

Given vector space V . Consider tensor algebra $T(V)$.

We now turn $T(V)$ into a bialgebra.

Thm 22 Given vector space V .

(i) \exists alg morphism

$$\Delta: T(V) \rightarrow T(V) \otimes T(V)$$

that sends

$$v \mapsto v \otimes 1 + 1 \otimes v \quad \forall v \in V$$

(ii) \exists alg morphism

$$\varepsilon: T(V) \rightarrow k$$

that sends

$$v \mapsto 0 \quad \forall v \in V$$

(iii) Δ, ε turn $T(V)$ into a coalgebra

(iv) the coalgebra $T(V)$ is co-commutative.

(v) the above algebra and coalgebra structures turn $T(V)$ into a bialgebra.

pt (i) \exists lin map

$$V \rightarrow T(V) \otimes T(V)$$

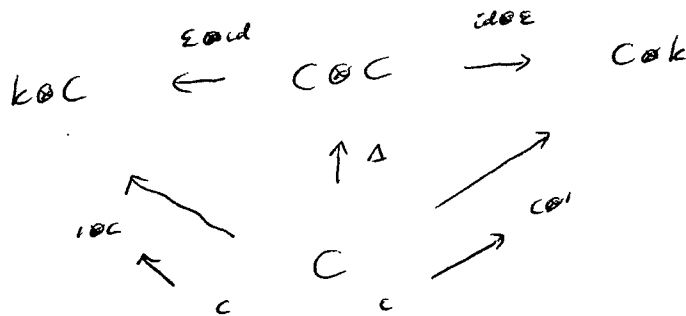
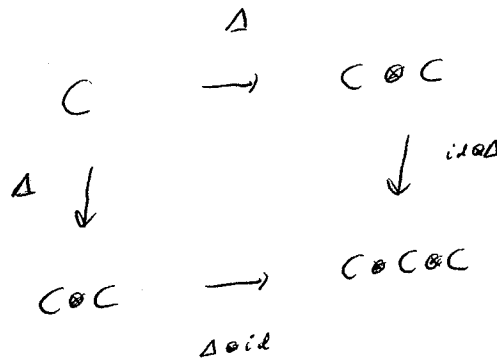
$$v \rightarrow v \otimes 1 + 1 \otimes v$$

this map induces the desired alg morphism.

(ii) sim to (i)

(iii) Write $C = T(V)$

Require these diagrams commute:



Ref to th 22,

For $n \in \mathbb{N}$,

For $v_1, v_2, \dots, v_n \in V$

Consider

$$w = v_1 \otimes v_2 \otimes \dots \otimes v_n \in T(V)$$

Then

$$\begin{aligned} \varepsilon(w) &= \varepsilon(v_1) \varepsilon(v_2) \dots \varepsilon(v_n) \\ &= \begin{cases} 1 & \text{if } n=0 \\ 0 & \text{if } n \geq 1 \end{cases} \end{aligned}$$

Describe $\Delta(w)$

Consider the set

$$\{1, 2, \dots, n\}$$

For a subset

$$S \subseteq \{1, 2, \dots, n\}$$

write

$$S = \{i_1 < i_2 < \dots < i_r\} \quad r = |S|$$

Define

$$w_S = v_{i_1} \otimes v_{i_2} \otimes \dots \otimes v_{i_r} \in T(V)$$

Then

$$\begin{aligned} \Delta(w) &= \Delta(v_1) \Delta(v_2) \dots \Delta(v_n) \\ &= \sum_{S \subseteq \{1, 2, \dots, n\}} w_S \otimes w_{\bar{S}} \end{aligned}$$

