Ex 33 Referring to Ex 32,

let J = 2-sided ideal of $T(V) = H$

generated by

$u \otimes v - v \otimes u \qquad u, v \in V$.

Then

(i) J is a coideal of H

(ii) Antipode S satisfies $S(J) \subseteq J$

[so sym algebra $S(V) = H/J$ inherits Hopf algebra str from H]

pf (i) Check

$$\Delta(J) \overset{?}{\subseteq} J \otimes H + H \otimes J \qquad *$$

For $u, v \in V$

$$\Delta(uv - vu) = (uv - vu) \otimes 1 + 1 \otimes (uv - vu)$$
$$\in J \otimes H + H \otimes J \quad \checkmark$$

$*$ follows since $J \otimes H + H \otimes J$ is a 2-sided ideal

of $H \otimes H$

Check
$$\varepsilon(J) \overset{?}{=} 0 \qquad **$$

For $u, v \in V$,

$$\varepsilon(uv - vu) = \underset{1}{\underset{\shortparallel}{\varepsilon(u)}}\,\underset{1}{\underset{\shortparallel}{\varepsilon(v)}} - \underset{1}{\underset{\shortparallel}{\varepsilon(v)}}\,\underset{1}{\underset{\shortparallel}{\varepsilon(u)}}$$

$$= 0 \quad \checkmark$$

** follows since ε is alg morphism

$\square$

(ii) USE the S action from Ex 32

Referring to Ex 32, 33 describe the

Hopf algebra $S(V)$.

For notational convenience assume $\dim(V) < \infty$
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ ″n

Fix a basis $\{v_i\}_{i=1}^n$ for V

Given commuting indets $\{x_i\}_{i=1}^n$

Recall algebra iso

$$\begin{aligned} S(V) &\longrightarrow k[x_1, \ldots, x_n] \\ v_i &\longrightarrow x_i \qquad 1 \le i \le n \end{aligned}$$

$k[x_1, \ldots, x_n]$ inherits Hopf alg str from $S(V)$

We have **algebra** morphisms

$$\Delta : \begin{aligned} k[x_1, \ldots, x_n] &\to k[x_1, \ldots, x_n] \otimes k[x_1, \ldots, x_n] \\ x_i &\longrightarrow x_i \otimes 1 + 1 \otimes x_i \qquad 1 \le i \le n \end{aligned}$$

$$\varepsilon : \begin{aligned} k[x_1, \ldots, x_n] &\to k \\ x_i &\longrightarrow 0 \qquad 1 \le i \le n \end{aligned}$$

$$S : \begin{aligned} k[x_1, \ldots, x_n] &\longrightarrow k[x_1, \ldots, x_n] \\ x_i &\longrightarrow -x_i \qquad 1 \le i \le n \end{aligned}$$

$\square$

Given f.d. Hopf algebra H with antipode S.

Recall

$$\text{alg } H \to \text{coalg } H^* \Rightarrow \text{bialg } H^*$$

$$\text{coalg } H \to \text{alg } H^*$$

LEM 34 Above bialg H^* is Hopf algebra with antipode S^*.

pf Given:

$$S \in \mathrm{End}(H)$$

$$\forall h \in H$$

$$\varepsilon(h) 1_H = \sum_{(h)} S(h') h'' = \sum_{(h)} h' S(h'')$$

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Recall bialg H^*:

<u>coproduct</u>: $\forall f \in H^*$

$$f(h_1 h_2) = \sum_{(f)} f'(h_1) f''(h_2) \qquad \forall h_1, h_2 \in H$$

<u>product</u>: $\forall f_1, f_2 \in H^*$

$$(f_1 f_2)(h) = \sum_{(h)} f_1(h') f_2(h'') \qquad \forall h \in H$$

<u>counit</u> $\forall f \in H^*$

$$\varepsilon^*(f) = f(1_H)$$

<u>unit</u>

$$1_{H^*} = \varepsilon$$

Require: $\forall f \in H^*$

$$\varepsilon^*(f)\, 1_{H^*} \overset{?}{=} \sum_{(f)} S^*(f')\, f'' \overset{?}{=} \sum_{(f)} f'\, S^*(f'')$$

$\uparrow$

Both sides in H^*. Apply each side to $h \in H$

LHS: $\varepsilon^*(f)\, 1_{H^*}(h) = f(1_H)\, \varepsilon(h)$

RHS:

$$\sum_{(f)} \left(S^*(f')\, f''\right)(h) = \sum_{(f)} \sum_{(h)} \underbrace{(S^*(f'))(h')}_{\substack{\parallel \\ \langle S^*(f'),\, h'\rangle \\ \parallel \\ \langle f',\, S(h')\rangle \\ \parallel \\ f'(S(h'))}}\, f''(h'')$$

$$= \sum_{(h)} \underbrace{\sum_{(f)} f'(S(h'))\, f''(h'')}_{\substack{\parallel \\ f(S(h')\, h'')}}$$

$$= f\Big(\underbrace{\sum_{(h)} S(h')\, h''}_{\substack{\parallel \\ \varepsilon(h)\, 1_H}}\Big)$$

$$= \varepsilon(h)\, f(1_H)$$

✓

□

Here is a handy principle

LEM 35 Given a Hopf algebra H.
Then for $T \in End(H)$ TFAE:

