

Bialgebras

Given a vector space A that is both an algebra and coalgebra

obs $A \otimes A$ is both ...

Coalg A

$$\Delta: A \rightarrow A \otimes A$$

$$\varepsilon: A \rightarrow k$$

Extra Conditions

Δ is algebra morphism I

ε is algebra morphism II

Alg A

$$\mu: A \otimes A \rightarrow A$$

$$\gamma: k \rightarrow A$$

μ is coalgebra morphism III

γ is coalgebra morphism IV

I requirement

$$\forall a, b \in A$$

$$\Delta(ab) = \Delta(a) \Delta(b)$$

$$= \left(\sum_{(a')} a' \otimes a'' \right) \left(\sum_{(b')} b' \otimes b'' \right)$$

$$= \sum_{(a')} \sum_{(b')} a' b' \otimes a'' b''$$

Also

$$\Delta(1) = 1 \otimes 1$$

II requirement

$$\forall a, b \in A$$

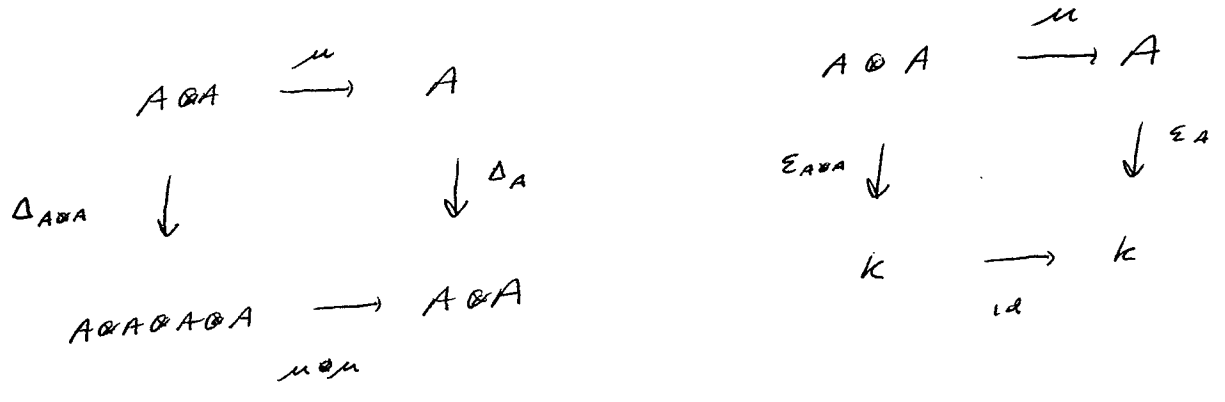
$$\varepsilon(ab) = \varepsilon(a)\varepsilon(b)$$

