Lecture 20 Monday Oct 19

Given algebra A and coalgebra C

Consider $Hom(C, A)$

Recall convolution product $\star$:

$\forall f, g \in Hom(C, A)$, $\forall c \in C$,

$$(f \star g)(c) = \sum_{(c)} f(c') g(c'').$$

Recall $\mathbb{1} \in Hom(C, A)$:

$$\mathbb{1}(c) = \varepsilon_C(c) 1_A \qquad \forall c \in C$$

Prop 27 Given an algebra A and coalgebra C. Then $Hom(C,A)$ is an algebra with product $\star$ and identity $\mathbb{1}$.

pf Show $\star$ is assoc:

Given $f,g,h \in Hom(C,A)$

show $(f \star g) \star h = f \star (g \star h)$.

For $c \in C$

$$((f \star g) \star h)(c) = \sum_{(c)} (f \star g)(c') \, h(c'')$$

$$= \sum_{(c)} f(c') \, g(c'') \, h(c''')$$

and

$$(f \star (g \star h))(c) = \sum_{(c)} f(c') \, (g \star h)(c'')$$

$$= \sum_{(c)} f(c') \, g(c'') \, h(c''')$$

ok

show $\mathbb{1}$ is ident:

Given $f \in Hom(C, A)$

show $\mathbb{1} \star f = f = f \star \mathbb{1}$

$\forall c \in C$

$$(\mathbb{1} \star f)(c) = \sum_{(c)} \mathbb{1}(c') f(c'')$$

$$= \sum_{(c)} \varepsilon_C(c') f(c'')$$

$$= f\Big(\underbrace{\sum_{(c)} \varepsilon_C(c') c''}_{= c \text{ since } \varepsilon_C \text{ is counit}} \Big)$$

$$= f(c)$$

ok

$\square$

Prop 28 Given an algebra A and coalgebra C. Then the linear map

$$\begin{array}{ccc} A \otimes C^* & \longrightarrow & \mathrm{Hom}(C, A) \\ a \otimes f & \longrightarrow & F \\ & & F(c) = f(c)\, a \end{array} \qquad \ast$$

is an algebra morphism.

pf Check the map $\ast$ sends

$$\begin{array}{ccc} 1_{A \otimes C^*} & \overset{?}{\longrightarrow} & \mathbb{1} \\ \parallel & & \\ 1_A \otimes 1_{C^*} & & \\ \parallel & & \\ 1_A \otimes \varepsilon_C & & \end{array}$$

$\forall c \in C$,

$$\mathbb{1}(c) \overset{?}{=} \varepsilon_C(c)\, 1_A \qquad \text{OK}$$

Check the map $*$ respects mult

For $a, b \in A$ and $f, g \in C^*$

$$*: \quad a \otimes f \longrightarrow F$$

$$*: \quad b \otimes g \longrightarrow G$$

check

$$*: \quad ab \otimes fg \overset{?}{\longrightarrow} F \star G$$

For $c \in C$

$$(F \star G)(c) \overset{?}{=} \underbrace{(fg)(c)}_{\substack{\shortparallel \\ \sum_{(c)} f(c')g(c'')}} \; ab$$

$$\shortparallel$$

$$\sum_{(c)} \underbrace{F(c')}_{\substack{\shortparallel \\ f(c')a}} \; \underbrace{G(c'')}_{\substack{\shortparallel \\ g(c'')b}}$$

$$\shortparallel$$

$$\left(\sum_{(c)} f(c')g(c'') \right) ab$$

ok

$\square$

Note 29 Referring to Prop 27,

Assume $A = k$

So
$$Hom(C, A) = Hom(C, k) = C^*$$

The algebra str on C^* from Prop 27 is the same one we found earlier.

Given algebra A and coalgebra C

Recall $Hom(C,A)$ is an algebra with

convolution product $\star$ and identity $\mathbb{1}$

Now assume $A = C$ is bialgebra

(call it H)

So

$$Hom(C,A) = End(H)$$

TWO algebra structures on $End(H)$:

(i) convolution product $\star$ and ident $\mathbb{1}$ "convolution algebra"

(ii) composition product $\circ$ and ident I

$\forall f, g \in End(H)$, $\forall h \in H$,

$$(f \star g)(h) = \sum_{(h)} f(h')g(h'') \qquad (f \circ g)(h) = f(g(h))$$

$$\mathbb{1}(h) = \varepsilon(h) 1_H \qquad I(h) = h$$

— ∘ —

Given bialgebra H with data

$\Delta, \mu, \varepsilon, \eta$

Get 4 bialgebras:

	Δ	Δ^{op}
μ	H	H^{cop}
μ^{op}	H^{op}	$H^{op,cop}$

all have same ε, η

Consider corresp convolution algebras

All have same $\mathbb{1}$:

$$\mathbb{1}(h) = \varepsilon(h)\, 1_H \qquad \forall h \in H.$$

For $f, g \in \mathrm{End}(H)$ and $h \in H$,

$(f \star g)(h) =$

	Δ	Δ^{op}
μ	$\sum_{(h)} f(h')\,g(h'')$	$\sum_{(h)} f(h'')\,g(h')$
μ^{op}	$\sum_{(h)} g(h'')\,f(h')$	$\sum_{(h)} g(h')\,f(h'')$

So

$$f \star g \ (\text{for } H) = g \star f \ (\text{for } H^{op,cop})$$

$$f \star g \ (\text{for } H^{op}) = g \star f \ (\text{for } H^{cop})$$

In other words,

- the convolution algebras for H, $H^{op,cop}$ are opposite
- the convolution algebras for H^{op}, H^{cop} are opposite.