↑
complement of S in $\{1, 2, \dots, n\}$

For instance if $n=3$,

and abbrev

$v_1 v_2 v_3 = v_1 \otimes v_2 \otimes v_3$ etc,

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$$\Delta(v_1 v_2 v_3) =$$

$$1 \otimes v_1 v_2 v_3$$

+

$$v_1 \otimes v_2 v_3 + v_2 \otimes v_1 v_3 + v_3 \otimes v_1 v_2$$

+

$$v_1 v_2 \otimes v_3 + v_1 v_3 \otimes v_2 + v_2 v_3 \otimes v_1$$

+

$$v_1 v_2 v_3 \otimes 1$$

Referring to Thm 22, Here is another view of Δ_0

Recall the symmetric group S_n

For $\sigma \in S_n$ and $0 \leq r \leq n$,

call σ an $(r, n-r)$ -shuffle whenever

$$\sigma(1) < \sigma(2) < \dots < \sigma(r)$$

and

$$\sigma(r+1) < \sigma(r+2) < \dots < \sigma(n)$$

Then for $v_1, v_2, \dots, v_n \in V$

$$\Delta(v_1, v_2, \dots, v_n) = \sum_{r=0}^n \sum_{\substack{\sigma \in S_n \\ \text{is an} \\ (r, n-r)\text{-shuffle}}} v_{\sigma(1)} v_{\sigma(2)} \dots v_{\sigma(r)} \otimes v_{\sigma(r+1)} v_{\sigma(r+2)} \dots v_{\sigma(n)}$$

Example

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$$\Delta(1) = 1 \otimes 1$$

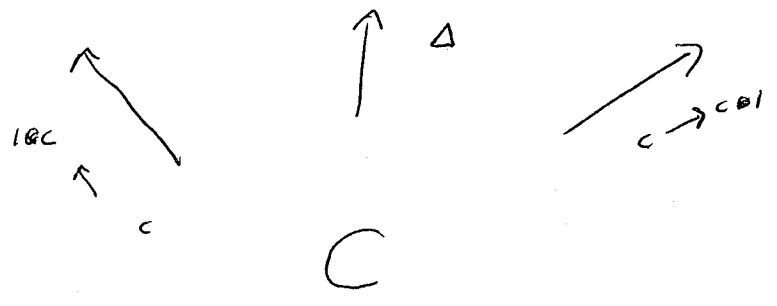
$$\Delta(a) = 1 \otimes a + a \otimes 1$$

