

Lec 21

Wed Oct 21

10/21/15

Ex 33

Referring to Ex 32,

1

Let  $J = 2$ -sided ideal of  $T(V) = H$

generated by

$$uv - vu$$

$$u, v \in V.$$

then

(i)  $J$  is a coideal of  $H$

(ii) Antipode  $S$  satisfies  $S(J) \subseteq J$

[so sym algebra  $S(V) = H/J$  inherits Hopf algebra structure from  $H$ ]

pf (i) check

$$\Delta(J) \stackrel{?}{\subseteq} J \otimes H + H \otimes J$$

\*

For  $u, v \in V$

$$\Delta(uv - vu) = (uv - vu) \otimes 1 + 1 \otimes (uv - vu) \in J \otimes H + H \otimes J$$

\* follows since  $J \otimes H + H \otimes J$  is a 2-sided ideal

of  $H \otimes H$

check

$\varepsilon(J) = 0$  ?

\*\*

For  $u, v \in V$ ,

$$\varepsilon(uv - vu) = \varepsilon(u) \varepsilon(v) - \varepsilon(v) \varepsilon(u)$$

$\quad \quad \quad \begin{matrix} u & u & u & u \\ | & | & | & | \end{matrix}$

$= 0 \quad \checkmark$

\*\* follows since  $\varepsilon$  is alg morphism

(ii) USE the  $S$  action from Ex 32

□

Referring to Ex 32, 33 describe the

Hopf algebra  $S(V)$ ,

For notational convenience assume  $\dim(V) < \infty$   
 $\mathbb{K}_n$

Fix a basis  $\{v_i\}_{i=1}^n$  for  $V$

Given commuting indets  $\{x_i\}_{i=1}^n$

Recall algebra iso

$$\begin{array}{lcl}
 S(V) & \longrightarrow & \mathbb{K}[x_1, \dots, x_n] \\
 v_i & \longrightarrow & x_i \quad \text{is an}
 \end{array}$$

$\mathbb{K}[x_1, \dots, x_n]$  inherits Hopf alg str from  $S(V)$

We have algebra morphisms

$$\begin{array}{lcl}
 \Delta : & \mathbb{K}[x_1, \dots, x_n] & \longrightarrow \mathbb{K}[x_1, \dots, x_n] \otimes \mathbb{K}[x_1, \dots, x_n] \\
 & x_i & \longrightarrow x_i \otimes 1 + 1 \otimes x_i \quad \text{is an}
 \end{array}$$

$$\begin{array}{lcl}
 \varepsilon : & \mathbb{K}[x_1, \dots, x_n] & \longrightarrow \mathbb{K} \\
 & x_i & \longrightarrow 0 \quad \text{is an}
 \end{array}$$

$$\begin{array}{lcl}
 S : & \mathbb{K}[x_1, \dots, x_n] & \longrightarrow \mathbb{K}[x_1^{-1}, \dots, x_n^{-1}] \\
 & x_i & \longrightarrow -x_i \quad \text{is an}
 \end{array}$$



Given f.d. Hopf algebra  $H$  with antipode  $S$ ,

Recall

$$\begin{aligned} \text{alg } H &\rightarrow \text{coalg } H^* && \Rightarrow \text{bialg } H^* \\ \text{coalg } H &\rightarrow \text{alg } H^* \end{aligned}$$

LEM 34 Above  $\text{bialg } H^*$  is Hopf algebra  
with antipode  $S^*$ .

pf

Given:

$$S \in \text{End}(H)$$

$$\forall h \in H$$

$$\sum_{(h)} \varepsilon(h') 1_H = \sum_{(h)} h' S(h'') = \sum_{(h)} h' S(h'')$$

Recall bialg  $H^*$ :Coproduct:  $\forall f \in H^*$ 

$$f(h_1, h_2) = \sum_{(f)} f'(h_1) f''(h_2) \quad \forall h_1, h_2 \in H$$

product:  $\forall f_1, f_2 \in H^*$ 

$$(f_1, f_2)(h) = \sum_{(h)} f_1(h') f_2(h'') \quad \forall h \in H$$

covast  $\forall f \in H^*$ 

$$\varepsilon^*(f) = f(h)$$

unit

$$1_{H^*} = \varepsilon$$

Requires:

$$\forall f \in H^*$$

$$\varepsilon^*(f) \mathbb{1}_{H^*} \stackrel{?}{=} \sum_{(f)} S^*(f') f'' \stackrel{?}{=} \sum_{(f)} f' S^*(f'')$$

↑

Both sides in  $H^*$ . Apply each side to  $h \in H$ 

LHS:  $\varepsilon^*(f) \mathbb{1}_{H^*}(h) = f(h) \varepsilon(h)$

RHS:

$$\sum_{(f)} (S^*(f') f'')(h) = \sum_{(f)} \sum_{(h')} \underbrace{(S^*(f'))(h')}_{\substack{= \\ \langle S^*(f'), h' \rangle \\ = \\ \langle f', S(h') \rangle \\ = \\ f'(S(h'))}}$$

$$= \sum_{(h')} \sum_{(f)} \underbrace{f'(S(h')) f''(h'')}_{=}$$

$$f(S(h') h'')$$

$$= f \left( \sum_{(h')} S(h') h'' \right)$$

$$= \varepsilon(h) \mathbb{1}_H f(h)$$

✓

□

Here is a handy principle

LEM 35 Given a Hopf algebra  $H$ .

