Lec 22 Fri Oct 23

(i) $\forall h \in H$ chase h around diagram.

$$\begin{array}{ccc} h & \longrightarrow & S(h) = \ell \\ & & \downarrow \\ & & \sum_{(\ell)} \ell'' \otimes \ell' \\ \downarrow & & \| ? \\ \sum_{(h)} h' \otimes h'' & \longrightarrow & \sum_{(h)} S(h') \otimes S(h'') \end{array}$$

show

$$\sum_{(\ell)} \ell'' \otimes \ell' = \sum_{(h)} S(h') \otimes S(h'') \qquad *$$

Give two proofs of $*$. The 1st is more natural but requires H to be f.d.

pf I Apply L36 to Hopf alg H^* with antipode S^*.

$\forall f, g \in H^*$

$$S^*(fg) = S^*(g) S^*(f)$$

Each side is in H^*.

Apply each side to $h \in H$

$$S^*(fg)(h) = \langle S^*(fg), h \rangle$$

$$= \langle fg, \underbrace{S(h)}_{\ell} \rangle$$

$$= (fg)(\ell)$$

$$= \sum_{(\ell)} \underset{k}{f(\ell')}\, \underset{k}{g(\ell'')}$$

$$= \sum_{(\ell)} g(\ell'')\, f(\ell')$$

$$= \mu_k \left(g \otimes f \left(\sum_{(\ell)} \ell'' \otimes \ell' \right) \right)$$

$$\left(S^*(g)\, S^*(f) \right)(h) = \sum_{(h)} \underbrace{S^*(g)(h')}_{\substack{\shortparallel \\ g(S(h'))}} \; \underbrace{S^*(f)(h'')}_{\substack{\shortparallel \\ f(S(h''))}}$$

$$= \sum_{(h)} g(S(h'))\, f(S(h''))$$

$$= \mu_K \left(g \otimes f \left(\sum_{(h)} S(h') \otimes S(h'') \right) \right)$$

Define

$$\theta = \sum_{(\ell)} \ell'' \otimes \ell' - \sum_{(h)} S(h') \otimes S(h'')$$

show $\theta = 0$

Have

$$\mu_K \left(g \otimes f (\theta) \right) = 0 \qquad \forall\ f, g \in H^*$$

Pick a basis $\{h_i\}$ for H

Dual basis $\{h^i\}$ for H^*.

Write

$$\theta = \sum_{i,j} \alpha_{ij}\, h_i \otimes h_j \qquad \alpha_{ij} \in K$$

$$0 = \mu_K\, g \otimes f \quad \theta$$

$$= \sum_{i,j} \alpha_{ij}\, g(h_i)\, f(h_j) \qquad \forall\, f, g \in H^*$$

For $\forall r, s$ show $\alpha_{rs} = 0$

Take $g = h^r \qquad f = h^s$

$$0 = \sum_{i,j} \alpha_{ij} \underbrace{h^r(h_i)}_{=\delta_{ir}} \underbrace{h^s(h_j)}_{=\delta_{js}}$$

$$= \alpha_{rs} \qquad \checkmark$$

Pf II:

Consider Convolution algebra $Hom(H, H \otimes H)$.

Elements:

$$\text{ident}: \quad \begin{aligned} H &\longrightarrow H \otimes H \\ h &\longmapsto \varepsilon(h)\, 1_H \otimes 1_H \end{aligned}$$

$$\Delta: \quad H \longrightarrow H \otimes H$$

$$H \xrightarrow{S} H \xrightarrow{\Delta} H \otimes H$$

show these are equal.
show they are both
the inverse of Δ

$$H \xrightarrow{\Delta^{op}} H \otimes H \xrightarrow{S \otimes S} H \otimes H$$

show $\Delta \star (\Delta \circ S) = \text{ident}$

$\forall h \in H$

$$\left(\Delta \star (\Delta \circ S)\right)(h)$$

$$= \sum_{(h)} \Delta(h') \, \Delta(S(h''))$$

$$= \Delta\left(\sum_{(h)} h' S(h'')\right) \qquad [\Delta \text{ is alg morph}]$$

$$= \Delta\left(\varepsilon(h) 1_H\right)$$

$$= \varepsilon(h) 1_{H \otimes H}$$

$$= \text{ident}(h)$$ ✓

show $(\Delta \circ S) \star \Delta = \text{ident}$

(sim)

