

Given bialg A with antipode S

Given A -module V

Recall lin map

$$\begin{aligned} \text{ev: } V^* \otimes V &\longrightarrow k \\ f \otimes v &\longrightarrow f(v) \end{aligned}$$

*

LEM. 53 * is an A -module morphism

pf

$$f \otimes v \xrightarrow{\text{ev}} f(v)$$

$$\begin{aligned} a(f \otimes v) &\xrightarrow{?} a f(v) \\ \parallel & \qquad \qquad \parallel \\ \sum_{(a)} a' f \otimes a'' v & \qquad \qquad \varepsilon(a) f(v) \end{aligned}$$

$\forall a \in A$

$$\begin{aligned} \text{ev} \left(\sum_{(a)} a' f \otimes a'' v \right) &= \sum_{(a)} (a' f)(a'' v) \\ &= \sum_{(a)} f(S(a'')(a'' v)) \\ &= f \left(\underbrace{\sum_{(a)} S(a'') a''}_{\varepsilon(a) 1_A} v \right) \\ &= f \left(\underbrace{\varepsilon(a)}_{\in k} v \right) \\ &= \varepsilon(a) f(v) \end{aligned}$$

ok

□

Given bialgebra A with antipode S

Given fin dim'l A -module V

Recall co-eval map

$$\begin{aligned}
 \mathcal{S}: \quad k &\longrightarrow V \otimes V^* \\
 1 &\longrightarrow \sum_i v_i \otimes v_i^*
 \end{aligned}$$

*

$\{v_i\}$ is basis for V
 $\{v_i^*\}$ is dual basis for V^*

LEM 54 * is an A -module morphism.

pf From LEM 51 get A -module ISO

$$\theta: \quad V \otimes V^* \longrightarrow \text{Hom}(V, V) = \text{End}(V)$$

Factor * :

$$\begin{array}{ccccc}
 k & \longrightarrow & \text{End}(V) & \longrightarrow & V \otimes V^* \\
 1 & \longrightarrow & I & & \\
 & & \uparrow & & \\
 & & \text{show this is } A\text{-module morphism} & &
 \end{array}$$

$$\forall a \in A$$

in A -module k ,

$$a \cdot 1 = \varepsilon(a)$$

in A -module $\text{End}(V)$, require

$$a \cdot I = \varepsilon(a) \cdot I$$

By LEM 50

$$a \cdot I = \sum_{(a')} a' \cdot I \cdot S(a'')$$

$$= \underbrace{\sum_{(a')} a' \cdot S(a'')}_{\varepsilon(a) \cdot 1_A} \cdot I$$

$$= \varepsilon(a) \cdot I$$

OK

□

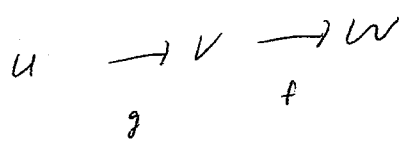
Given bialg A with antipode S

Given A -modules U, V, W

Recall the composition linear map

Comp: $\text{Hom}(V, W) \otimes \text{Hom}(U, V) \rightarrow \text{Hom}(U, W)$ *

$f \quad \otimes \quad g \qquad \qquad \qquad f \circ g$



LEM 55. The map * is an A -module morphism.

pf I (assume U, V, W fd)

* is the composition of A -module morphisms

$$\begin{aligned}
 \text{Hom}(V, W) \otimes \text{Hom}(U, V) &\rightarrow (W \otimes V^*) \otimes (V \otimes U^*) \\
 &\rightarrow W \otimes (V^* \otimes V) \otimes U^* \\
 &\rightarrow W \otimes K \otimes U^* \\
 &\rightarrow W \otimes U^* \\
 &\rightarrow \text{Hom}(U, W)
 \end{aligned}$$

pf II

10/30/15
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*

$$f \circ g \longrightarrow f \circ g$$

$$a(f \circ g) \xrightarrow{?} a(f \circ g)$$

$\forall a \in A$

||

$$\sum_{(a)} a' f \circ a'' g$$

$$\sum_{(a)} a' f g S(a'')$$

|| ?

||

$$\sum_{(a)} a' f S(a'') \circ a''' g S(a''') \longrightarrow \sum_{(a)} a' f \underbrace{S(a'') a'''}_{\text{shrink}} g S(a''')$$

$$\parallel \left[\sum_{(b)} S(b') b'' = \varepsilon(b) 1_A \right]$$

$$\sum_{(a)} a' f \underbrace{\varepsilon(a'') 1_A}_{\uparrow k} g S(a''')$$

||

$$\sum_{(a)} \underbrace{a' \varepsilon(a'')}_{\text{shrink}} f g S(a''')$$

$$\parallel \left[\sum_{(b)} b' \varepsilon(b'') = b \right]$$

$$\sum_{(a)} a' f g S(a'')$$

OK.

