Lec 23 Monday Oct 26 10/26/15
1

LEM 40 Given a Hopf algebra H with antipode S.

TFAE:

(i) $S^2 = I$

(ii) $\forall h \in H$,
$$\varepsilon(h) 1_H = \sum_{(h)} h'' S(h')$$

(iii) $\forall h \in H$,
$$\varepsilon(h) 1_H = \sum_{(h)} S(h'') h'$$

pf (i) → (ii) By *

(ii) → (i) $\forall h \in H$
$$\varepsilon(h) 1_H = \sum_{(h)} h'' S(h')$$

Apply S:
$$\varepsilon(h) 1_H = \sum_{(h)} S^2(h') S(h'')$$

So $\mathbb{1} = S^2 \star S$

So $$\underbrace{\mathbb{1} \star I}_{= I} = \underbrace{S^2 \star \underbrace{S \star I}_{= \mathbb{1}}}_{= S^2}$$

(i) ⟺ (iii) Sim. □

COR 41 Given a Hopf algebra H with antipode S

Assume H is commutative or cocommutative

then $S^2 = I$.

pf Use LEM 40 $\square$

LEM 42 Given a Hopf algebra H with antipode S. Then $\forall\, \tilde{S} \in End(H)$

TFAE:

(i) $S\tilde{S} = \tilde{S}S = I$

(ii) H^{op} is Hopf alg with antipode $\tilde{S}$

(iii) H^{cop} is Hopf alg with antipode $\tilde{S}$

pf (i) $\to$ (ii) We have, $\forall h \in H$

$$\varepsilon(h)1_H = \sum_{(h)} S(h')h'' = \sum_{(h)} h' S(h'')$$

Apply $\tilde{S}$:

$$\varepsilon(h)\,1_H = \sum_{(h)} \tilde{S}(h'')\,\tilde{S}S(h') = \sum_{(h)} \tilde{S}S(h'')\,\tilde{S}(h')$$

$$= \sum_{(h)} \tilde{S}(h'')\,h' = \sum_{(h)} h''\,\tilde{S}(h')$$

Result follows.

(ii) $\to$ (i) $\quad \forall h \in H,$

$$\varepsilon(h) 1_H = \sum_{(h)} h'' \tilde{S}(h')$$

Apply S:

$$\varepsilon(h) 1_H = \sum_{(h)} S\tilde{S}(h') \, S(h'')$$

So

$$\mathbb{1} = S\tilde{S} \star S$$

So

$$\underbrace{\mathbb{1} \star I}_{= I} = \underbrace{S\tilde{S} \star \underbrace{S \star I}_{= \mathbb{1}}}_{= S\tilde{S}}$$

So

$$I = S\tilde{S}$$

Interchanging roles of H, H^{op} get

$$I = \tilde{S} S$$

(i) $\leftrightarrow$ (iii) Similar. $\square$

COR 43 Given a Hopf algebra H with antipode S. Assume $S: H \to H$ is a bijection. Then:

(i) S is a Hopf algebra isomorphism $H \to H^{op,cop}$

(ii) ... $H^{op,cop} \to H$

(iii) ... $H^{op} \to H^{cop}$

(iv) ... $H^{cop} \to H^{op}$

pf (i), (ii) By Cor 38.

(iii), (iv) By Cor 38 and LEM 42,

S^{-1} is Hopf alg iso $H^{op} \leftrightarrow H^{cop}$.

But then S is Hopf alg iso $H^{op} \leftrightarrow H^{cop}$ $\square$

Given bialg H

For $x \in H$ call x <u>group like</u> whenever

$$\Delta(x) = x \otimes x, \qquad \varepsilon(x) = 1$$

let G = set of grouplike elements of H.

obs $1_H \in G$

$xy \in G \quad \forall\ x, y \in G$

Now assume H is Hopf alg with antipode S

$\forall\ x \in G$

$$\Delta(x) = x \otimes x$$

So
$$\underset{\overset{\shortparallel}{1}}{\varepsilon(x)}\, 1_H = S(x)\, x = x\, S(x)$$

$S(x)$ is the inverse of x in alg H.

So $\Delta(S(x)) = S(x) \otimes S(x)$ (Since Δ is alg morphism)

$S(x) \in G$

G is a group

$\square$

Given bialgebra H.

Next consider how to recognize an antipode in H.

LEM 44 Given bialg H.

Given a generating set X for the algebra H.

Given $S \in End(H)$ s.t.

