

LEM 6 Given algebra  $A$  and f.d.  $A$ -module  $V$ 

TFAE

(i) For each  $A$ -submodule  $V' \subseteq V$  there exists an  $A$ -submodule  $V'' \subseteq V$  s.t.  $V = V' + V''$  (ds)(ii) For each simple  $A$ -submodule  $V' \subseteq V$  there exists an  $A$ -submodule  $V'' \subseteq V$  s.t.  $V = V' + V''$  (ds)(iii) For each  $A$ -submodule  $V' \subseteq V$  there exists an  $A$ -linear map  $p: V \rightarrow V'$  s.t.  $p^2 = p$ (iv) For each simple  $A$ -submodule  $V' \subseteq V$  there exists an  $A$ -linear map  $p: V \rightarrow V'$  s.t.  $p^2 = p$ 

(v)

 $V$  is s.s.

pf are

— 0 —

Next goal: Given algebras  $A, A'$ 

describe the set

 $\text{Homalg}(A, A')$

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Given a set  $X$

We define the free algebra  $k\{X\}$

For an integer  $n \geq 0$ , a word of length  $n$  over  $X$

is a sequence  $x_1 x_2 \dots x_n$  with  $x_i \in X$  for  $i=1, \dots, n$ .

view  $\phi$  = word of length 0

The words over  $X$  form a basis for a vector space  $k\{X\}$  over  $k$ .

Given words  $x_1 x_2 \dots x_r$  and  $x'_1 x'_2 \dots x'_s$  their

product is the word

$$x_1 x_2 \dots x_r x'_1 \dots x'_s$$

(concatenation)

This product turns  $k\{X\}$  into an algebra

with mult identity  $1 = \phi$

Call  $k\{X\}$  the free algebra over  $X$

the set  $X$  generates  $k\{X\}$ .

DEF 7 Given sets  $X, Y$

$\text{Hom}_{\text{set}}(X, Y) =$  set of all functions from  $X$  to  $Y$ .

Ex  $F_n$  finite  $X = \{x_1, x_2, \dots, x_n\}$

we get a bijection

$\text{Hom}_{\text{set}}(X, Y) \longrightarrow Y \times Y \times \dots \times Y \quad (n \text{ copies})$   
 $f \longrightarrow (f(x_1), f(x_2), \dots, f(x_n))$

Note

$Y^0$  contains unique element  $\emptyset$

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LEM 8 Given algebra  $A$  and set  $X$ .

For  $f \in \text{Hom}_{\text{set}}(X, A) \quad \exists$  unique

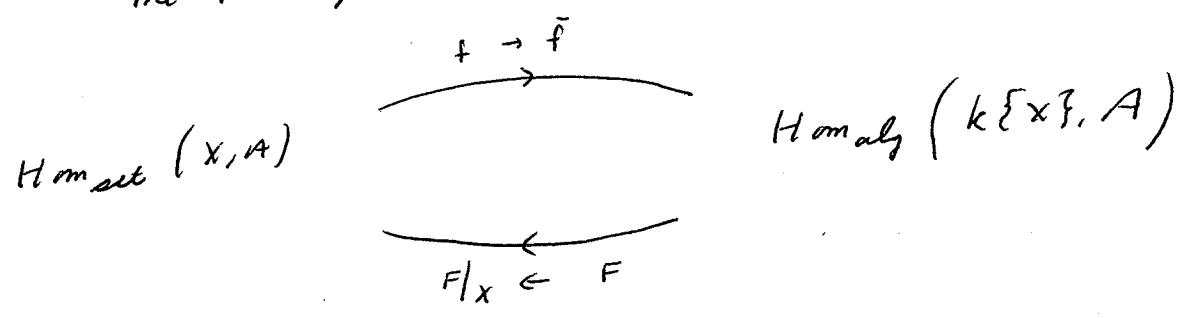
algebra morphism  $\bar{f} : k\{X\} \rightarrow A$

that sends  $x \rightarrow f(x) \quad \forall x \in X$ .

pf (ex).