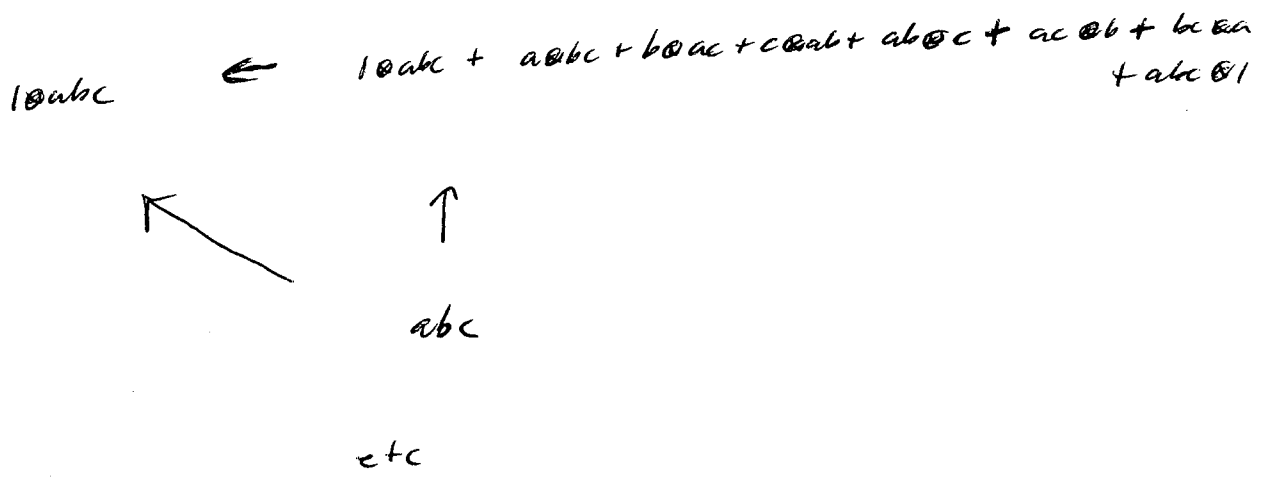
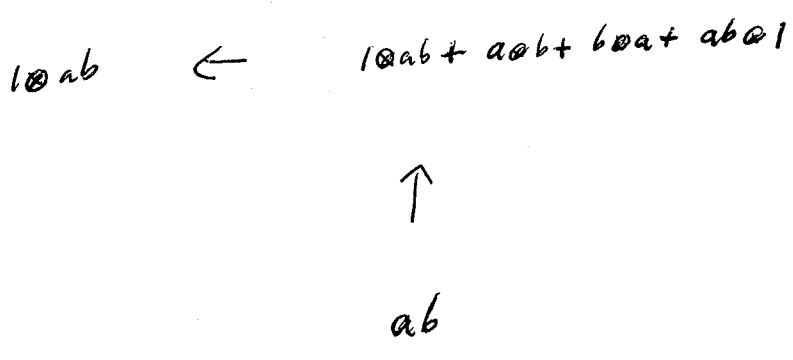
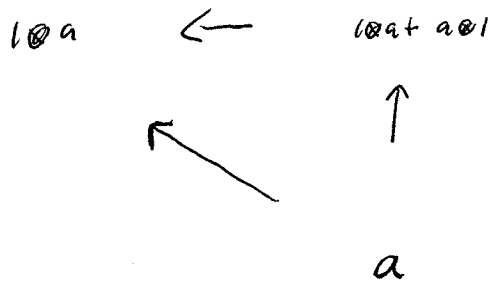
$$\Delta(ab) = 1 \otimes ab + a \otimes b + b \otimes a + ab \otimes 1$$

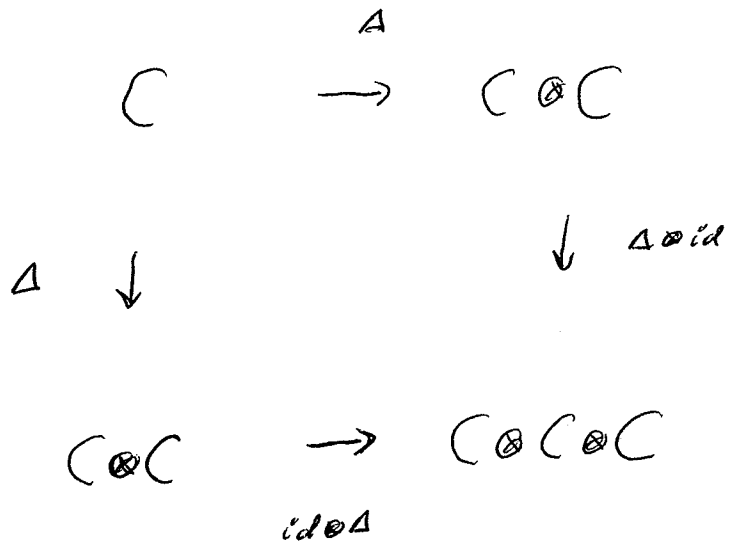
$$\Delta(abc) = 1 \otimes abc + a \otimes bc + b \otimes ac + c \otimes ab \\ + ab \otimes c + ac \otimes b + bc \otimes a + abc \otimes 1$$

check counit, coassoc directly

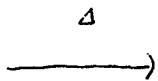
$$k \otimes C \xleftarrow{\varepsilon \otimes \text{id}} C \otimes C \xrightarrow{\text{id} \otimes \varepsilon} C \otimes k$$