— o —

In the convolution algebra $End(H)$ the element I might not be invertible. If it is, we call the inverse the antipode of H.

Def 30 For $S \in End(H)$, S is an <u>antipode</u> whenever

$$S \circledast I = \mathbb{1} = I \circledast S$$

Obs S is an antipode iff $\forall h \in H$,

$$\varepsilon(h) 1_H = \sum_{(h)} S(h') h'' = \sum_{(h)} h' S(h'')$$

<u>Note</u> Referring to Def 30,

antipode S is unique if it exists.

Indeed for an antipode S'

$$
\begin{array}{ccc}
(S \star I) \star S' & = & S \star (I \star S') \\
\| & & \| \\
\mathbb{1} \star S' & & S \star \mathbb{1} \\
\| & & \| \\
S' & & S
\end{array}
$$

Def A <u>Hopf algebra</u> is a bialgebra that has an antipode.

A <u>morphism of Hopf algebras</u> is a bialgebra morphism that commutes with the antipodes.

Ex 31 Let $G =$ group.

Consider group algebra $H = kG$ with

coalg structure

$$\Delta(x) = x \otimes x \quad \forall x \in G$$

$$\varepsilon(x) = 1 \quad \text{''}$$

Then antipode S exists. We have

$$S(x) = x^{-1} \quad \forall x \in G.$$

pf $\forall x \in G$

$$\underbrace{\varepsilon(x)}_{=1} \underbrace{1_H}_{=e} = \underbrace{S(x)\, x}_{} \overset{?}{=} x\, S(x)$$

$S(x) = x^{-1}$, $S(x) = x^{-1}$

ok

□

Ex 32 let V = vector space

Consider tensor algebra $H = T(V)$

with coalgebra structure

$\Delta(v) = 1 \otimes v + v \otimes 1 \quad \forall v \in V$

$\varepsilon(v) = 0 \quad \forall v \in V$

Then antipode S exists.

For $n \in \mathbb{N}$ and $v_1, v_2, \ldots, v_n \in V$,

$$S(v_1 \otimes v_2 \otimes \cdots \otimes v_n) = (-1)^n v_n \otimes v_{n-1} \otimes \cdots \otimes v_1$$

pf For $h \in H$ show

$$\varepsilon(h) 1_H = \sum_{(h)} S(h') h'' = \sum_{(h)} h' S(h'') \quad *$$

let $\tilde{H}$ = set of elements h that satisfy $*$.

Show $\tilde{H} = H$.

claim $1_H \in \tilde{H}$: Since

$\Delta(1_H) = 1_H \otimes 1_H \quad S(1_H) = 1_H, \quad \varepsilon(1_H) = 1$

ok

claim $V \subseteq \tilde{H}$:

For $v \in V$,

$\Delta(v) = 1 \otimes v + v \otimes 1$

$\varepsilon(v) = 0$

$S(v) = -v$

$$\underset{\substack{\shortparallel \\ 0}}{\varepsilon(v)\, 1_H} \overset{?}{=} \underset{\substack{\shortparallel \\ 1}}{S(1)}\, v + \underset{\substack{\shortparallel \\ -v}}{S(v)}\, 1 \qquad \text{ok}$$

claim $hg \in \tilde{H}$ $\forall h, g \in H$:

$$\Delta(h) = \sum_{(h)} h' \otimes h''$$

$$\Delta(g) = \sum_{(g)} g' \otimes g''$$

$$\Delta(hg) = \sum_{(h)} \sum_{(g)} h'g' \otimes h''g''$$

Require

$$\varepsilon(hg)\,1_H \stackrel{?}{=} \sum_{(h)}\sum_{(g)} \underset{\underset{S(g')\,S(h')}{\|}}{S(h'g')}\, h''g''$$

$$\begin{aligned}
\text{RHS} &= \sum_{(g)}\sum_{(h)} S(g')\,S(h')\,h''g'' \\
&= \sum_{(g)} S(g')\Big(\underbrace{\sum_{(h)} S(h')\,h''}_{\varepsilon(h)\,1_H}\Big)\,g'' \\
&= \varepsilon(h)\,1_H \underbrace{\sum_{(g)} S(g')\,g''}_{\varepsilon(g)\,1_H} \\
&= \varepsilon(h)\,\varepsilon(g)\,1_H \\
&= \varepsilon(hg)\,1_H \quad ✓
\end{aligned}$$

□

Now $\tilde{H} = H$ since V generates $T(V) = H$