(i) $S(1_H) = 1_H$

(ii) $S(xy) = S(y)S(x) \quad \forall x,y \in H$

(iii) $\forall h \in X$

$$\varepsilon(h)1_H = \sum_{(h)} S(h')h'' = \sum_{(h)} h' S(h'')$$

Then S is an antipode for H.

pf (Similar to pf of Ex 32)

Define

$\tilde{H}$ = set of elements $h \in H$ s.t.

$$\varepsilon(h)1_H = \sum_{(h)} S(h')h'' = \sum_{(h)} h' S(h'')$$

Show $\tilde{H} = H$

By constr and since $\Delta(1_H) = 1_H \otimes 1_H$,

$$1_H \in \tilde{H}$$

By assumption,

$$X \subseteq \tilde{H}$$

For $x, y \in \tilde{H}$ show $xy \in \tilde{H}$:

$$\varepsilon(x) 1_H = \sum_{(x)} S(x')x'' = \sum_{(x)} x'S(x'')$$

$$\varepsilon(y) 1_H = \sum_{(y)} S(y')y'' = \sum_{(y)} y'S(y'')$$

$$\Delta(xy) = \sum_{(x)} \sum_{(y)} x'y' \otimes x''y''$$

Require

$$\underbrace{\varepsilon(xy) 1_H}_{= \varepsilon(x)\varepsilon(y) 1_H} \overset{?}{=} \sum_{(x)} \sum_{(y)} \underbrace{S(x'y')}_{= S(y')S(x')} x''y''$$

$$\text{RHS} = \sum_{(y)} \sum_{(x)} S(y') S(x') x'' y''$$

$$= \sum_{(y)} S(y') \Big(\underbrace{\sum_{(x)} S(x') x''}_{= \varepsilon(x) 1_H} \Big) y''$$

$$= \varepsilon(x) \underbrace{\sum_{(y)} S(y') y''}_{= \varepsilon(y) 1_H}$$

$$= \varepsilon(x) \varepsilon(y) 1_H \checkmark$$

$\square$

Recall bialgebra $M(2)$

<u>algebra str</u>:

$$M(2) = k[x_{ij} \mid 1 \le i,j \le 2] \qquad \text{poly algebra}$$

<u>coalg str</u>:

$$\Delta : \quad M(2) \longrightarrow M(2) \otimes M(2) \qquad \text{alg morph}$$

$$x_{ij} \longrightarrow \sum_{l=1}^{2} x_{il} \otimes x_{lj}$$

$$\varepsilon : \quad M(2) \longrightarrow k \qquad \text{alg morph}$$

$$x_{ij} \longrightarrow \delta_{ij}$$

$M(2)$ is commutative but not cocommutative

Define $\quad d = x_{11} x_{22} - x_{12} x_{21}$

Get $\quad \Delta(d) = d \otimes d$

$\qquad \varepsilon(d) = 1$

In $M(2)$, search for antipode S:

Require: $\forall h \in M(2)$,

$$\varepsilon(h) 1_H = \sum_{(h)} S(h') h'' = \sum_{(h)} h' S(h'')$$

Take $h = x_{ij}$ $(1 \le i, j \le 2)$

Requirement becomes

$$\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \begin{pmatrix} S(x_{11}) & S(x_{12}) \\ S(x_{21}) & S(x_{22}) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} S(x_{11}) & S(x_{12}) \\ S(x_{21}) & S(x_{22}) \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

take det:

Require $d\, S(d) = 1$

Require d is invertible in $M(2)$ cnt

S does not exist in $M(2)$.

Recall bialgebra $GL(2)$

algebra str.

algebra $GL(2)$ is commutative with gens

x_{ij} $(1 \le i,j \le 2)$, t

and relation

$$(x_{11}x_{22} - x_{12}x_{21})\, t = 1$$

Coalg str.

$$\Delta : \quad GL(2) \longrightarrow GL(2) \otimes GL(2)$$

$$x_{ij} \longrightarrow \sum_{l=1}^{2} x_{il} \otimes x_{lj} \qquad \text{alg morph}$$

$$t \longrightarrow t \otimes t$$

$$\varepsilon : \quad GL(2) \longrightarrow k$$

$$x_{ij} \longrightarrow \delta_{ij} \qquad \text{alg morph}$$

$$t \longrightarrow 1$$

$GL(2)$ is commutative, not cocommutative.

Define
$$d = x_{11}x_{22} - x_{12}x_{21} \qquad \in GL(2)$$

So
$$dt = 1,$$
$$\Delta(d) = d \otimes d,$$
$$\varepsilon(d) = 1.$$

In $GL(2)$, search for antipode S

As in $M(2)$ require
$$\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}, \quad \begin{pmatrix} S(x_{11}) & S(x_{12}) \\ S(x_{21}) & S(x_{22}) \end{pmatrix} \qquad \text{inverses}$$

So
$$\begin{pmatrix} S(x_{11}) & S(x_{12}) \\ S(x_{21}) & S(x_{22}) \end{pmatrix} = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}^{-1}$$
$$= \cancel{d}\, t \begin{pmatrix} x_{22} & -x_{12} \\ -x_{21} & x_{11} \end{pmatrix}$$

Also since
$$\Delta(t) = t \otimes t,$$

require
$$\underset{\substack{\parallel \\ 1}}{\varepsilon(t)} = S(t)\,t = t\,S(t)$$
$$S(t) = t^{-1} = d$$

S must be an alg morphism

(since $S(xy) = S(y)S(x)$ $\forall x, y$ and $GL(2)$ is commutative)

LEM 45 The bialg $GL(2)$ has an antipode S that sends

$$
\begin{aligned}
x_{11} &\to t\, x_{22} \\
x_{12} &\to -t\, x_{12} \\
x_{21} &\to -t\, x_{21} \\
x_{22} &\to t\, x_{11} \\
t &\to x_{11}x_{22} - x_{12}x_{21}
\end{aligned}
$$

(alg morphism)

Moreover $S^2 = I$.

pf Alg morphism S exists. To show S is an antipode use LEM 44.

