Given vector space V. Consider tensor algebra $T(V)$.

We now turn $T(V)$ into a bialgebra.

thm 22 Given vector space V.

(i) $\exists$ alg morphism

$$\Delta : T(V) \longrightarrow T(V) \otimes T(V)$$

that sends

$$v \longrightarrow v \otimes 1 + 1 \otimes v \qquad \forall v \in V.$$

(ii) $\exists$ alg morphism

$$\varepsilon : T(V) \longrightarrow k$$

that sends

$$v \longrightarrow 0 \qquad \forall v \in V$$

(iii) Δ, ε turn $T(V)$ into a coalgebra

(iv) The coalgebra $T(V)$ is co-commutative.

(v) The above algebra and coalgebra structures turn $T(V)$ into a bialgebra.

pf (i) $\exists$ lin map

$$V \longrightarrow T(V) \otimes T(V)$$
$$v \longrightarrow v \otimes 1 + 1 \otimes v$$

This map induces the desired alg morphism.

(ii) Sim to (i)

(iii) Write $C = T(V)$

Require these diagrams commute:

$$\begin{tikzcd}
C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes C \arrow[d, "id \otimes \Delta"] \\
C \otimes C \arrow[r, "\Delta \otimes id"'] & C \otimes C \otimes C
\end{tikzcd}$$

$$\begin{tikzcd}
k \otimes C & C \otimes C \arrow[l, "\varepsilon \otimes id"'] \arrow[r, "id \otimes \varepsilon"] & C \otimes k \\
& C \arrow[u, "\Delta"] \arrow[ul, "1 \otimes c \,\leftarrow\!\shortmid\, c"] \arrow[ur, "c \,\shortmid\!\!\rightarrow\, c \otimes 1"'] &
\end{tikzcd}$$

For $c \in C$ chase c around the diagrams.

Each map involved is an alg morphism.

So WLOG $c = v \in V$ is a generator for C

Obs:

$$\begin{tikzcd}
v \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & v\otimes 1 + 1\otimes v \arrow[d, "id\otimes\Delta"] \\
v\otimes 1 + 1\otimes v \arrow[r, "\Delta\otimes id"'] & (v\otimes 1 + 1\otimes v)\otimes 1 + (1\otimes 1)\otimes v
\end{tikzcd}$$

$$\begin{aligned}
v\otimes(1\otimes 1) + 1\otimes(v\otimes 1 + 1\otimes v) &= v\otimes 1\otimes 1 + 1\otimes v\otimes 1 + 1\otimes 1\otimes v \\
&= (v\otimes 1 + 1\otimes v)\otimes 1 + (1\otimes 1)\otimes v
\end{aligned}$$

ok

$$\begin{tikzcd}
\underset{=\,1\otimes v}{\underbrace{\varepsilon(v)}_{0}\otimes 1 + \underbrace{\varepsilon(1)}_{1}\otimes v} & v\otimes 1 + 1\otimes v \arrow[l] \arrow[r] & \underset{=\,v\otimes 1}{v\otimes\underbrace{\varepsilon(1)}_{1} + 1\otimes\underbrace{\varepsilon(v)}_{0}} \\
& v \arrow[u, "\Delta"] \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

ok

(iv) clear from (i)

(v) Since Δ, ε are algebra morphisms, and Th14 $\square$

Ref to Th 22,

For $n \in \mathbb{N}$,

For $v_1, v_2, \ldots, v_n \in V$

Consider

$$w = v_1 \otimes v_2 \otimes \cdots \otimes v_n \in T(V)$$

Then

$$\varepsilon(w) = \varepsilon(v_1)\,\varepsilon(v_2) \cdots \varepsilon(v_n)$$

$$= \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n \geq 1 \end{cases}$$

Describe $\Delta(w)$

Consider the set $\{1, 2, \ldots, n\}$

For a subset $S \subseteq \{1, 2, \ldots, n\}$

write $S = \{ i_1 < i_2 < \cdots < i_r \}$ $\quad r = |S|$

Define $w_S = v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_r} \in T(V)$