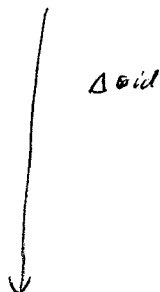




ab



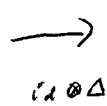
$$1 \otimes ab + a \otimes b + b \otimes a + ab \otimes 1$$



$$\begin{aligned} & (1 \otimes) \otimes ab \\ & + (1 \otimes a + a \otimes 1) \otimes b \\ & + (1 \otimes b + b \otimes 1) \otimes a \\ & + (1 \otimes ab + a \otimes b + b \otimes a + ab \otimes 1) \otimes 1 \end{aligned}$$

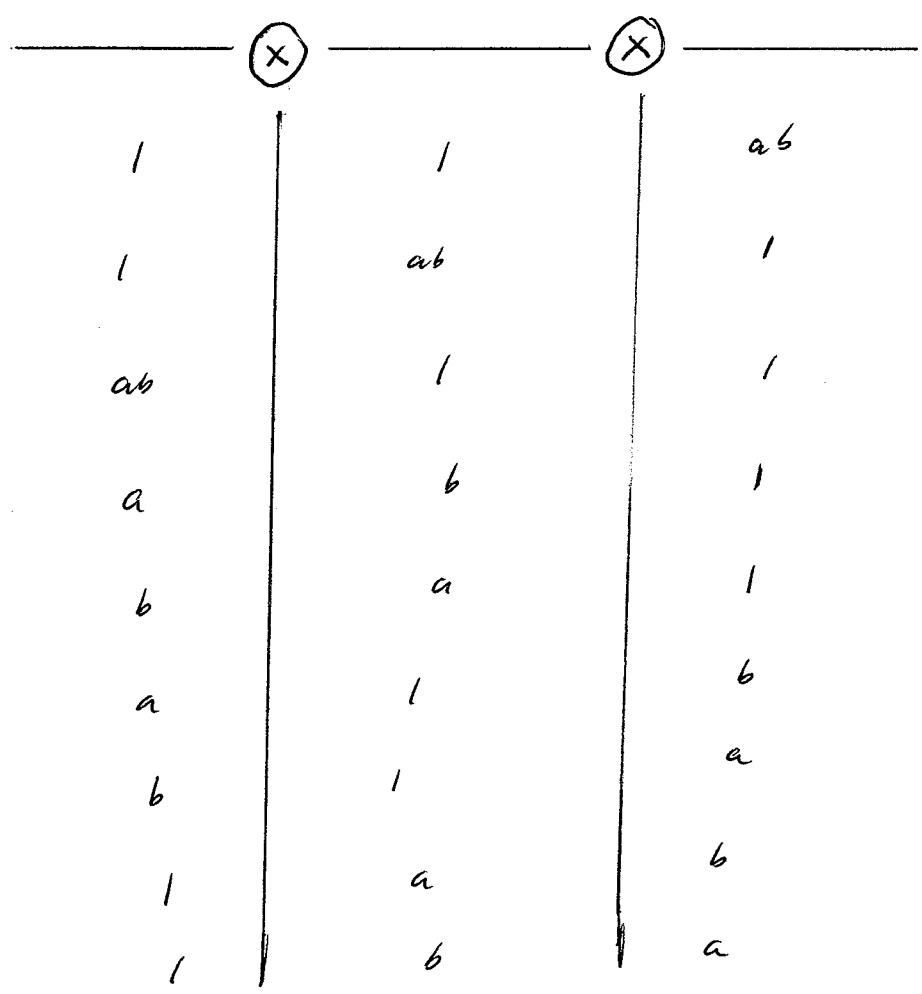


$$1 \otimes ab + a \otimes b + b \otimes a + ab \otimes 1$$



$$\begin{aligned} & 1 \otimes (1 \otimes ab + a \otimes b + b \otimes a + ab \otimes 1) \\ & + a \otimes (1 \otimes b + b \otimes 1) \\ & + b \otimes (1 \otimes a + a \otimes 1) \\ & + ab \otimes (1 \otimes 1) \end{aligned}$$

Both sides equal the sum of the following terms:



this equals the product

$$(a^2 + 1 + ab + ba + 1) (ab + 1 + 1 + 1 + 1 + b + a + b + a)$$

Def 23

Given a bialgebra A

For $a \in A$

a is primitive whenever $\Delta(a) = a \otimes 1 + 1 \otimes a$

Define

$$\text{Prim}(A) = \{ a \in A \mid a \text{ prim} \}$$

subspace of A .

