Recall

$$\varepsilon : A \to k$$

is alg morphism

So k becomes an A-module with action

$$A \otimes k \to k$$
$$a \otimes 1 \mapsto \varepsilon(a)$$

More generally, any vector space W becomes an A-module with action

$$A \otimes W \to W$$
$$a \otimes w \mapsto \varepsilon(a) w$$

Given bialg A.

LEM 47 Given A-modules U, V, W

then the linear map

$$\theta: (U \otimes V) \otimes W \to U \otimes (V \otimes W)$$

$$(u \otimes v) \otimes w \to u \otimes (v \otimes w)$$

is an A-module iso.

pf Recall θ is VS iso

Show θ commutes with A action.

For $a \in A$ $u \in U$ $v \in V$ $w \in W$

show

$$\theta\Big(a\big((u \otimes v) \otimes w\big)\Big) \overset{?}{=} a\Big(\theta\big((u \otimes v) \otimes w\big)\Big)$$

$$LHS = \theta\Big(\sum_{(a)} \big(a'(u \otimes v)\big) \otimes a''w \Big)$$

$$= \theta\Big(\sum_{(a)} \big(a'u \otimes a''v\big) \otimes a'''w \Big)$$

$$= \sum_{(a)} a'u \otimes \big(a''v \otimes a'''w\big)$$

$$RHS = a\big(u \otimes (v \otimes w)\big)$$

$$= \sum_{(a)} a'u \otimes a''(v \otimes w)$$

$$= \sum_{(a)} a'u \otimes \big(a''v \otimes a'''w\big) \qquad \text{ok} \qquad \square$$

LEM 48 Given bialg A and A-module V.

Then the linear maps

$$\varphi : V \to k \otimes V \qquad \phi : V \to V \otimes k$$
$$v \to 1 \otimes v \qquad v \to v \otimes 1$$

are A-module isomorphisms

pf Consider φ

φ is vs iso

show φ commutes with A action.

$\forall a \in A \ \forall v \in V$

$$\varphi\, a v \overset{?}{=} a\, \varphi v$$

$$\varphi\, a v = 1 \otimes a v$$

$$a\, \varphi v = a(1 \otimes v) = \sum_{(a)} a' 1 \otimes a'' v = \sum_{(a)} \varepsilon(a') 1 \otimes a'' v = 1 \otimes \sum_{(a)} \varepsilon(a') a'' v = 1 \otimes a v$$

$$a = \sum_{(a)} \varepsilon(a') a''$$

ok

$\square$

Given bialg A

Given A-modules U, V

$\rightarrow$ A-modules $U \otimes V$, $V \otimes U$

Recall vs iso

$$T_{U,V} \quad U \otimes V \longrightarrow V \otimes U \qquad *$$
$$u \otimes v \longrightarrow v \otimes u$$

Then $*$ is A module iso provided that Δ is cocommutive

check: $\forall a \in A$

$$\Delta(a) = \sum_{(a)} a' \otimes a'' = \sum_{(a)} a'' \otimes a'$$

$\forall u \in U \quad \forall v \in V$

$$u \otimes v \xrightarrow{T_{U,V}} v \otimes u$$

$$a(u \otimes v) \xrightarrow[?]{} a(v \otimes u)$$

$$a(u \otimes v) = \sum_{(a)} a'u \otimes a''v = \sum_{(a)} a''u \otimes a'v$$

ok

$$a(v \otimes u) = \sum_{(a)} a'v \otimes a''u$$

$\square$

We now bring on antipode S

LEM 49 Given bialgebra A with antipode S

Given A-module V.

Then V^* becomes an A module with action

$$A \otimes V^* \to V^*$$
$$a \otimes f \to af$$

$$af(v) = f(S(a)v) \qquad \forall v \in V$$

pf check: $\forall a, b \in A \quad \forall f \in V^*$

$$(ab)f \overset{?}{=} a(bf)$$

Apply each side to $v \in V$:

$$((ab)f)(v) = f(S(ab)v) \qquad S(ab) = S(b)S(a)$$
$$= f((S(b)S(a))v)$$

$$a(bf)(v) = bf(S(a)v)$$
$$= f(S(b)(S(a)v))$$
$$= f((S(b)S(a))v) \qquad \text{ok}$$

check $1_A f \overset{?}{=} f$

For $v \in V$ $\quad (1_A f)(v) = f(\underbrace{S(1_A)}_{=1_A}v) = f(v) \qquad \text{ok}$

$\square$

Given bialgebra A with antipode S

Given f.d. A-modules U, V

$\longrightarrow$ A-module U^*

$\longrightarrow$ A-module $V \otimes U^*$

Recall vs iso

$$V \otimes U^* \longrightarrow \mathrm{Hom}(U, V)$$

$\mathrm{Hom}(U,V)$ inherits A-module str.