Then

$$\Delta(w) = \Delta(v_1)\,\Delta(v_2) \cdots \Delta(v_n)$$

$$= \sum_{S \subseteq \{1, 2, \ldots, n\}} w_S \otimes w_{\bar{S}}$$

$\uparrow$ complement of S in $\{1, 2, \ldots, n\}$

For instance if $n=3$, and abbrev $v_1 v_2 v_3 = v_1 \otimes v_2 \otimes v_3$ etc,

$$\Delta(v_1 v_2 v_3) =$$

$$1 \otimes v_1 v_2 v_3$$

$$+$$

$$v_1 \otimes v_2 v_3 + v_2 \otimes v_1 v_3 + v_3 \otimes v_1 v_2$$

$$+$$

$$v_1 v_2 \otimes v_3 + v_1 v_3 \otimes v_2 + v_2 v_3 \otimes v_1$$

$$+ \quad v_1 v_2 v_3 \otimes 1$$

Referring to Thm 22, Here is another view of Δ.

Recall the symmetric group S_n.

For $\sigma \in S_n$ and $0 \le r \le n$,

call σ an <u>$(r, n-r)$-shuffle</u> whenever

$$\sigma(1) < \sigma(2) < \cdots < \sigma(r)$$

and

$$\sigma(r+1) < \sigma(r+2) < \cdots < \sigma(n)$$

Then for $v_1, v_2, \ldots, v_n \in V$

$$\Delta(v_1 v_2 \cdots v_n) = \sum_{r=0}^{n} \sum_{\substack{\sigma \in S_n \\ \text{is an} \\ (r,n-r)\text{-shuffle}}} v_{\sigma(1)} v_{\sigma(2)} \cdots v_{\sigma(r)} \otimes v_{\sigma(r+1)} v_{\sigma(r+2)} \cdots v_{\sigma(n)}$$

10/16/15
7A

Example

$$\Delta(1) = 1\otimes 1$$

$$\Delta(a) = 1\otimes a + a\otimes 1$$

$$\Delta(ab) = 1\otimes ab + a\otimes b + b\otimes a + ab\otimes 1$$

$$\Delta(abc) = 1\otimes abc + a\otimes bc + b\otimes ac + c\otimes ab$$
$$+ ab\otimes c + ac\otimes b + bc\otimes a + abc\otimes 1$$

check counit, coassoc directly

$$\begin{tikzcd}
k \otimes C & C \otimes C \arrow[l, "\varepsilon \otimes id"'] \arrow[r, "id \otimes \varepsilon"] & C \otimes k \\
& C \arrow[ul, "c \mapsto 1 \otimes c"] \arrow[u, "\Delta"'] \arrow[ur, "c \mapsto c \otimes 1"'] &
\end{tikzcd}$$

10/16/15
9A

$$
\begin{tikzcd}
1\otimes a & 1\otimes a + a\otimes 1 \arrow[l] \\
& a \arrow[u] \arrow[ul]
\end{tikzcd}
$$

$$
\begin{tikzcd}
1\otimes ab & 1\otimes ab + a\otimes b + b\otimes a + ab\otimes 1 \arrow[l] \\
& ab \arrow[u]
\end{tikzcd}
$$

$$
\begin{tikzcd}
1\otimes abc & 1\otimes abc + a\otimes bc + b\otimes ac + c\otimes ab + ab\otimes c + ac\otimes b + bc\otimes a + abc\otimes 1 \arrow[l] \\
& abc \arrow[u] \arrow[ul]
\end{tikzcd}
$$

etc

10/16/15
10A

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes C \arrow[d, "\Delta \otimes id"] \\ C \otimes C \arrow[r, "id \otimes \Delta"'] & C \otimes C \otimes C \end{tikzcd}$$

