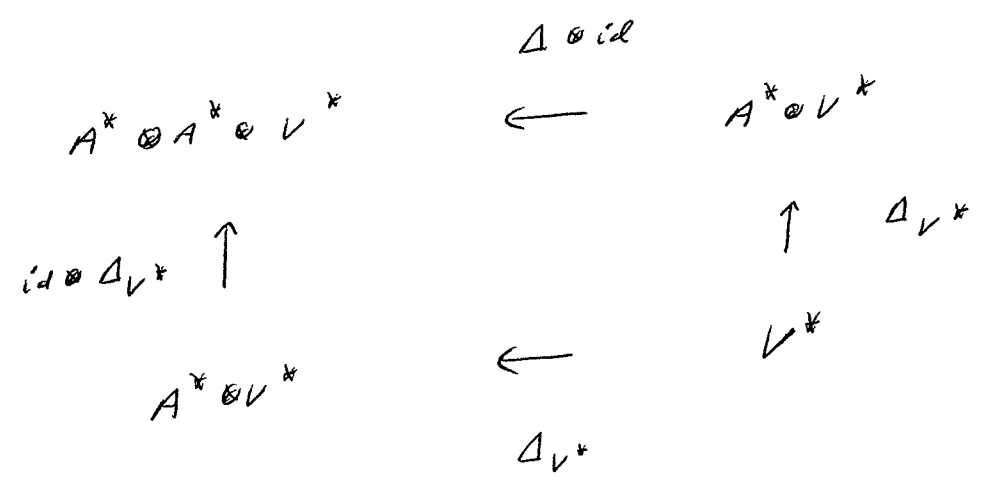
$$\begin{aligned} F(a \otimes v) &= \langle F, a \otimes v \rangle \\ &= \langle (f_{V^*})^* f, a \otimes v \rangle \\ &= \langle f, \mu_V(a \otimes v) \rangle \\ &= \langle f, av \rangle \\ &= f(av) \end{aligned}$$

$$\text{Also } F(a \otimes v) = \sum_{(f)} \underset{\uparrow k}{f_{V^*}(v)} \underset{\uparrow k}{f_{A^*}(a)} = \sum_{(f)} f_{A^*}(a) f_{V^*}(v)$$

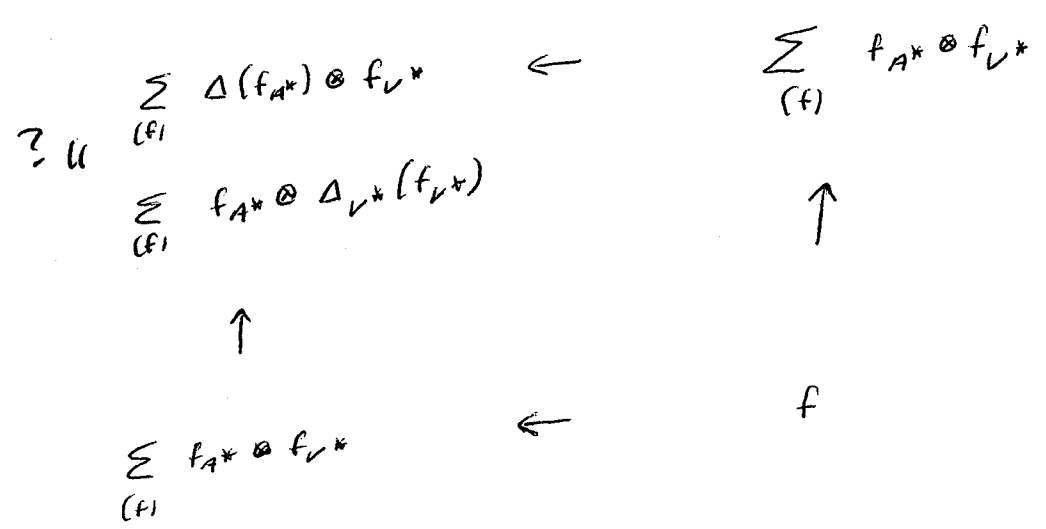
So

$$f(av) = \sum_{(f)} f_{A^*}(a) f_{V^*}(v) \quad \forall a \in A, \forall v \in V$$

check diagrams



$\forall f \in V^*$



write

$$\sum_{(f)} \Delta(f_{A^*}) \otimes f_{V^*} = \sum_{(f)} (f_{A^*})' \otimes (f_{A^*})'' \otimes f_{V^*} \quad *$$