LEM 9 For an alg  $A$  and a set  $X$

the following maps are inverse bijections:



pf Since  $X$  generates  $k\langle X \rangle$ .

□

Given alg  $A$

$X =$  gen set for  $A$

$\exists$  alg morphism

$$\begin{aligned} k\{X\} &\rightarrow A \\ x &\rightarrow x \end{aligned}$$

\*

let  $I =$  kernel of \*

$I$  is 2-sided ideal of  $k\{X\}$

\* induces alg iso

$$k\{X\}/I \rightarrow A$$

Given an algebra  $A'$ ,

to describe

$$\text{Hom}_{\text{alg}}(A, A')$$

it suffices to describe

$$\text{Hom}_{\text{alg}}\left(k\{X\}/I, A'\right)$$

Given set  $X$  and alg  $A'$

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Prop 10 For each  $f \in \text{Hom}_{\text{alg}}(X, A')$   
and each 2-sided ideal  $I$  of  $k\{X\}$  TFAE

(i)  $\bar{f}(x) = 0$

(ii)  $\exists$  alg hom  $f^v: k\{X\}/I \rightarrow A'$   
that sends  $x+I \rightarrow f(x) \quad \forall x \in X$

Suppose (i), (ii) hold, then  $f^v$  is unique.

Pf (i)  $\rightarrow$  (ii) the map

$$\begin{aligned} k\{X\}/I &\rightarrow A' \\ w+I &\rightarrow \bar{f}(w) \end{aligned}$$

is an alg hom that sends  $x+I \rightarrow f(x) \quad \forall x \in X$ .

(ii)  $\rightarrow$  (i) By const

$$f^v(w+I) = \bar{f}(w) \quad \forall w \in k\{X\}$$

So

$$0 = f^v(I) = \bar{f}(I)$$

Suppose (i), (ii) hold. Then  $f^v$  is unique since

$\{x+I \mid x \in X\}$  generates  $k\{X\}/I$

□

Cor 11 Referring to Prop 10,

the map  $f \rightarrow f^\vee$  gives a bijection between

$$(i) \quad \left\{ f \in \text{Hom}_{\text{alg}}(X, A') \mid \bar{f}(I) = 0 \right\} = *$$

$$(ii) \quad \text{Hom}_{\text{alg}}(k\{x\}/I, A') = *$$

pf injective:  $\forall f_1, f_2 \in *$  s.t.  $f_1^\vee = f_2^\vee$

$$\forall x \in X \quad \begin{array}{ccc} f_1^\vee(x+I) & = & f_2^\vee(x+I) \\ \text{"} & & \text{"} \\ f_1(x) & & f_2(x) \end{array}$$

so  $f_1 = f_2$

surj: Given  $F \in *$

define  $f \in \text{Hom}_{\text{alg}}(X, A')$  s.t.

$$f(x) = F(x+I) \quad \forall x \in X$$

Prop 10 (ii) holds with  $f^\vee = F$

so Prop 10 (i) holds i.e.  $\bar{f}(I) = 0$

now  $f \in *$  and  $f^\vee = F$

□



EXAMPLE

Given finite set  $X = \{x_1, x_2, \dots, x_n\}$ Let  $I =$  2-sided ideal of  $k\{X\}$  gen by

$$xy - yx$$

$$x \cdot y \in X$$

Algebra  $k\{X\}/I$  iso  $k[X]$  view  $x_i$  as commuting indets (ex)For any algebra  $A$  we have bijections:

$$\text{Hom}_{\text{alg}}(k[X], A) \leftrightarrow \left\{ f \in \text{Hom}_{\text{set}}(X, A) \mid \begin{array}{l} f(x_i) f(y_j) = \\ f(y_j) f(x_i) \quad \forall x_i, y_j \in X \end{array} \right\}$$

$$\leftrightarrow \left\{ (x_1, x_2, \dots, x_n) \in A^n \mid a_i a_j = a_j a_i \text{ for } 1 \leq i, j \leq n \right\}$$