10/16/15

VA

$$
\begin{tikzcd}
ab \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & 1\otimes ab + a\otimes b + b\otimes a + ab\otimes 1 \arrow[d, "\Delta\otimes id"] \\
1\otimes ab + a\otimes b + b\otimes a + ab\otimes 1 \arrow[r, "id\otimes\Delta"'] & \begin{array}{l} 1\otimes(1\otimes ab + a\otimes b + b\otimes a + ab\otimes 1) \\ + a\otimes(1\otimes b + b\otimes 1) \\ + b\otimes(1\otimes a + a\otimes 1) \\ + ab\otimes(1\otimes 1) \end{array}
\end{tikzcd}
$$

where the right vertical arrow gives

$$
\begin{array}{l} (1\otimes 1)\otimes ab \\ + (1\otimes a + a\otimes 1)\otimes b \\ + (1\otimes b + b\otimes 1)\otimes a \\ + (1\otimes ab + a\otimes b + b\otimes a + ab\otimes 1)\otimes 1 \end{array}
$$

10/16/15
12A

Both sides equal the sum of the following terms:

$\otimes$		$\otimes$	
1	1	ab
1	ab	1
ab	1	1
a	b	1
b	a	1
a	1	b
b	1	a
1	a	b
1	b	a

this equals the product

$$(a\otimes 1\otimes 1 + 1\otimes a\otimes 1 + 1\otimes 1\otimes a)(b\otimes 1\otimes 1 + 1\otimes b\otimes 1 + 1\otimes 1\otimes b)$$

10/16/15

Def 23

Given a bialgebra A

For $a \in A$

a is <u>primitive</u> whenever $\Delta(a) = a \otimes 1 + 1 \otimes a$

Define

$$\mathrm{Prim}(A) = \{ a \in A \,/\, a \text{ prim} \}$$

subspace of A.

LEM 24 For a bialgebra A,

each subspace of $\mathrm{Prim}(A)$ is a coideal of A

$$\underset{I}{\underbrace{\text{subspace}}}$$

pf show

$$\Delta(I) \subseteq I \otimes A + A \otimes I$$

$\forall a \in I$

$$\Delta(a) = a \otimes 1 + 1 \otimes a$$
$$\in I \otimes A + A \otimes I \quad \checkmark$$

show $\varepsilon(I) = 0$

Given $a \in I$. show $\varepsilon(a) = 0$.

this diag commutes:

$$\begin{tikzcd}
k \otimes A & A \otimes A \arrow[l, "\varepsilon \otimes id"'] \\
 & A \arrow[u, "\Delta"'] \arrow[ul]
\end{tikzcd}
\qquad 1 \otimes a \mapsfrom a$$

$$\begin{tikzcd}
\varepsilon(a) \otimes 1 + \underset{1}{\underset{\parallel}{\varepsilon(1)}} \otimes a & a \otimes 1 + 1 \otimes a \arrow[l] \\
1 \otimes a & a \arrow[u] \arrow[ul]
\end{tikzcd}$$

So $\varepsilon(a) \otimes 1 = 0$

so $\varepsilon(a) = 0$

$\square$

LEM 25 For a bialgebra A and $a, b \in Prim(A)$,

$$ab - ba \in Prim(A)$$

pf We have

$$\Delta(a) = a \otimes 1 + 1 \otimes a$$

$$\Delta(b) = b \otimes 1 + 1 \otimes b$$

$$\Delta(ab - ba) = \Delta(a)\Delta(b) - \Delta(b)\Delta(a)$$

$$= ab \otimes 1 + a \otimes b + b \otimes a + 1 \otimes ab$$

$$- (ba \otimes 1 + b \otimes a + a \otimes b + 1 \otimes ba)$$

$$= (ab - ba) \otimes 1 + 1 \otimes (ab - ba) \quad \checkmark$$

$\square$

10/16/15

10

Note For a bialgebra A and for $P = Prim(A)$ define

$$[\,,\,] : \begin{array}{ccc} P \times P & \longrightarrow & P \\ a \quad b & \longmapsto & ab - ba \end{array}$$

Then

(i) $[\,,\,]$ is bilinear

(ii) $[a, a] = 0 \quad \forall a \in P$

(iii) $\forall\, a, b, c \in P,$

$$[a,[b,c]] + [b,[c,a]] + [c,[a,b]] = 0$$

In other words, $[\,,\,]$ turns P into a Lie algebra. $\square$

pf routine.