LEM 24 For a bialgebra A ,

each subspace I of $\text{Prim}(A)$ is a coideal of A

pf

show

$$\Delta(I) \subseteq I \otimes A + A \otimes I$$

$\forall a \in I$

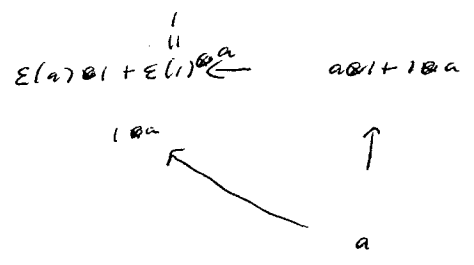
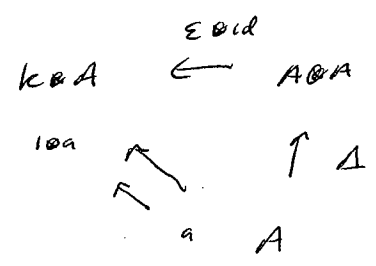
$$\begin{aligned} \Delta(a) &= a \otimes 1 + 1 \otimes a \\ &\in I \otimes A + A \otimes I \end{aligned}$$

show

$$\varepsilon(I) = 0$$

Given $a \in I$ show $\varepsilon(a) = 0$.

this diag commutes:



so $\varepsilon(a) \otimes 1 = 0$

so $\varepsilon(a) = 0$

□

LEM 25 For a bialgebra A and
 $a, b \in \text{Prim}(A)$,

$$ab - ba \in \text{Prim}(A)$$

pf We have

$$\Delta(a) = a \otimes 1 + 1 \otimes a$$

$$\Delta(b) = b \otimes 1 + 1 \otimes b$$

$$\Delta(ab - ba) = \Delta(a)\Delta(b) - \Delta(b)\Delta(a)$$

$$= ab \otimes 1 + a \otimes b + b \otimes a + 1 \otimes ab$$

$$- (ba \otimes 1 + b \otimes a + a \otimes b + 1 \otimes ba)$$

$$= (ab - ba) \otimes 1 + 1 \otimes (ab - ba)$$

✓

□

Note

For a bialgebra A and for
 $P = \text{Prim}(A)$ define

$$[\cdot, \cdot] : \begin{array}{l} P \times P \rightarrow P \\ a \ b \rightarrow ab - ba \end{array}$$

Then

(i) $[\cdot, \cdot]$ is bilinear

(ii) $[a, a] = 0 \quad \forall a \in P$

(iii) $\forall a, b, c \in P,$

$$[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0$$

In other words, $[\cdot, \cdot]$ turns P into a Lie algebra.

pf routine.