For  $A$  commutative, these bijections become

$$\text{Hom}_{\text{alg}}(k[X], A) \leftrightarrow \text{Hom}_{\text{set}}(X, A)$$

$$\leftrightarrow A^n$$

(Aside)

LEM let  $X = \text{set}$

let  $I = 2\text{-sided ideal of } k\{X\} \text{ gen by}$   
 $xy - yx \quad x, y \in X$

then the algebras

$$k\{X\}/I, \quad k[X]$$

are isomorphic.

pf Consider alg hom

$$\begin{array}{ccc} k\{X\} & \longrightarrow & k[X] \\ \theta & & \\ x & \longrightarrow & x \end{array}$$

$$I \subseteq \ker(\theta)$$

(1)

show  $I = \ker(\theta)$

let  $B = \text{set of words over } X$

$B$  is basis for vector space  $k\{X\}$

Define binary relation  $\sim$  on  $B$ :

words  $x_1 x_2 \dots x_n \sim y_1 y_2 \dots y_n \iff y_1, y_2, \dots, y_n \text{ is perm of } x_1, x_2, \dots, x_n$

$\sim$  is equiv rel.

For each equiv class of  $\sim$ , pick out one word and call it "good"

$\forall w \in B$  let  $\bar{w}$  denote the good word with  $w \sim \bar{w}$

Obs

$$w - \bar{w} \in I$$

\*

Let  $G =$  subspace of  $k\{x\}$  spanned by the good words.

By \*

$$k\{x\} = G + I$$

(2)

By the def of  $\theta$ ,

$$G \cap \ker(\theta) = 0$$

(3)

By (1) - (3),

$$I = \ker(\theta)$$

Result follows.

□

Until further notice

$A =$  commutative algebra

$X = \{x_1, \dots, x_n\}$  is finite set

Recall bijections

$$\begin{aligned} \text{Hom}_{\text{alg}}(k[X], A) &\longleftrightarrow \text{Hom}_{\text{sets}}(X, A) \longleftrightarrow A^n \\ F &\longleftrightarrow F[X] \quad f \longleftrightarrow (f(x_1), f(x_2), \dots, f(x_n)) \end{aligned}$$

Cases  
 $n=0$

$X = \emptyset$

$$\begin{aligned} \text{Hom}_{\text{alg}}(k, A) &\xleftrightarrow{\text{bij}} A^{\emptyset} = \emptyset \\ &\text{"} \\ &\cong A \end{aligned}$$

$n=1$

$X = \{x\}$

$$\text{Hom}_{\text{alg}}(k[x], A) \xleftrightarrow{\text{bij}} A$$

$n=2$

$X = \{x^1, x^2\}$

$$\text{Hom}_{\text{alg}}(k[x^1, x^2], A) \xleftrightarrow{\text{bij}} A^2 = A \times A$$

Consider addition operation on  $A$ :

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$$\begin{array}{l} \text{Plus :} \\ A \times A \quad \rightarrow A \\ a, b \quad \rightarrow a+b \end{array}$$

Under above bijections, Plus corresponds to some map

$$\overline{\text{Plus}} : \text{Homalg}(k[x_1, x_2], A) \rightarrow \text{Homalg}(k[x], A)$$

Find  $\overline{\text{Plus}}$

Consider alg hom

$$\Delta: \begin{array}{ccc} k[x] & \rightarrow & k[x', x''] \\ x & \rightarrow & x' + x'' \end{array}$$