Prop 26 Given a bialgebra A

Given $n \in \mathbb{N}$ and primitive

$a_i \in A$ $\quad 1 \le i \le n$

Given vector space V of dim n

Given basis $v_1, v_2 \ldots v_n$ for V

Then $\exists$ bialgebra morphism

$\varphi : T(V) \longrightarrow A$

that sends $v_i \longrightarrow a_i$ for $1 \le i \le n$.

pf $\exists$ lin map

$$V \longrightarrow A$$
$$v_i \longrightarrow a_i \qquad 1 \le i \le n$$

Extend this to algebra morphism

$\varphi : T(V) \to A$

Check φ is coalgebra morphism

10/16/15

12

Require these commute:

$$\begin{array}{ccc} T(V) & \xrightarrow{\varphi} & A \\ {\scriptstyle \Delta_{T(V)}}\downarrow & & \downarrow{\scriptstyle \Delta_A} \\ T(V)\otimes T(V) & \xrightarrow[\varphi\otimes\varphi]{} & A\otimes A \end{array} \qquad \begin{array}{ccc} T(V) & \xrightarrow{\varphi} & A \\ {\scriptstyle \varepsilon_{T(V)}}\downarrow & & \downarrow{\scriptstyle \varepsilon_A} \\ k & \xrightarrow[id]{} & k \end{array}$$

For $w \in T(V)$ chase w around diagrams

All maps above are algebra morphisms.

WLOG $w = v_i$ is a generator of $T(V)$

$$\begin{array}{ccc} v_i & \longrightarrow & \varphi(v_i) = a_i \\ \downarrow & & \downarrow \\ v_i\otimes 1 + 1\otimes v_i & \longrightarrow & a_i\otimes 1 + 1\otimes a_i \end{array} \qquad \begin{array}{ccc} v_i & \longrightarrow & a_i \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & 0 \end{array}$$

✓ ✓

□

The convolution product $\star$

Motivation:

Start with fin dim'l

algebra A, coalgebra C

Get algebra C^*

Get algebra $A \otimes C^*$

Recall vector space iso

$$A \otimes C^* \simeq \mathrm{Hom}(C, A)$$

$\uparrow$ inherits algebra structure

Describe ~~the~~ algebra $\mathrm{Hom}(C, A)$

Given $f, g \in \mathrm{Hom}(C, A)$,

describe the product $f \star g \in \mathrm{Hom}(C, A)$

Guess:

$$f \star g : \quad C \xrightarrow{\Delta_C} C \otimes C \xrightarrow{f \otimes g} A \otimes A \xrightarrow{\mu_A} A$$

$$c \longrightarrow \sum_{(c)} c' \otimes c'' \longrightarrow \sum_{(c)} f(c') \otimes g(c'') \to \sum_{(c)} f(c') g(c'')$$

So

$$(f \star g)(c) = \sum_{(c)} f(c') g(c'') \qquad \forall c \in C$$

"Convolution product".

Describe the identity $\mathbb{1} \in \mathrm{Hom}(C, A)$

Guess:

$$\mathbb{1} : \quad C \xrightarrow{\varepsilon_C} k \xrightarrow{\eta_A} A$$

$$c \longrightarrow \varepsilon_C(c)\, 1 \longrightarrow \varepsilon_C(c)\, 1_A$$

So

$$\mathbb{1}(c) = \varepsilon_C(c)\, 1_A \qquad \forall c \in C$$

We will show the guesses are correct.

Turns out the algebra structure on $\mathrm{Hom}(C, A)$ does not require A, C to be fin dim'l.