Prop 26 Given a bialgebra A

Given $n \in \mathbb{N}$ and primitive

$$a_i \in A \quad i \in \{1, \dots, n\}$$

Given vector space V of dim n

Given basis v_1, v_2, \dots, v_n for V

Then \exists bialgebra morphism

$$\varphi: T(V) \rightarrow A$$

that sends

$$v_i \rightarrow a_i \quad \text{for } i \in \{1, \dots, n\}.$$

pf \exists lin map

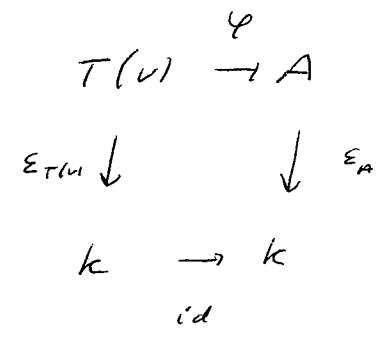
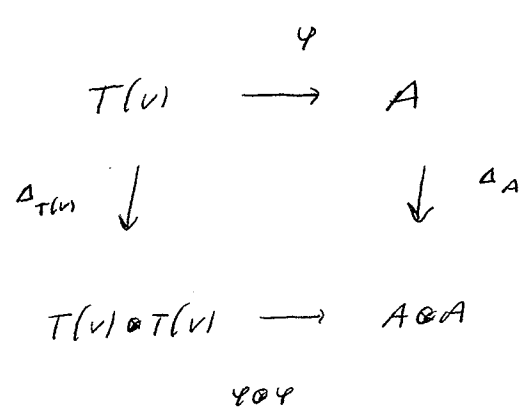
$$\begin{aligned} V &\rightarrow A \\ v_i &\rightarrow a_i \quad i \in \{1, \dots, n\} \end{aligned}$$

Extend this to algebra morphism

$$\varphi: T(V) \rightarrow A$$

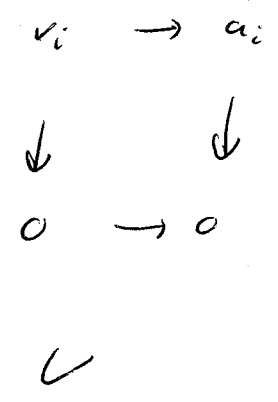
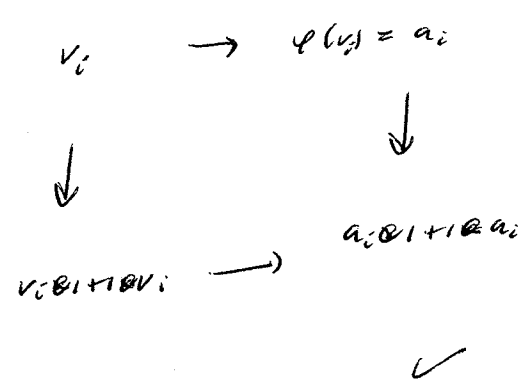
Check φ is coalgebra morphism

Require these comute:



For $w \in T(V)$ chase w around diagrams
 All maps above are algebra morphisms.

$w \in k \subset T(V)$ $w = v_i$ is a generator of $T(V)$



□

The convolution product ★

Motivation:

Start with fin dim'l

algebra A , coalgebra C

Get algebra C^*

Get algebra $A \otimes C^*$

Recall vector space iso

$$A \otimes C^* \cong \text{Hom}(C, A)$$

↑
inherits algebra structure

Describe the algebra $\text{Hom}(C, A)$

Given $f, g \in \text{Hom}(C, A)$,

describe the product $f \star g \in \text{Hom}(C, A)$

Guess:

$$f \star g : C \xrightarrow{\Delta_C} C \otimes C \xrightarrow{f \otimes g} A \otimes A \xrightarrow{\mu_A} A$$

$$c \longrightarrow \sum_{(c')} c' \otimes c'' \longrightarrow \sum_{(c)} f(c') \otimes g(c'') \longrightarrow \sum_{(c)} f(c') g(c'')$$

So

$$(f \star g)(c) = \sum_{(c)} f(c') g(c'') \quad \forall c \in C$$

"convolution product"

Describe the identity $\mathbb{1} \in \text{Hom}(C, A)$

Guess:

$$\mathbb{1} : C \xrightarrow{\epsilon_C} k \xrightarrow{\eta_A} A$$

$$c \longrightarrow \epsilon_C(c) \mathbb{1} \longrightarrow \epsilon_C(c) \eta_A$$

So

$$\Delta(c) = \varepsilon_C(c) 1_A \quad \forall c \in C$$

We will show the guesses are correct.

Turns out the algebra structure on $\text{Hom}(C, A)$ does not require A, C to be fin dim'l.