One checks
$$S^2(x_{ij}) = x_{ij} \qquad 1 \le i, j \le 2$$
$$S^2(t) = t$$

So $S^2 = I$. $\square$

Recall bialg $SL(2)$

Alg str

alg $SL(2)$ is commutative with gens

$$x_{ij} \quad (1 \le i,j \le 2)$$

and rel

$$x_{11} x_{22} - x_{12} x_{21} = 1$$

Coalg str

$$\Delta : \begin{array}{ccc} SL(2) & \longrightarrow & SL(2) \otimes SL(2) \\ x_{ij} & \longrightarrow & \sum_{l} x_{il} \otimes x_{lj} \end{array} \qquad 1 \le i,j \le 2$$

alg morph

$$\varepsilon : \begin{array}{ccc} SL(2) & \longrightarrow & k \\ x_{ij} & \longrightarrow & \delta_{ij} \end{array} \qquad 1 \le i,j \le 2$$

alg morph.

$SL(2)$ is commutative, not cocommutative.

LEM 46 The bialgebra $SL(2)$ has an antipode S that sends

$$X_{11} \longrightarrow X_{22}$$
$$X_{12} \longrightarrow -X_{12}$$
$$X_{21} \longrightarrow -X_{21}$$
$$X_{22} \longrightarrow X_{11}$$

(alg morphism)

Moreover $S^2 = I$

pf Alg morphism S exists. To show S is antipode use LEM 44:

For $h = X_{ij}$ check

$$\varepsilon(h)\, 1_H \overset{?}{=} \sum_{(h)} S(h')\, h'' \overset{?}{=} \sum_{(h)} h' S(h'') \qquad H = SL(2)$$

$\uparrow$

X_{11} $\quad 1 \overset{?}{=} \underset{\overset{\shortparallel}{X_{22}}}{S(X_{11})}\, X_{11} + \underset{\overset{\shortparallel}{-X_{12}}}{S(X_{12})}\, X_{21}$ ✓

X_{12} $\quad 0 \overset{?}{=} \underset{\overset{\shortparallel}{X_{22}}}{S(X_{11})}\, X_{12} + \underset{\overset{\shortparallel}{-X_{12}}}{S(X_{12})}\, X_{22}$ ✓

X_{21} $\quad 0 \overset{?}{=} \underset{\overset{\shortparallel}{-X_{21}}}{S(X_{21})}\, X_{11} + \underset{\overset{\shortparallel}{X_{11}}}{S(X_{22})}\, X_{21}$ ✓

X_{22} $\quad 1 \overset{?}{=} \underset{\overset{\shortparallel}{-X_{21}}}{S(X_{21})}\, X_{12} + \underset{\overset{\shortparallel}{X_{11}}}{S(X_{22})}\, X_{22}$ ✓

So S is antipode. One checks $S^2 = I$

$\square$

Modules over a Hopf algebra

Given algebras A, B

yields algebra $A \otimes B$

Given A-module U with action

$$m_U : A \otimes U \to U$$

Given B-module V with action

$$m_V : B \otimes V \to V$$

then $U \otimes V$ becomes an $(A \otimes B)$-module with action

$$(A \otimes B) \otimes (U \otimes V) \xrightarrow[id \otimes T_{B,U} \otimes id]{} A \otimes U \otimes B \otimes V \xrightarrow[m_U \otimes m_V]{} U \otimes V$$

So for

$a \in A$, $b \in B$, $u \in U$, $v \in V$

$a \otimes b$ sends $u \otimes v \to au \otimes bv$

Also, given an algebra morphism $\varphi : A \to B$

the B-module V becomes an A-module with action

$$A \otimes V \xrightarrow[\varphi \otimes id]{} B \otimes V \xrightarrow[m_V]{} V$$

"pull the B-action back to A via φ"

Next assume

$$A = B \quad \text{is a bialgebra}$$

Recall

$$\Delta : A \longrightarrow A \otimes A$$

is algebra morphism

Given A-modules U, V

$$\longrightarrow \quad (A \otimes A)\text{-module } U \otimes V$$

$$\xrightarrow{\text{pullback via } \Delta} \quad A\text{-module } U \otimes V$$

For $a \in A$, $u \in U$, $v \in V$

$$a(u \otimes v) = \sum_{(a)} a'u \otimes a''v$$