□

Given bialgebra A with antipode S

Given A -modules

$$u, u', v, v'$$

Recall linear map

$$\lambda : \begin{array}{ccc} \text{Hom}(u, u') \otimes \text{Hom}(v, v') & \rightarrow & \text{Hom}(v \otimes u, u' \otimes v') \\ f \otimes g & \rightarrow & H \\ & & H(v \otimes u) = f(u) \otimes g(v) \end{array}$$

Consider when is λ an A -module morphism

To simplify things, assume both

- (i) u or u' is fin dim'l, so $\text{Hom}(u, u') \cong u' \otimes u^*$
- (ii) v or v' is fin dim'l, so $\text{Hom}(v, v') \cong v' \otimes v^*$

Now λ is the composition

$$\begin{aligned}
\text{Hom}(U, U') \otimes \text{Hom}(V, V') &\xrightarrow{\gamma} (U' \otimes U^*) \otimes (V' \otimes V^*) \\
&\xrightarrow{\gamma} U' \otimes (U^* \otimes V') \otimes V^* \\
&\rightarrow U' \otimes (V' \otimes U^*) \otimes V^* \\
&\xrightarrow{\gamma} (U' \otimes V') \otimes (U^* \otimes V^*) \\
&\xrightarrow{\gamma} U' \otimes V' \otimes (V \otimes U)^* \\
&\xrightarrow{\gamma} \text{Hom}(V \otimes U, U' \otimes V^*)
\end{aligned}$$

LEM 56 With the above notation and assumptions,
 λ is an A -module morphism provided that

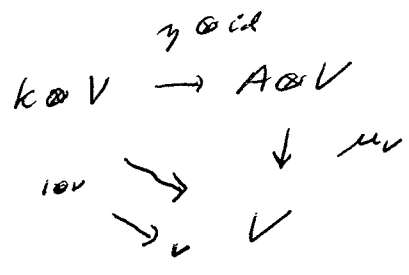
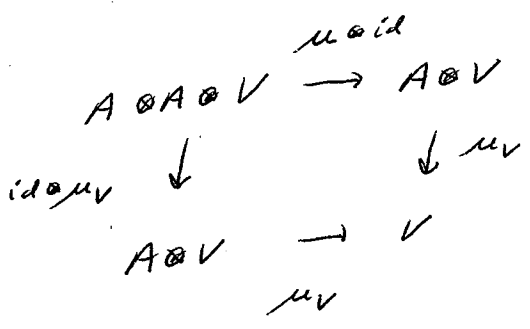
$$\tau_{U^*, V'} : U^* \otimes V' \rightarrow V' \otimes U^*$$

is an A -module morphism.

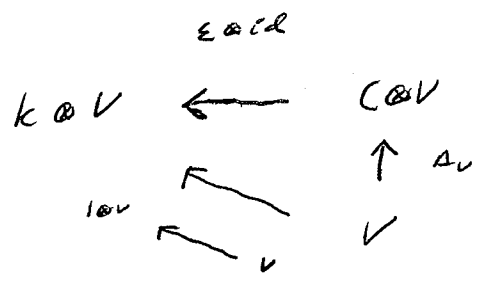
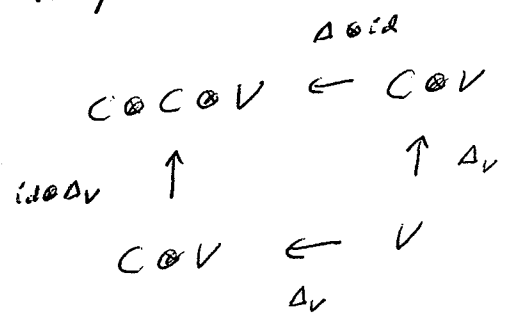


Comodules

Motivation: Given an algebra A with data μ, γ
 View an A -module as a vector space V together with
 a linear map $\mu_V: A \otimes V \rightarrow V$ s.t. these
 diagrams commute:



Given a coalgebra C with data Δ, ϵ
 A (left) C -comodule is a vector space V together
 with a linear map $\Delta_V: V \rightarrow C \otimes V$ s.t. these
 diagrams commute:



A right C -comodule is similarly defined.

Extending the Sweedler notation, write

$$\Delta_V : \begin{array}{ccc} V & \longrightarrow & C \otimes V \\ x & \longrightarrow & \sum_{(x)} x_C \otimes x_V \end{array}$$

The commuting diagrams require that for $x \in V$,

$$\sum_{(x)} (x_C)' \otimes (x_C)'' \otimes x_V = \sum_{(x)} x_C \otimes (x_V)' \otimes (x_V)''$$

and

$$x = \sum_{(x)} \epsilon(x_C) x_V$$

— o —

Given fd algebra A get coalg A^*

Given A -module V .

expect V^* is A^* -comodule with action

$$\Delta_{V^*} : \begin{array}{ccc} V^* & \longrightarrow & (A \otimes V)^* \cong V^* \otimes A^* \longrightarrow A^* \otimes V^* \\ (x_V)^* & & \tau_{V^*, A^*} \end{array}$$

Describe Δ_{V^*}