?, ((

$$\sum_{(f)} f_{A^*} \otimes \Delta_{V^*}(f_{V^*}) = \sum_{(f)} f_{A^*} \otimes (f_{V^*})_{A^*} \otimes (f_{V^*})_{V^*} \quad **$$

$$\forall a, b \in A \quad \forall v \in V$$

$$(ab)v = a(bv)$$

So

$$f((ab)v) = f(a(bv))$$

||

$$\sum_{(f)} f_{A^*}(ab) f_{V^*}(v)$$

$$\sum_{(f)} f_{A^*}(a) f_{V^*}(bv)$$

||

$$\sum_{(f)} (f_{A^*})'(a) (f_{A^*})''(b) f_{V^*}(v)$$

$$\sum_{(f)} f_{A^*}(a) (f_{V^*})_{A^*}(b) (f_{V^*})_{V^*}(v)$$

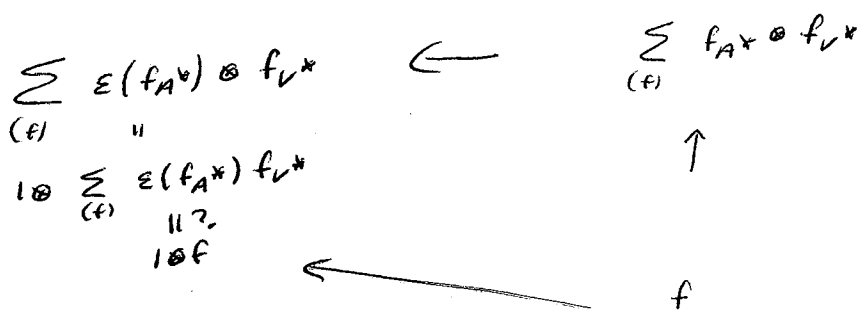
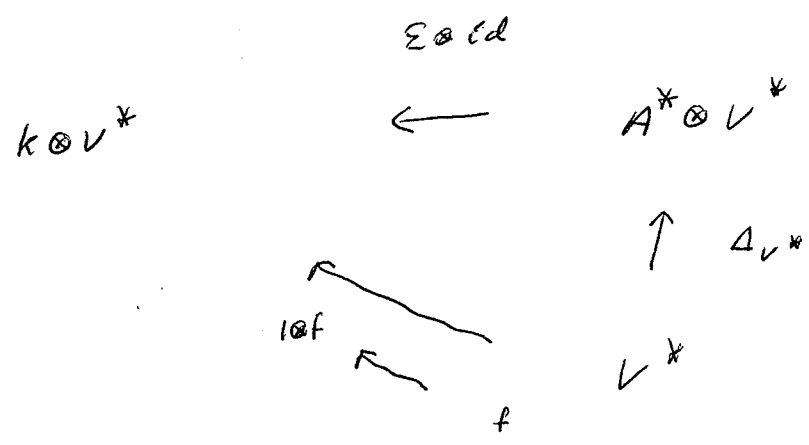
Since A is f.d. we have vs iso

$$\theta \quad A^* \otimes A^* \otimes V^* \cong (V \otimes A \otimes A)^*$$

Under θ , images of $*$, $**$ are equal.

$$\text{So } * = **$$

check diagrams, cont.



Require

$$f = \sum_{(f)} \varepsilon(f_{A^*}) f_{V^*}$$

$\forall v \in V$

$$f(v) = \sum_{(f)} \varepsilon(f_{A^*}) f_{V^*}(v)$$

$$= \underbrace{\varepsilon(f_{A^*})}_{f_{A^*}(1_A)} f_{V^*}(v)$$

$$= f(1_A v) = f(v) \quad \text{OK}$$

We have shown that Δ_{V^*} turns V^* into an A^* -comodule.



Given Coalgebra C get algebra C^* .

Given C -comodule V , expect V^* is C^* -module.

with action

$$\mu_{C^*}: C^* \otimes V^* \xrightarrow{\tau_{C^*V^*}} V^* \otimes C^* \xrightarrow{\Delta_V^*} (C \otimes V)^* \xrightarrow{\Delta_V^*} V^*$$

Describe μ_{C^*} , $\forall g \in C^* \forall f \in V^*$

$$g \circ f \longrightarrow f \circ g \longrightarrow H \longrightarrow \Delta_V^* H$$

$H(c \otimes v) = g(c)f(v)$

$$\begin{aligned} \forall x \in V \\ \Delta_V^* H(x) &= \langle \Delta_V^* H, x \rangle \\ &= \langle H, \Delta_V x \rangle \\ &= \langle H, \sum_{\alpha_i} x_{c \otimes v} \rangle \\ &= \sum_{(x)} g(x_c) f(x_v) \end{aligned}$$