Show

$$\Delta \star (S \otimes S \circ \Delta^{op}) = \text{ident.}$$

$\forall h \in H$,

$$\left(\Delta \star (S \otimes S \circ \Delta^{op})\right)(h)$$

$$= \sum_{(h)} \Delta(h') \, (S \otimes S \circ \Delta^{op})(h'')$$

$$= \mu\,(1 \otimes (S \otimes S)) \sum_{(h)} \Delta(h') \otimes \Delta^{op}(h'')$$

$$= \mu\,(1 \otimes (S \otimes S)) \sum_{(h)} h' \otimes h'' \otimes h'''' \otimes h'''$$

$$= \sum_{(h)} (h' \otimes h'')\,(S(h'''') \otimes S(h'''))$$

$$= \sum_{(h)} h' S(h'''') \otimes h'' S(h''')$$

$$= \sum_{(h)} h' S(h''') \otimes \varepsilon(h'') 1_H$$

$$= \left(\sum_{(h)} h' S(h''') \varepsilon(h'')\right) \otimes 1_H$$

and

$$\sum_{(h)} h' \, S(h''') \, \varepsilon(h'')$$

$$= \sum_{(h)} h' \, \varepsilon(h'') \, S(h''')$$

$$\left[\begin{array}{l} \text{since} \\ h = \sum_{(h)} \varepsilon(h') h'' \\ S(h) = \sum_{(h)} \varepsilon(h') S(h'') \end{array} \right]$$

$$= \sum_{(h)} h' \, S(h'')$$

$$= \varepsilon(h) \, 1_H$$

ok

show

$$(S \otimes S \circ \Delta^{op}) \star \Delta = \text{ident}$$

(sim).

Pf II completed

(ii) $\forall h \in H$ show

$$\varepsilon(S(h)) = \varepsilon(h)$$

We have

$$h = \sum_{(h)} \varepsilon(h') h''$$

$$S(h) = \sum_{(h)} \varepsilon(h') S(h'')$$

$$\varepsilon(S(h)) = \sum_{(h)} \varepsilon(h') \varepsilon(S(h''))$$

Also

$$\varepsilon(h) 1_H = \sum_{(h)} h' S(h'')$$

Apply ε

$$\varepsilon(h) \underset{1}{\underset{\shortparallel}{\varepsilon(1_H)}} = \sum_{(h)} \varepsilon(h') \varepsilon(S(h'')) = \varepsilon(S(h))$$

✓

COR 38 Given a Hopf alg H with antipode S.

(i) The bialgebra $H^{op,cop}$ is a Hopf algebra with antipode S

(ii) S is a Hopf algebra morphism $H \to H^{op,cop}$
$H^{op,cop} \to H$

(iii) ...

pf (i) Since the convolution algebras for H, $H^{op,cop}$ are opposite.

(ii) By LEM 36, 37.

(iii) By (i), (ii) $\square$

COR 39 Given a Hopf alg H with antipode S, $S^2 : H \to H$ is a Hopf alg morphism.

— o —

Given a Hopf alg H with antipode S.

$\forall h \in H$

$$\varepsilon(h) 1_H = \sum_{(h)} h' S(h'') = \sum_{(h)} S(h') h''$$

Apply S and use $S(1_H) = 1_H$:

$$\varepsilon(h) 1_H = \sum_{(h)} S^2(h'') S(h') = \sum_{(h)} S(h'') S^2(h') \qquad *$$

Next consider, when is $S^2 = I$?

LEM 40 Given a Hopf algebra with antipode S.

TFAE:

(i) $S^2 = I$

(ii) $\forall h \in H$,

$$\varepsilon(h) 1_H = \sum_{(h)} h'' S(h')$$

(iii) $\forall h \in H$,

$$\varepsilon(h) 1_H = \sum_{(h)} S(h'') h'$$

pf (i) → (ii) By *

(ii) → (i) $\forall h \in H$

$$\varepsilon(h) 1_H = \sum_{(h)} h'' S(h')$$

Apply S:

$$\varepsilon(h) 1_H = \sum_{(h)} S^2(h') S(h'')$$

So $\mathbb{1} = S^2 \star S$

So $\underbrace{\mathbb{1} \star I}_{= I} = \underbrace{S^2 \star \underbrace{S \star I}_{= \mathbb{1}}}_{= S^2}$

(i) ⟺ (iii) Sim. $\square$