Consider algebra homs

$$\begin{array}{ccc} & \Delta & \\ k[x] & \longrightarrow & k[x', x''] \\ & \searrow & \swarrow \\ & A & \end{array}$$

$$\text{For } F \in \text{Hom}_{\text{alg}}(k[x', x''], A),$$

$$F \circ \Delta \in \text{Hom}_{\text{alg}}(k[x], A)$$

Consider map

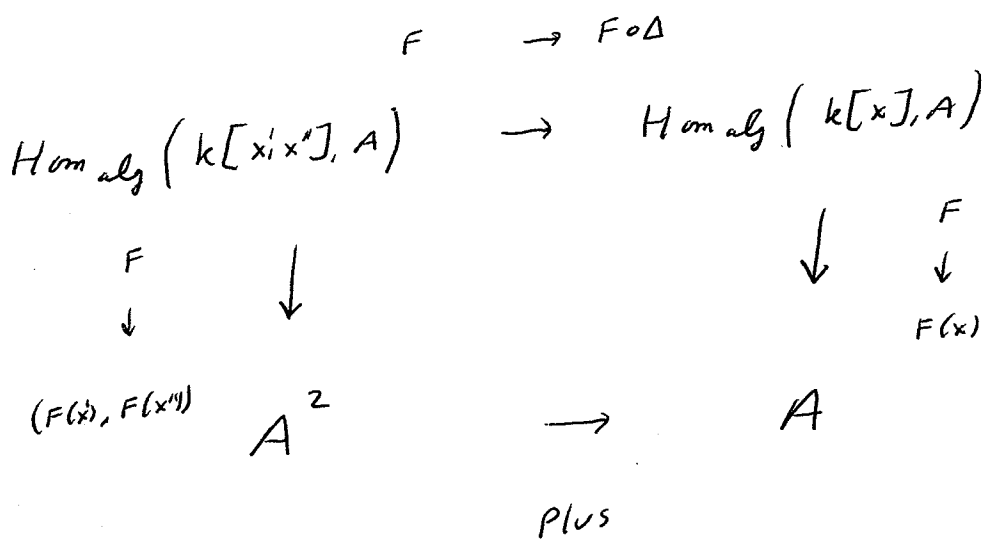
$$\text{Hom}_{\text{alg}}(k[x', x''], A) \rightarrow \text{Hom}_{\text{alg}}(k[x], A)$$

$$F \rightarrow F \circ \Delta$$

\*

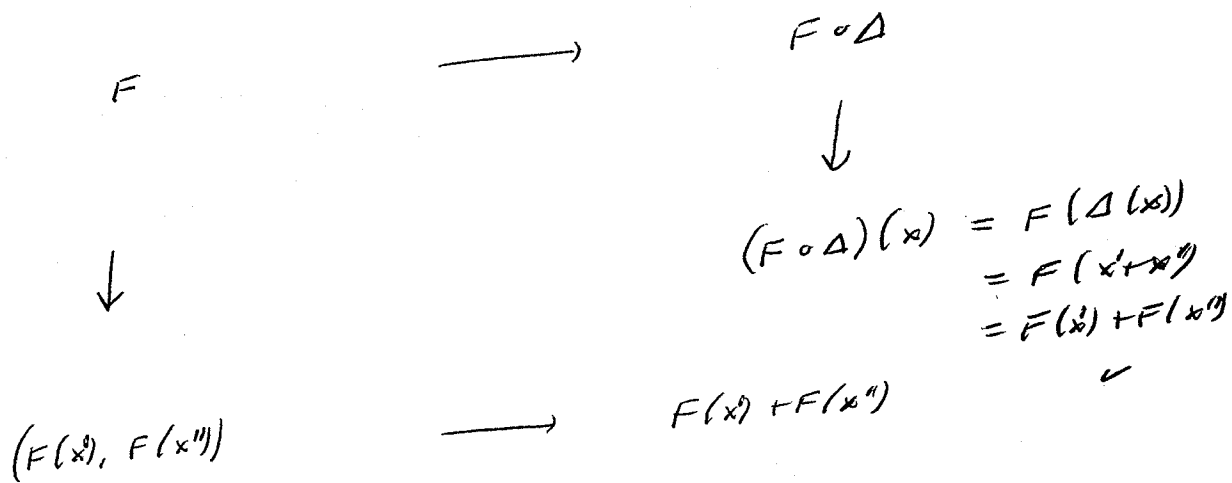
Prop 12 the following diagram commutes:

