Lec 25 Friday Oct 30 10/30/15

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Given bialg $A$ with antipode $S$

Given $A$-module $V$

Recall lin map

$$ev: \quad V^* \otimes V \longrightarrow k \qquad (*)$$
$$f \otimes v \longrightarrow f(v)$$

LEM 53 $*$ is an $A$-module morphism

pf

$$f \otimes v \xrightarrow{ev} f(v)$$

$$a(f \otimes v) \xrightarrow{?} a\, f(v) \qquad \forall a \in A$$

$$a(f\otimes v) = \sum_{(a)} a'f \otimes a''v$$

$$a\,f(v) = \varepsilon(a)\, f(v)$$

$$\begin{aligned}
ev\left(\sum_{(a)} a'f \otimes a''v\right) &= \sum_{(a)} (a'f)(a''v) \\
&= \sum_{(a)} f\left(S(a')(a''v)\right) \\
&= f\Big(\underbrace{\sum_{(a)} S(a')\, a''}_{=\varepsilon(a) 1_A}\, v\Big) \\
&= f\big(\underbrace{\varepsilon(a)}_{\in k}\, v\big) \\
&= \varepsilon(a)\, f(v) \qquad \text{ok}
\end{aligned}$$

$\square$

Given bialgebra $A$ with antipode $S$

Given fin dim'l $A$-module $V$

Recall co-eval map

$$\delta: \quad k \longrightarrow V \otimes V^* \qquad (*)$$
$$1 \longrightarrow \sum_i v_i \otimes v^i$$

$\{v_i\}$ is basis for $V$
$\{v^i\}$ is dual basis for $V^*$

LEM 54 $(*)$ is an $A$-module morphism.

pf From LEM 51 get $A$-module iso

$$\theta: \quad V \otimes V^* \longrightarrow Hom(V, V) = End(V)$$

Factor $(*)$:

$$k \longrightarrow End(V) \xrightarrow{\theta^{-1}} V \otimes V^*$$
$$1 \longrightarrow I$$

↑ show this is $A$-module morphism

$\forall a \in A$

in $A$-module $k$,

$$a\,1 = \varepsilon(a)$$

in $A$-module $\mathrm{End}(V)$, require

$$a\,I \overset{?}{=} \varepsilon(a)\,I$$

By LEM 50

$$a\,I = \sum_{(a)} a'\,I\,S(a'')$$

$$= \underbrace{\sum_{(a)} a'\,S(a'')}_{=\ \varepsilon(a)\,1_A}\,I$$

$$= \varepsilon(a)\,I \qquad \text{ok}$$

$\square$

Given bialg $A$ with antipode $S$

Given $A$-modules $U, V, W$

Recall the composition linear map

$$\text{Comp}: \quad \begin{array}{ccc} \operatorname{Hom}(V,W) \otimes \operatorname{Hom}(U,V) & \longrightarrow & \operatorname{Hom}(U,W) \\ f \otimes g & & f \circ g \end{array} \qquad *$$

$$U \xrightarrow{g} V \xrightarrow{f} W$$

LEM 55 the map $*$ is an $A$-module morphism.

pf I (assume $U, V, W$ fd)

$*$ is the composition of $A$-module morphisms

$$\begin{aligned} \operatorname{Hom}(V,W) \otimes \operatorname{Hom}(U,V) &\to (W \otimes V^*) \otimes (V \otimes U^*) \\ &\to W \otimes (V^* \otimes V) \otimes U^* \\ &\to W \otimes k \otimes U^* \\ &\to W \otimes U^* \\ &\to \operatorname{Hom}(U,W) \end{aligned}$$

pf II



$*$

$$f \otimes g \longrightarrow f \circ g$$

$\forall a \in A$

$$a(f \otimes g) \xrightarrow{?} a(f \circ g)$$

$$a(f \otimes g) = \sum_{(a)} a' f \otimes a'' g = \sum_{(a)} a' f S(a'') \otimes a''' g S(a'''')$$

$$a(f \circ g) = \sum_{(a)} a' f g S(a'') \overset{?}{=}$$

$$\sum_{(a)} a' f S(a'') \otimes a''' g S(a'''') \longrightarrow \sum_{(a)} a' f \underbrace{S(a'') a'''}_{\text{shrink}} g S(a'''')$$

$$= \sum_{(a)} a' f \underbrace{\varepsilon(a'')}_{\in k} 1_A g S(a''') \qquad \left[ \sum_{(b)} S(b') b'' = \varepsilon(b) 1_A \quad \forall b \in A \right]$$

$$= \sum_{(a)} \underbrace{a' \varepsilon(a'')}_{\text{shrink}} f g S(a''')$$

$$= \sum_{(a)} a' f g S(a'') \qquad \left[ \sum_{(b)} b' \varepsilon(b'') = b \quad \forall b \in A \right]$$

ok.