Also

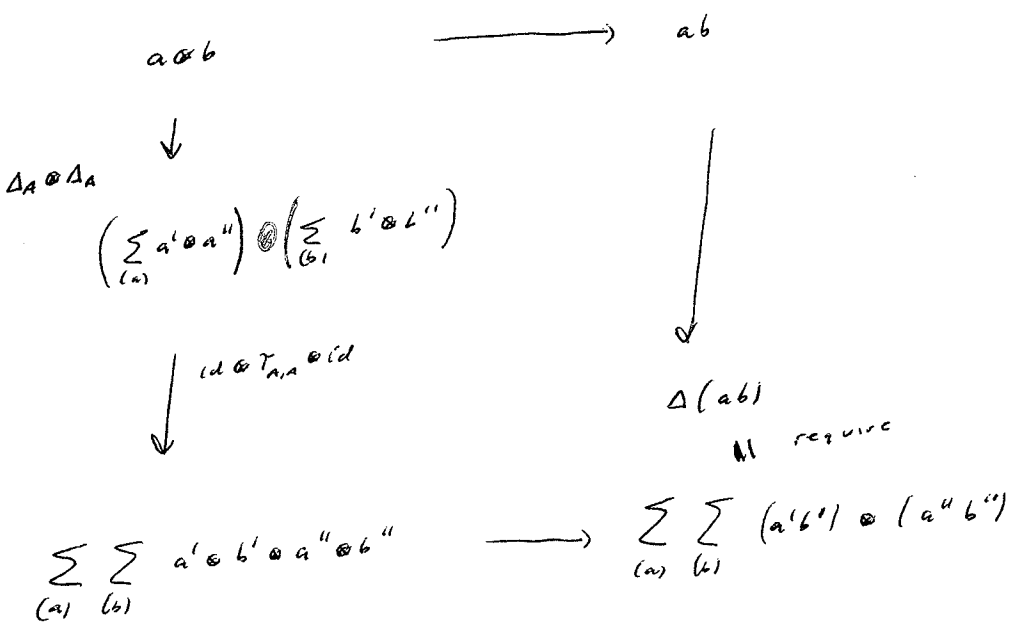
$$\varepsilon(1) = 1$$

III requirement:

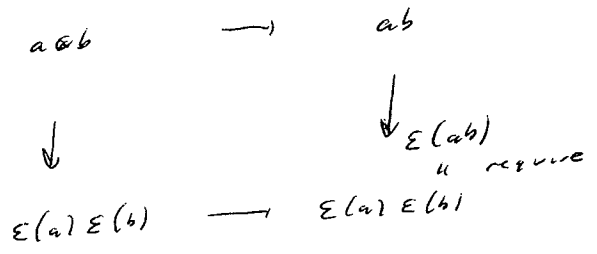
these commute



$\forall a, b \in A$

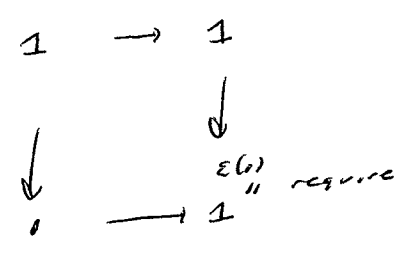
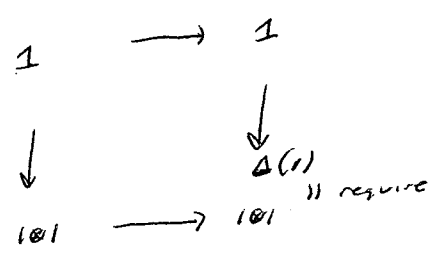
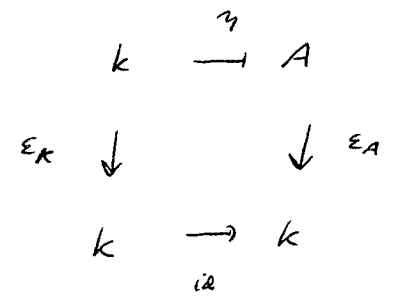
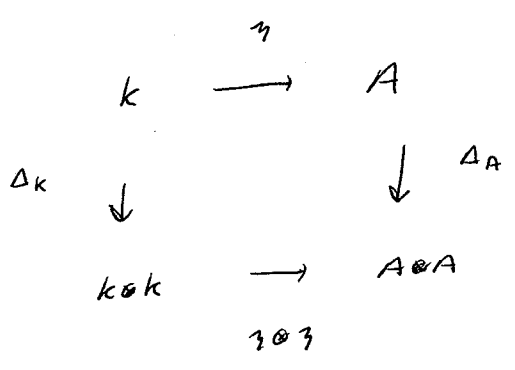


Also



IV requirement :

these commute :



The above calculations show:

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Thm. 14 Given a vector space A that is both an algebra and coalgebra

TFAE:

(i) I and II hold

(ii) III and IV hold

(iii) $\Delta(1) = 1 \otimes 1,$

$\varepsilon(1) = 1$

$$\Delta(ab) = \sum_{(a')} \sum_{(b')} a' b' \otimes a'' b'' \quad \forall a, b \in A$$

$$\varepsilon(ab) = \varepsilon(a) \varepsilon(b)$$

□

DEF 15 Ref to Thm 14, A is called a
bialgebra whenever (i) - (iii) hold.