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Moreover the map  $*$  is  $\overline{\text{Plus}}$

pf Chase  $F$  around diagram



□

Next consider (additive) inverse operation on  $A$ : 9/4/15  
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$$\begin{array}{l} \text{inv} : \\ A \rightarrow A \\ a \rightarrow -a \end{array}$$

Under earlier bijections,  $\text{inv}$  corresponds to  
a map

$$\overline{\text{inv}} : \text{Hom}_{\text{alg}}(k[x], A) \rightarrow \text{Hom}_{\text{alg}}(k[x], A)$$

Find  $\overline{\text{inv}}$



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Consider algebra hom

$$S: \begin{aligned} k[x] &\rightarrow k[x] \\ x &\rightarrow -x \end{aligned}$$

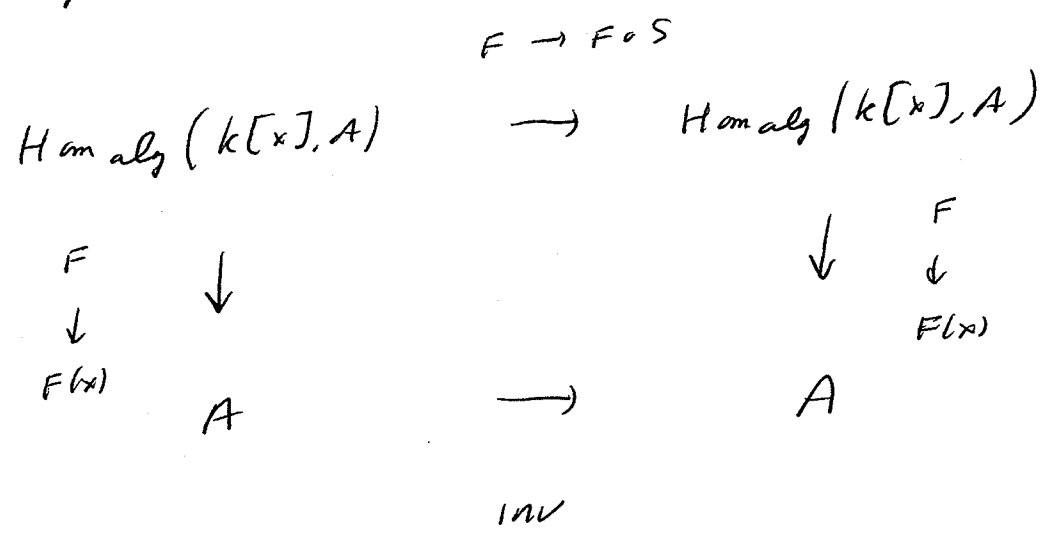
Gives map

$$\text{Hom}_{\text{alg}}(k[x], A) \rightarrow \text{Hom}_{\text{alg}}(k[x], A)$$

