Lec 2 Friday Sept 4 9/4/15

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LEM 6 Given algebra A and f.d. A-module V

TFAE

(i) For each A-submodule $V' \subseteq V$ there exists an A-submodule $V'' \subseteq V$ s.t $V = V' + V''$ (ds)

(ii) For each simple A-submodule $V' \subseteq V$ there exists an A-submodule $V'' \subseteq V$ s.t $V = V' + V''$ (ds)

(iii) For each A-submodule $V' \subseteq V$ there exists an A-linear map $p : V \to V'$ s.t $p^2 = p$

(iv) For each simple A-submodule $V' \subseteq V$ there exists an A-linear map $p : V \to V'$ s.t $p^2 = p$

(v) V is s.s.

pf ex

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Next goal: Given algebras A, A'

describe the set

$\mathrm{Hom}_{alg}(A, A')$

Given a set X

We define the free algebra $k\{X\}$

For an integer $n \geq 0$, a <u>word of length n</u> over X

is a sequence $x_1 x_2 \cdots x_n$ with $x_i \in X$ for $1 \leq i \leq n$.

View ϕ = word of length 0

The words over X form a basis for a vector space $k\{X\}$

over k.

Given words $x_1 x_2 \cdots x_r$ and $x_1' x_2' \cdots x_s'$ their

<u>product</u> is the word

$x_1 x_2 \cdots x_r x_1' \cdots x_s'$ (concatenation)

This product turns $k\{X\}$ into an algebra

with mult identity $1 = \phi$

Call $k\{X\}$ the <u>free algebra over X</u>

The set X generates $k\{X\}$.

DEF 7 Given sets X, Y

$$\mathrm{Hom}_{set}(X, Y) = \text{set of all functions from } X \text{ to } Y.$$

Ex For finite $X = \{x_1, x_2, \dots, x_n\}$

we get a bijection

$$\begin{aligned} \mathrm{Hom}_{set}(X, Y) &\longrightarrow \underbrace{Y \times Y \times \dots \times Y}_{=: Y^n} \quad (n \text{ copies}) \\ f &\longrightarrow (f(x_1), f(x_2), \dots, f(x_n)) \end{aligned}$$

<u>Note</u>

Y^0 contains unique element $\emptyset$

LEM 8 Given algebra A and set X.
For $f \in \mathrm{Hom}_{set}(X, A)$ $\exists$ unique
algebra morphism $\bar{f} : k\{X\} \to A$
that sends $x \to f(x)$ $\forall x \in X$.

pf (ex).

LEM 9 For an alg A and a set X

the following maps are inverse bijections:

$$\begin{tikzcd} \mathrm{Hom}_{set}(X, A) \arrow[r, bend left, "f \to \tilde{f}"] & \mathrm{Hom}_{alg}(k\{X\}, A) \arrow[l, bend left, "F|_X \leftarrow F"] \end{tikzcd}$$

$\square$

pf Since X generates $k\{X\}$.

Given alg A

$X =$ gen set for A

$\exists$ alg morphism

$$k\{X\} \to A \qquad (*)$$
$$x \to x$$

let $I =$ kernel of $*$

I is 2-sided ideal of $k\{X\}$

$*$ induces alg iso

$$k\{X\}/I \to A$$

Given an algebra A',

to describe

$$\mathrm{Hom}_{alg}(A, A')$$

it suffices to describe

$$\mathrm{Hom}_{alg}\left(k\{X\}/I,\ A'\right)$$

Given set X and alg A'

Prop 10 For each $f \in \mathrm{Hom}_{set}(X, A')$ and each 2-sided ideal I of $k\{X\}$ TFAE

(i) $\bar{f}(I) = 0$

(ii) $\exists$ alg hom $f^\vee : k\{X\}/I \to A'$ that sends $x + I \to f(x)$ $\forall x \in X$

Suppose (i), (ii) hold, then $f^\vee$ is unique.

pf (i) $\to$ (ii) the map

$$k\{X\}/I \longrightarrow A'$$
$$w + I \longrightarrow \bar{f}(w)$$

is an alg hom that sends $x + I \to f(x)$ $\forall x \in X$.