So $gf \in V^*$ satisfies

$$(gf)(x) = \sum_{(x)} g(x_c) f(x_v) \quad \forall x \in V$$

check diagrams

$$\forall g, h \in C^* \quad \forall f \in V^*$$

$$(gh)f \stackrel{?}{=} g(hf)$$

Both sides in V^* . Apply each side to $x \in V$:

$$\begin{aligned} ((gh)f)(x) &= \sum_{(x)} (gh)(x_c) f(x_v) \\ &= \sum_{(x)} g(x'_c) h(x''_c) f(x_v) \\ &=_{MK} g \otimes h \otimes f \text{ at } \sum_{(x)} x'_c \otimes x''_c \otimes x_v \end{aligned}$$

$$\begin{aligned} (g(hf))(x) &= \sum_{(x)} g(x_c) (hf)(x_v) \\ &= \sum_{(x)} g(x_c) h((x_v)_c) f((x_v)_v) \\ &=_{MK} g \otimes h \otimes f \text{ at } \sum_{(x)} x_c \otimes (x_v)_c \otimes (x_v)_v \end{aligned}$$

OK

check $1_{C^*} f \stackrel{?}{=} f \quad \forall f \in V^*$

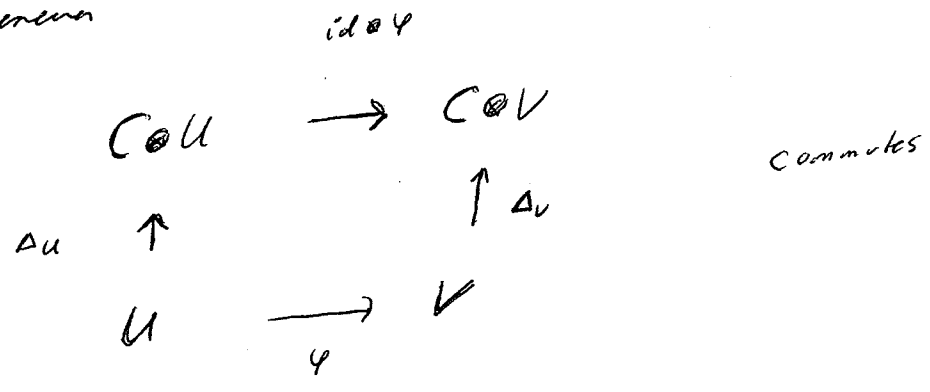
$$\begin{aligned} \forall x \in V, \quad (1_{C^*} f)(x) &= \sum_{(x)} 1_{C^*}(x_c) f(x_v) \\ &= f\left(\sum_{(x)} \varepsilon(x_c) x_v\right) \\ &= f(x) \end{aligned}$$

We have shown that M_{C^*} turns V^* into a C^* -module.

Given coalg C and C -comodules U, V

Algebra map $\varphi: U \rightarrow V$ is a comodule morphism

whereas



So $\forall x \in U$

$$\begin{array}{ccc}
 \sum_{(x)} x_C \otimes x_U & \longrightarrow & \sum_{(x)} x_C \otimes \varphi(x_U) \\
 \uparrow & & \uparrow \\
 x & \longrightarrow & \varphi(x)
 \end{array}
 \Rightarrow \sum_{(\varphi(x))} \varphi(x)_C \otimes (\varphi(x))_V$$

Require

$$\sum_{(x)} x_C \otimes \varphi(x_U) = \sum_{(\varphi(x))} \varphi(x)_C \otimes \varphi(x)_V \quad \forall x \in U$$

LEM 57 Given f.d. algebra A and A -modules U, V .

Given A -module morphism $\varphi: U \rightarrow V$

then $\varphi^*: V^* \rightarrow U^*$ is an A^* -comodule morphism.

pf Show this diag commutes:

$$\begin{array}{ccc}
 A^* \otimes V^* & \xrightarrow{\text{id} \otimes \varphi^*} & A^* \otimes U^* \\
 \Delta_{V^*} \uparrow & & \uparrow \Delta_{U^*} \\
 V^* & \xrightarrow{\varphi^*} & U^*
 \end{array}$$

$\forall f \in V^*$ require

$$\begin{aligned}
 & \sum_{(f)} f_{A^*} \otimes \varphi^*(f_{V^*}) \quad * \\
 \stackrel{?}{=} & \sum_{(\varphi^*(f))} (\varphi^*(f))_{A^*} \otimes (\varphi^*(f))_{U^*} \quad * *
 \end{aligned}$$

Have: $\forall a \in A \quad \forall u \in U$

$$a \varphi(u) = \varphi(au)$$

So $\forall f \in V^*$

$$f(a \varphi(u)) = f(\varphi(au))$$

$$\parallel$$

$$\parallel$$

$$\varphi^*(f)(au)$$

$$\sum_{(f)} f_{A^*}(a) f_{V^*}(\varphi(u))$$

$$\parallel$$

$$\parallel$$

$$\sum_{(\varphi^*(f))} (\varphi^*(f))_{A^*}(a) \varphi^*(f)_{U^*}(u)$$

$$\sum_{(f)} f_{A^*}(a) \varphi^*(f_{V^*})(u)$$

Recall vs 150

$$\theta: A^* \otimes U^* \rightarrow (U \otimes A)^*$$

Under θ , images of $*$, $**$ are equal.

So $* = **$