(i) $\forall h \in H$
$$0 = \sum_{(h)} T(h')h''$$

(ii) $\forall h \in H$
$$0 = \sum_{(h)} h' T(h'')$$

(iii) $T = 0$

pf Recall the antipode $S \in End(H)$ satisfies
$$S \star I = \mathbb{1} = I \star S$$

(i) $\to$ (iii) $\forall h \in H$
$$(T \star I)(h) = \sum_{(h)} T(h')h'' = 0$$

so $T \star I = 0$

So $\underbrace{T \star \underbrace{I \star S}_{\mathbb{1}}}_{T} = 0$ ✓

Other assertions are clear. $\square$

LEM 36 Given Hopf alg H with antipode S.

then

(i) $S(xy) = S(y)S(x) \quad \forall x,y \in H$

(ii) $S(1_H) = 1_H$

In other words

S is an algebra morphism from the algebra H to the algebra H^{op}

pf (i) Define a bilin map

$$\theta : \begin{array}{l} H \times H \to H \\ x, y \to S(xy) - S(y)S(x) \end{array}$$

show $\theta = 0$

Recall

$$\varepsilon(x) 1_H = \sum_{(x)} S(x') x''$$

$$\varepsilon(y) 1_H = \sum_{(y)} S(y') y''$$

$$\Delta(xy) = \sum_{(x)} \sum_{(y)} x'y' \otimes x''y''$$

So

$$\varepsilon(xy) 1_H = \sum_{(x)} \sum_{(y)} S(x'y') x''y'' \qquad *$$

Also

$$\varepsilon(xy)\, 1_H = \varepsilon(x)\,\varepsilon(y)\, 1_H$$

$$= \varepsilon(x) \sum_{(y)} S(y')\, y''$$

$$= \sum_{(y)} S(y') \Big(\sum_{(x)} S(x')\, x'' \Big)\, y''$$

$$= \sum_{(x)} \sum_{(y)} S(y')\, S(x')\, x''\, y'' \qquad **$$

By *, **

$$0 = \sum_{(x)} \sum_{(y)} \Big(S(x'y') - S(y')S(x') \Big)\, x''y''$$

$$= \sum_{(x)} \sum_{(y)} \theta(x', y')\, x''\, y''$$

$\forall x \in H$ Define

$$T_x : \quad H \longrightarrow H$$
$$z \longrightarrow \sum_{(x)} \theta(x', z)\, x''$$

Obs $\quad T_x \in End(H)$

Obs

$$\sum_{(y)} T_x(y')y'' = \sum_{(y)} \sum_{(x)} \theta(x', y')\, x'' y''$$

$$= \sum_{(x)} \sum_{(y)} \theta(x', y')\, x'' y''$$

$$= 0$$

So

$$T_x = 0$$

So $\forall\, x, y \in H$

$$0 = \sum_{(x)} \theta(x', y)\, x''$$

Fn $y \in H$ the map

$$H \longrightarrow H$$
$$x \longmapsto \theta(x, y)$$

is linear and

$$0 = \sum_{(x)} \theta(x', y)\, x'' \qquad \forall\, x \in H$$

so

$$\theta(x, y) = 0 \qquad \forall\, x \in H$$

We have shown $\theta(x, y) = 0 \quad \forall\, x, y \in H$

Result follows $\square$

(ii) In the eqn $\varepsilon(h)\, 1_H = \sum_{(h)} S(h')h''$ set $h = 1_H$

LEM 36 is about the algebra structure for the Hopf algebra H

Now consider the coalgebra structure for H.

We expect S is a coalgebra morphism from the coalg H to the coalg H^{cop}.

This is correct.

LEM 37 Given a Hopf alg H with antipode S. then these diagrams commute:

$$(i) \quad \begin{array}{ccc} H & \xrightarrow{S} & H \\ {\scriptstyle \Delta}\downarrow & & \downarrow{\scriptstyle \Delta^{op}} \\ H\otimes H & \xrightarrow[S\otimes S]{} & H\otimes H \end{array}$$

$$(ii) \quad \begin{array}{ccc} H & \xrightarrow{S} & H \\ {\scriptstyle \varepsilon}\downarrow & & \downarrow{\scriptstyle \varepsilon} \\ k & \xrightarrow[id]{} & k \end{array}$$

In other words, S is a coalgebra morphism from coalg H to coalg H^{cop}.