(ii) $\to$ (i) By constr

$$f^\vee(w + I) = \bar{f}(w) \quad \forall w \in k\{X\}$$

So

$$0 = f^\vee(I) = \bar{f}(I)$$

Suppose (i), (ii) hold. Then $f^\vee$ is unique since

$\{x + I \mid x \in X\}$ generates $k\{X\}/I$

$\square$

Cor 11 Referring to Prop 10,

the map $f \to f^\vee$ gives a bijection between

(i) $\{ f \in Hom_{set}(X, A') \mid \bar{f}(I) = 0 \} = *$

(ii) $Hom_{alg}(k\{X\}/I, A') = \star$

pf injective: For $f_1, f_2 \in *$ s.t $f_1^\vee = f_2^\vee$

$\forall x \in X$ $\quad f_1^\vee(x+I) = f_2^\vee(x+I)$

$$\begin{array}{ccc} f_1^\vee(x+I) & = & f_2^\vee(x+I) \\ \| & & \| \\ f_1(x) & & f_2(x) \end{array}$$

so $f_1 = f_2$

surj: Given $F \in \star$

Define $f \in Hom_{set}(X, A')$ s.t

$f(x) = F(x+I)$ $\quad \forall x \in X$

Prop 10 (ii) holds with $f^\vee = F$

So Prop 10 (i) holds ie $\bar{f}(I) = 0$

Now $f \in *$ and $f^\vee = F$ $\square$

EXAMPLE

Given finite set $X = \{x_1, x_2, \ldots, x_n\}$

let I = 2-sided ideal of $k\{X\}$ gen by

$$xy - yx \qquad x, y \in X$$

Algebra $k\{X\}/I$ iso $k[X]$ ← view x_i as commuting indets (ex)

For any algebra A we have bijections:

$$\mathrm{Hom}_{alg}(k[X], A) \leftrightarrow \left\{ f \in \mathrm{Hom}_{set}(X, A) \,/\, \begin{matrix} f(x)f(y) = \\ f(y)f(x) \quad \forall x, y \in X \end{matrix} \right\}$$

$$\leftrightarrow \left\{ (x_1, x_2, \ldots, x_n) \in A^n \,/\, \begin{matrix} a_i a_j = a_j a_i \text{ for} \\ 1 \le i, j \le n \end{matrix} \right\}$$

For A commutative, these bijections become

$$\mathrm{Hom}_{alg}(k[X], A) \leftrightarrow \mathrm{Hom}_{set}(X, A)$$

$$\leftrightarrow A^n$$

(Aside)

LEM let X = set

let I = 2-sided ideal of $k\{X\}$ gen by

$xy - yx \qquad x, y \in X$

then the algebras

$$k\{X\}/I, \qquad k[X]$$

are isomorphic.

pf Consider alg hom

$$\theta \quad \begin{array}{ccc} k\{X\} & \longrightarrow & k[X] \\ x & \longmapsto & x \end{array}$$

$$I \subseteq \ker(\theta) \qquad (1)$$

show $I = \ker(\theta)$

let B = set of words over X

B is basis for vector space $k\{X\}$

Define binary relation $\sim$ on B:

words $x_1 x_2 \cdots x_n \sim y_1 y_2 \cdots y_n$ iff $y_1, y_2, \ldots, y_n$ is perm of $x_1, x_2, \ldots, x_n$

$\sim$ is equiv rel.

For each equiv class of $\sim$, pick out one word and call it "good"

$\forall w \in B$ let $\bar{w}$ denote the goodword with $w \sim \bar{w}$

Obs

$$w - \bar{w} \in I \qquad *$$

Let G = subspace of $k\{X\}$ spanned by the good words.

By $*$,

$$k\{X\} = G + I \qquad (2)$$

By the def of θ,

$$G \cap \ker(\theta) = 0 \qquad (3)$$

By (1) – (3),

$$I = \ker(\theta)$$

Result follows. $\square$

Until further notice

$A =$ commutative algebra

$X = \{x_1 \dots x_n\}$ is finite set

Recall bijections

$$\mathrm{Hom}_{alg}(k[X], A) \longleftrightarrow \mathrm{Hom}_{set}(X, A) \longleftrightarrow A^n$$

$$F \longleftrightarrow F|X \qquad f \longleftrightarrow (f(x_1), f(x_2), \dots, f(x_n))$$

Cases

$n=0$ $\quad X = \emptyset$

$$\underset{\substack{\shortparallel \\ \exists A}}{\mathrm{Hom}_{alg}(k, A)} \overset{bij}{\longleftrightarrow} A^0 = \emptyset$$

$n=1$ $\quad X = \{x\}$

$$\mathrm{Hom}_{alg}(k[x], A) \overset{bij}{\longleftrightarrow} A$$

$n=2$ $\quad X = \{x', x''\}$

$$\mathrm{Hom}_{alg}(k[x', x''], A) \overset{bij}{\longleftrightarrow} A^2 = A \times A$$

Consider addition operation on A:

$$\text{Plus}: \begin{array}{ccc} A \times A & \longrightarrow & A \\ a, b & \longrightarrow & a+b \end{array}$$

Under above bijections, Plus corresponds to some map

$$\overline{\text{Plus}}: \quad \text{Hom}_{alg}(k[x', x''], A) \longrightarrow \text{Hom}_{alg}(k[x], A)$$

Find $\overline{\text{Plus}}$

Consider alg hom

$$\Delta : \quad \begin{aligned} k[x] &\to k[x', x''] \\ x &\to x' + x'' \end{aligned}$$