$\square$

Given bialgebra $A$ with antipode $S$

Given $A$-modules

$U, U', V, V'$

Recall linear map

$$\lambda : \quad Hom(U,U') \otimes Hom(V,V') \to Hom(V \otimes U, U' \otimes V')$$

$$f \otimes g \mapsto H$$

$$H(v \otimes u) = f(u) \otimes g(v)$$

Consider when is $\lambda$ an $A$-module morphism

To simplify things, assume both

(i) $U$ or $U'$ is fin dim'l, so $Hom(U,U') \cong U' \otimes U^*$

(ii) $V$ or $V'$ is fin dim'l so $Hom(V,V') \cong V' \otimes V^*$

Now $\lambda$ is the composition

A-module morphism?

$$
\begin{aligned}
\mathrm{Hom}(U,U')\otimes\mathrm{Hom}(V,V') &\xrightarrow{\gamma} (U'\otimes U^*)\otimes(V'\otimes V^*)\\
&\xrightarrow{\gamma} U'\otimes(U^*\otimes V')\otimes V^*\\
&\longrightarrow U'\otimes(V'\otimes U^*)\otimes V^*\\
&\xrightarrow{\gamma} (U'\otimes V')\otimes(U^*\otimes V^*)\\
&\xrightarrow{\gamma} U'\otimes V'\otimes(V\otimes U)^*\\
&\xrightarrow{\gamma} \mathrm{Hom}(V\otimes U,\, U'\otimes V')
\end{aligned}
$$

LEM 56 With the above notation and assumptions, $\lambda$ is an $A$-module morphism provided that

$$\tau_{U^*,V'} : U^*\otimes V' \to V'\otimes U^*$$

is an $A$-module morphism. $\square$

## Comodules

Motivation: Given an algebra $A$ with data $\mu, \eta$

View an $A$-module as a vector space $V$ together with a linear map $\mu_V : A \otimes V \to V$ s.t. these diagrams commute:

$$\begin{tikzcd} A \otimes A \otimes V \arrow[r, "\mu \otimes id"] \arrow[d, "id \otimes \mu_V"'] & A \otimes V \arrow[d, "\mu_V"] \\ A \otimes V \arrow[r, "\mu_V"'] & V \end{tikzcd} \qquad \begin{tikzcd} k \otimes V \arrow[r, "\eta \otimes id"] \arrow[dr, "1 \otimes v \mapsto v"'] & A \otimes V \arrow[d, "\mu_V"] \\ & V \end{tikzcd}$$

Given a coalgebra $C$ with data $\Delta, \varepsilon$

A (left) <u>$C$-comodule</u> is a vector space $V$ together with a linear map $\Delta_V : V \to C \otimes V$ s.t. these diagrams commute:

$$\begin{tikzcd} C \otimes C \otimes V & C \otimes V \arrow[l, "\Delta \otimes id"'] \\ C \otimes V \arrow[u, "id \otimes \Delta_V"] & V \arrow[l, "\Delta_V"] \arrow[u, "\Delta_V"'] \end{tikzcd} \qquad \begin{tikzcd} k \otimes V & C \otimes V \arrow[l, "\varepsilon \otimes id"'] \\ & V \arrow[ul, "1 \otimes v \mapsfrom v"] \arrow[u, "\Delta_V"'] \end{tikzcd}$$

A right $C$-comodule is similarly defined.

Extending the Sweedler notation, write

$$\Delta_V : \quad \begin{array}{ccc} V & \longrightarrow & C \otimes V \\ x & \longmapsto & \sum_{(x)} x_C \otimes x_V \end{array}$$

The commuting diagrams require that for $x \in V$,

$$\sum_{(x)} (x_C)' \otimes (x_C)'' \otimes x_V = \sum_{(x)} x_C \otimes (x_V)_C \otimes (x_V)_V$$

and

$$x = \sum_{(x)} \varepsilon(x_C)\, x_V$$

—— o ——

Given fd algebra $A$ get coalg $A^*$

Given $A$ module $V$.

expect $V^*$ is $A^*$-comodule with action

$$\Delta_{V^*} : \quad V^* \xrightarrow[(\mu_V)^*]{} (A \otimes V)^* \simeq V^* \otimes A^* \xrightarrow[\tau_{V^*, A^*}]{} A^* \otimes V^*$$

Describe $\Delta_{V^*}$