<u>Describe A-module $\mathrm{Hom}(U,V)$:</u>

vs iso

$$\begin{aligned} V \otimes U^* &\longrightarrow \mathrm{Hom}(U,V) \\ v \otimes f &\longrightarrow F \end{aligned}$$

$$F(u) = f(u)v \qquad \forall u \in U$$

$\forall a \in A$ find

$$aF \in \mathrm{Hom}(U,V)$$

$\forall u \in U$ find

$$(aF)(u)$$

$$v \otimes f \xrightarrow{\theta} F$$

$$a(v \otimes f) \xrightarrow{\theta} aF$$

$$\| $$

$$\sum_{(a)} a'v \otimes a''f$$

$$(aF)(u) = \sum_{(a)} \theta(a'v \otimes a''f)(u)$$

$$= \sum_{(a)} (a''f)(u)\, a'v$$

$$= \sum_{(a)} \underbrace{f(S(a'')u)}_{k} \; a'v$$

$$= \sum_{(a)} a' f(S(a'')u)\, v$$

$$= \sum_{(a)} a' F(S(a'')u)$$

$$= \left(\sum_{(a)} a' F S(a'')\right)(u)$$

So $$aF = \sum_{(a)} a' F S(a'')$$

$$U \xrightarrow{S(a'')} U \xrightarrow{F} V \xrightarrow{a'} V$$

LEM50 Given bialgebra A with antipode S

Given A-modules U, V

Then $Hom(U,V)$ becomes an A-module such that

$$aF = \sum_{(a)} a' F S(a'') \qquad \forall a \in A \quad \forall F \in Hom(U,V)$$

pf check $\forall a, b \in A$

$$(ab)F \stackrel{?}{=} a(bF)$$

$$(ab)F = \sum_{(a)} \sum_{(b)} a'b' F S(a''b'')$$

$$a(bF) = a\left(\sum_{(b)} b' F S(b'') \right)$$

$$= \sum_{(a)} \sum_{(b)} a'b' F \underbrace{S(b'') S(a'')}_{\substack{\shortparallel \\ S(a''b'')}} \qquad \text{ok}$$

check $\quad 1_A F \stackrel{?}{=} F \qquad \Delta(1_A) = 1_A \otimes 1_A$

$$1_A F = 1_A F \underbrace{S(1_A)}_{\substack{\shortparallel \\ 1_A}} = F \qquad \text{ok}$$

$\square$

<u>Note</u> Referring to LEM 50

if we set $V = k$ so $Hom(U, V) = U^*$

we get the A-module U^* from LEM 49

(ex)

LEM 51 Given bialg A with antipode S

Given A-modules U, V.

Then the lin map

$$V \otimes U^* \longrightarrow Hom(U, V)$$

$$v \otimes f \rightarrow F \qquad F(u) = f(u)v$$

is an A-module morphism. $\square$

pf From how we defined the A-action on $Hom(U, V)$

Given bialg A with antipode S

Given A-modules U, V.

Get A-modules

$U \otimes V$, $(U \otimes V)^*$, U^*, V^*, $V^* \otimes U^*$

Recall linear map

$$V^* \otimes U^* \longrightarrow (U \otimes V)^* \qquad (*)$$

$$g \otimes f \longrightarrow H$$

$$H(u \otimes v) = f(u) g(v)$$

LEM 52 $(*)$ is an A-module morphism.

pf

$$g \otimes f \xrightarrow{*} H$$

$$a(g \otimes f) \underset{?}{\xrightarrow{*}} aH \qquad \forall a \in A$$

$$\| $$

$$\sum_{(a)} a' g \otimes a'' f$$

Require $\forall\, u \in U \quad \forall v \in V$

$$(aH)(u \otimes v) \overset{?}{=} \sum_{(a)} \underbrace{(a''f)(u)}_{\substack{\shortparallel \\ f(S(a'')u)}} \; \underbrace{(a'g)(v)}_{\substack{\shortparallel \\ g(S(a')v)}}$$

$$(aH)(u \otimes v) = H(\underbrace{S(a)}_{\substack{\shortparallel \\ b}}(u \otimes v))$$

$$= H\left(\sum_{(b)} b'u \otimes b''v\right)$$

$$\Delta(b) = \Delta S(a)$$
$$= (S \otimes S)\,\Delta^{op}(a)$$
$$= (S \otimes S) \sum_{(a)} a'' \otimes a'$$
$$= \sum_{(a)} S(a'') \otimes S(a')$$

$$= H\left(\sum_{(a)} S(a'')u \otimes S(a')v\right)$$

$$= \sum_{(a)} f(S(a'')u)\; g(S(a')v)$$

OK

$\square$