Ex 16

Recall $M(2)$ is both alg + coalg.

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By const Δ, ϵ are algebra morphisms so $M(2)$

is a bialgebra.

Ex 17 Given a group G ident e

Consider the group algebra KG

So G is a basis for vector space KG .

Via Ex 7, KG becomes coalgebra.

$$\Delta(x) = x \otimes x \quad x \in G$$

$$\varepsilon(x) = 1$$

Now KG is both alg + coalg.

This KG is a bialg.

pt Require

$$\Delta(e) = e \otimes e \quad \text{yes since } e \in G$$

$$\varepsilon(e) = 1 \quad \text{yes}$$

$\forall x, y \in G$

$$\Delta(xy) = xy \otimes xy \quad \text{yes since } xy \in G$$

$$\varepsilon(xy) = \varepsilon(x) \varepsilon(y)$$

$$\begin{matrix} \text{"} & \text{"} & \text{"} \\ \text{"} & \text{"} & \text{"} \\ \text{"} & \text{"} & \text{"} \end{matrix}$$

ok

□

Note 18 In Ex 17, never used the fact that x^{-1} exists

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$\forall x \in G.$

Need only the mult

$$\begin{array}{ccc} G \times G & \rightarrow & G \\ x & \cdot & y \end{array}$$

s.t. $(xy)z = x(yz) \quad \forall x, y, z \in G$ and $\exists e \in G$ s.t.

$ex = xe = x \quad \forall x \in G.$ " G is monoid "

Ex 19 Given set X

Consider free alg $k\{X\}$

Via Ex 7 $k\{X\}$ has coalg str set

$$\Delta(w) = w \otimes w \quad \forall \text{ words } w \text{ on } X$$

$$\epsilon(w) = 1 \quad \dots$$

this turns $k\{X\}$ into a bialgebra.

pf Apply Note 18 to

$G =$ set of all words on X

mult is concatenation.

$$e = \phi.$$

□

Given fin dim'l bialgebra A

<u>A</u>		<u>A^*</u>	
alg	\Rightarrow	coalg	
coalg	\Rightarrow	alg	

Prop 20 the above A^* is a bialgebra

pf Given

$$\Delta(1_A) = 1_A \otimes 1_A \tag{1}$$

$$\varepsilon(1_A) = 1 \tag{2}$$

$\forall a, b \in A$

$$\Delta(ab) = \sum_{(a)} \sum_{(b)} a'b'' \otimes a''b' \tag{3}$$

$$\varepsilon(ab) = \varepsilon(a)\varepsilon(b) \tag{4}$$

Given from constr:

$$\forall f, g \in A^* \quad \forall a \in A,$$

$$(fg)(a) = \sum_a f(a') g(a'')$$

(5)

$$\forall a, b \in A \quad \forall f \in A^*,$$

$$f(ab) = \sum_{(f)} f'(a) f''(b)$$

(6)

$$\text{where } \Delta^*(f) = \sum_f f' \otimes f''$$

$$1_{A^*} = \varepsilon$$

(7)

$$\forall f \in A^*$$

$$\varepsilon^*(f) = f(1_A)$$

(8)

Need:

$$\Delta^*(1_{A^*}) = 1_{A^*} \otimes 1_{A^*}$$

I

$$\varepsilon^*(1_{A^*}) = 1$$

II

$\forall f, g \in A^*$

$$\Delta^*(fg) = \sum_{(f)} \sum_{(g)} f'g' \otimes f''g''$$

III

$$\varepsilon^*(fg) = \varepsilon^*(f) \varepsilon^*(g)$$

IV

I:

$$\Delta^*(\epsilon) \stackrel{?}{=} \epsilon \cup \epsilon$$

(7)

$\forall a, b \in A$

$$\epsilon(ab) \stackrel{?}{=} \epsilon(a) \cup \epsilon(b)$$

(4), (6)