Consider algebra homs

$$\begin{tikzcd} k[x] \arrow[rr, "\Delta"] \arrow[dr] & & k[x', x''] \arrow[dl] \\ & A & \end{tikzcd}$$

For $F \in Hom_{alg}(k[x', x''], A)$,

$$F \circ \Delta \in Hom_{alg}(k[x], A)$$

Consider map

$$\begin{aligned} Hom_{alg}(k[x', x''], A) &\longrightarrow Hom_{alg}(k[x], A) \qquad (*) \\ F &\to F \circ \Delta \end{aligned}$$

Prop 12 The following diagram commutes:

$$F \mapsto F \circ \Delta$$

$$\begin{tikzcd}
\mathrm{Hom}_{alg}(k[x', x''], A) \arrow[r] \arrow[d] & \mathrm{Hom}_{alg}(k[x], A) \arrow[d] \\
A^2 \arrow[r, "\text{Plus}"'] & A
\end{tikzcd}$$

$$F \mapsto (F(x'), F(x'')) \qquad F \mapsto F(x)$$

Moreover the map $*$ is $\overline{\text{Plus}}$

pf Chase F around diagram

$$\begin{tikzcd}
F \arrow[r] \arrow[d] & F \circ \Delta \arrow[d] \\
(F(x'), F(x'')) \arrow[r] & F(x') + F(x'')
\end{tikzcd}$$

$$\begin{aligned}
(F \circ \Delta)(x) &= F(\Delta(x)) \\
&= F(x' + x'') \\
&= F(x') + F(x'') \checkmark
\end{aligned}$$

$\square$

Next consider (additive) inverse operation on A:

$$\mathrm{inv}: \begin{array}{ccc} A & \to & A \\ a & \to & -a \end{array}$$

Under earlier bijections, inv corresponds to a map

$$\overline{\mathrm{inv}}: \mathrm{Hom}_{alg}(k[x], A) \longrightarrow \mathrm{Hom}_{alg}(k[x], A)$$

Find $\overline{\mathrm{inv}}$

Consider algebra hom

$$S: \quad k[x] \to k[x]$$
$$x \to -x$$

Gives map

$$\mathrm{Hom}_{alg}(k[x], A) \to \mathrm{Hom}_{alg}(k[x], A) \qquad \text{**}$$
$$F \to F \circ S$$

Prop 13 the following diagram commutes:

$$\begin{tikzcd} \mathrm{Hom}_{alg}(k[x],A) \arrow[r, "F \mapsto F\circ S"] \arrow[d, "F \mapsto F(x)"'] & \mathrm{Hom}_{alg}(k[x],A) \arrow[d, "F \mapsto F(x)"] \\ A \arrow[r, "\mathrm{inv}"] & A \end{tikzcd}$$

Moreover ~~[illegible]~~ is $\overline{\mathrm{inv}}$

pf Chase F around diagram!

$$\begin{tikzcd} F \arrow[r] \arrow[d] & F\circ S \arrow[d] \\ F(x) \arrow[r] & (F\circ S)(x) = F(S(x)) = F(-x) \overset{\checkmark}{=} -F(x) \end{tikzcd}$$

$\square$

Consider the element $0 \in A$

View as a map

zero $\quad A^0 \to A$

$\phi \to 0$

Under earlier bijections, zero corresponds to a map

$\overline{\text{zero}}: \quad Hom_{alg}(k, A) \to Hom_{alg}(k[x], A)$

Find $\overline{\text{zero}}$

Consider alg hom

$$\varepsilon: \begin{array}{ccc} k[x] & \to & k \\ x & \to & 0 \end{array}$$

Gives map

$$\begin{array}{ccc} Hom_{alg}(k, A) & \to & Hom_{alg}(k[x], A) \\ F & \to & F \circ \varepsilon \end{array} \qquad \text{***}$$

Prop 14 The following diag commutes

$$F \mapsto F \circ \varepsilon$$

$$\begin{tikzcd}
\mathrm{Homalg}(k, A) \arrow[r] \arrow[d] & \mathrm{Homalg}(k[x], A) \arrow[d] \\
A^{\circ} \arrow[r, "\text{zero}"'] & A
\end{tikzcd}$$

$\zeta_A \mapsto \emptyset$ $\qquad$ $F \mapsto F(x)$

Moreover ~~[illegible]~~ is $\overline{\text{zero}}$

pf Chase ζ_A around diag

$$\begin{tikzcd}
\zeta_A \arrow[r] \arrow[d] & \zeta_A \circ \varepsilon \arrow[d] \\
\emptyset \arrow[r] & 0
\end{tikzcd}$$

$$(\zeta_A \circ \varepsilon)(x) = \zeta_A(\varepsilon(x)) = \zeta_A(0) = 0 \quad ✓$$