~~FX~~

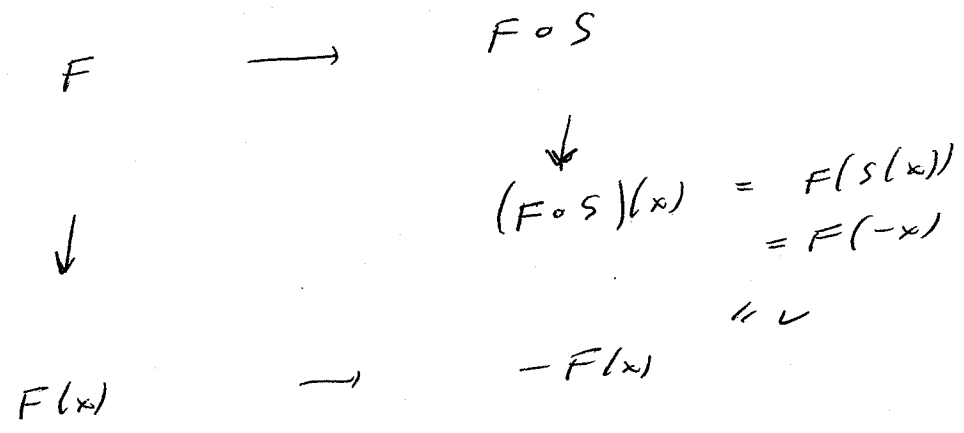
$$F \rightarrow F \circ S$$

Prop 13 the following diagram commutes:



More over ~~is~~ is INV

pf Chase F around diagram!



□

Consider the element  $0 \in A$

View as a map

$$\begin{array}{l} \text{zero} \quad A^0 \rightarrow A \\ \quad \quad \quad \emptyset \rightarrow 0 \end{array}$$

Under earlier bijections, zero corresponds to a map

$$\overline{\text{zero}} : \text{Hom}_{\text{alg}}(k, A) \rightarrow \text{Hom}_{\text{alg}}(k[x], A)$$

Find  $\overline{\text{zero}}$

Consider alg hom

$$\varepsilon : \begin{array}{l} k[x] \rightarrow k \\ x \rightarrow 0 \end{array}$$

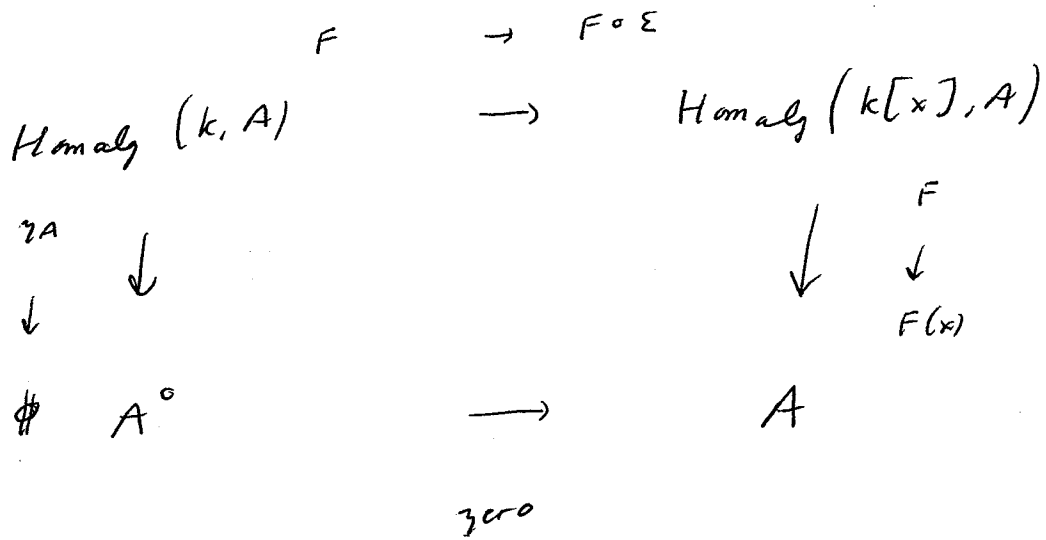
Gives map

$$\begin{array}{l} \text{Hom}_{\text{alg}}(k, A) \rightarrow \text{Hom}_{\text{alg}}(k[x], A) \\ F \rightarrow F \circ \varepsilon \end{array}$$

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Prop 14 The following diagram commutes

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Moreover ~~zero~~ is  $\overline{\text{zero}}$

pf chase  $\eta_A$  around diag