Then for  $T \in \text{End}(H)$  TFAE:

(i)  $\forall h \in H$

$$0 = \sum_{(h)} T(h') h''$$

(ii)  $\forall h \in H$

$$0 = \sum_{(h)} h' T(h'')$$

(iii)  $T = 0$

pf Recall the antipode  $S \in \text{End}(H)$  satisfies

$$S \star I = \mathbb{1} = I \star S$$

(i)  $\rightarrow$  (iii)  $\forall h \in H$

$$(T \star I)(h) = \sum_{(h)} T(h') h'' = 0$$

so  $T \star I = 0$

so  $T \star \underbrace{I \star S}_{\mathbb{1}} = 0$

$\underbrace{\hspace{2cm}}_T$  ✓

Other assertions are clear.

□

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LEM 36 Given Hopf alg  $H$  with antipode  $S$ .

Then

$$(i) \quad S(xy) = S(y)S(x) \quad \forall x, y \in H$$

$$(ii) \quad S(1_H) = 1_H$$

In other words

$S$  is an algebra morphism from the algebra  $H$  to the algebra  $H^{\text{op}}$

pf (i) Define a bilinear map

$$\theta : H \times H \rightarrow H$$

$$x, y \rightarrow S(xy) - S(y)S(x)$$

show  $\theta = 0$

Recall

$$\varepsilon(x) 1_H = \sum_{(x)} S(x') x''$$

$$\varepsilon(y) 1_H = \sum_{(y)} S(y') y''$$

$$\Delta(xy) = \sum_{(x)} \sum_{(y)} x' y' \otimes x'' y''$$

So

$$\varepsilon(xy) 1_H = \sum_{(x)} \sum_{(y)} S(x' y') x'' y'' \quad *$$



Also

$$\begin{aligned}
 E(xy) | H &= E(x) E(y) | H \\
 &= E(x) \sum_{(y)} S(y') y'' \\
 &= \sum_{(y)} S(y') \left( \sum_{(x)} S(x') x'' \right) y'' \\
 &= \sum_{(x)} \sum_{(y)} S(y') S(x') x'' y''
 \end{aligned}$$

\*\*

$B_1$  \*\* \*\*

$$\begin{aligned}
 0 &= \sum_{(x)} \sum_{(y)} \left( S(x'y') - S(y')S(x') \right) x'' y'' \\
 &= \sum_{(x)} \sum_{(y)} \theta(x'y') x'' y''
 \end{aligned}$$

$\forall x \in H$  Define

$$\begin{aligned}
 T_x : H &\longrightarrow H \\
 z &\longrightarrow \sum_{(x)} \theta(x'z) x''
 \end{aligned}$$

Obs  $T_x \in \text{End}(H)$

Obs

$$\begin{aligned} \sum_{(y)} T_x(y') y'' &= \sum_{(y)} \sum_{(x)} \theta(x', y') x'' y'' \\ &= \sum_{(x)} \sum_{(y)} \theta(x', y') x'' y'' \\ &= 0 \end{aligned}$$

So

$$T_x = 0$$

So  $\forall x, y \in H$

$$0 = \sum_{(x)} \theta(x', y) x''$$

For  $y \in H$  the map

$$\begin{array}{ccc} H & \rightarrow & H \\ x & \rightarrow & \theta(x, y) \end{array}$$

is linear and

$$0 = \sum_{(x)} \theta(x', y) x''$$

$\forall x \in H$

so

$$\theta(x, y) = 0$$

$\forall x \in H$

We have shown

$$\theta(x, y) = 0 \quad \forall x, y \in H$$

Result follows

(ii) In the eqn  $\varepsilon(h) |_{H} = \sum_{(h)} \varepsilon(h' | h'') \text{ wt } h = 1_H$



LEM 36 is about the algebra structure for the Hopf algebra  $H$

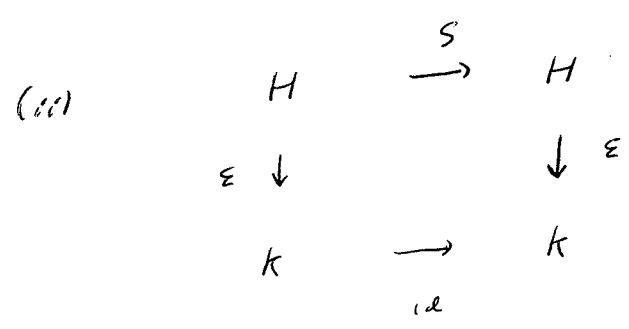
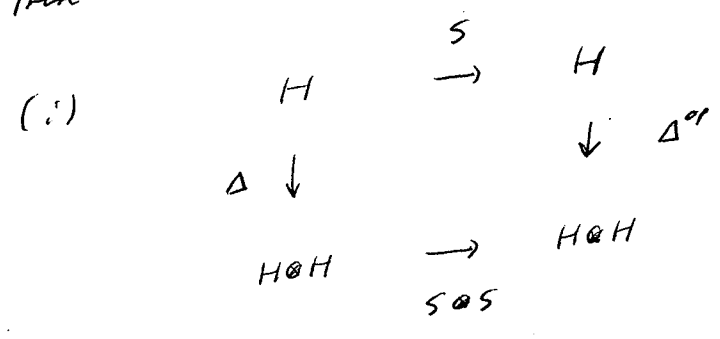
Now consider the coalgebra structure for  $H$ .

We expect  $S$  is a coalgebra morphism from the coalg  $H$  to the coalg  $H^{cop}$ .

This is correct.

LEM 37 Given a Hopf alg  $H$  with antipode  $S$ ,

then these diagrams commute:



In other words,  $S$  is a coalgebra morphism from coalg  $H$  to coalg  $H^{cop}$ .