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II

$$\varepsilon^*(\varepsilon) = 1 \quad ?$$

|| (8)

$$\varepsilon(1A)$$

|| (2)

1 ✓

III $\forall f, g \in A^*$

$$\Delta^*(fg) = \sum_{(f')} \sum_{(g')} f' g' @ f'' g''$$

$\forall a, b \in A$

$$(fg)(ab) = \sum_{(f')} \sum_{(g')} \underbrace{(f' g')}(a) \underbrace{(f'' g'')}(b)$$

// (3), (5)

$$\sum_{(a)} f'(a') g'(a'') \quad \sum_{(b)} f''(b') g''(b'')$$

$$\sum_{(a')} \sum_{(b'')} \underbrace{f(a' b'')}_{\parallel (6)} \underbrace{g(a'' b'')}_{\parallel (6)}$$

$$\sum_{(g')} g'(a'') g''(b'')$$

$$\sum_{(f')} f'(a') f''(b')$$

OK.

IV

$\forall f, g \in A^*$

$$\begin{array}{l}
 \varepsilon^*(fg) \stackrel{?}{=} \varepsilon^*(f) \varepsilon^*(g) \\
 \parallel (8) \qquad \qquad \parallel (8) \\
 (fg)(1A) \qquad \qquad f(1A) \quad g(1A) \\
 \parallel (1), (5) \\
 f(1A) g(1A) \qquad \qquad \text{ok}
 \end{array}$$



LEM 21

Given bialgebra A

(i) If we replace the algebra A by its opposite, then the result is a bialgebra (called A^{op})

(ii) If we replace the coalgebra A by its opposite, then the result is a bialgebra (called A^{cop})

pf

Given

$$\Delta(1_A) = 1_A \otimes 1_A$$

$$\varepsilon(1_A) = 1$$

$\forall a, b \in A$

$$\Delta(ab) = \sum_{(a)} \sum_{(b)} a' b' \otimes a'' b''$$

$$\varepsilon(ab) = \varepsilon(a) \varepsilon(b)$$

(i)

In A^{op}

$$1_{A^{op}} = 1_A$$

Δ, ε unchanged

need

$$\Delta \left(\underset{op}{\underset{\uparrow}{a}} \underset{op}{\underset{\uparrow}{b}} \right) = \sum_{(a)} \sum_{(b)} \underset{op}{\underset{\uparrow}{a'}} \underset{op}{\underset{\uparrow}{b'}} \otimes \underset{op}{\underset{\uparrow}{a''}} \underset{op}{\underset{\uparrow}{b''}}$$

$$\Delta(ba) = \sum_{(a)} \sum_{(b)} b' a' \otimes b'' a'' \quad \text{in orig bialg } A$$

this is * with $a \leftrightarrow b$

$$\varepsilon(ab) \stackrel{?}{=} \varepsilon(a) \varepsilon(b)$$

\uparrow
op

$$\varepsilon(ba) \stackrel{?}{=} \varepsilon(a) \varepsilon(b) \quad \text{in any bialg } A$$

this is ** with $a \leftrightarrow b$

(iii) In A^{cop}

$$1_{A^{cop}} = 1_A$$

ε unchanged

$$\Delta^{op}(1_A) = 1_A \otimes 1_A \quad \checkmark$$

$\forall a, b \in A$

need

$$\Delta^{op}(ab) \stackrel{?}{=} \sum_{(a)} \sum_{(b)} a'' b'' \otimes a' b'$$

yes by constr.

where

$$\Delta^{op}(a) = \sum_{(a)} a'' \otimes a'$$

$$\Delta^{op}(b) = \sum_{(b)} b'' \otimes b'$$

\square

Given monoid G as in Note 18,
assume $|G| < \infty$.

Recall $V = kG$ is bialgebra.
Describe the bialgebra V^* .

Algebra V^* : We saw

$$\forall f, g \in V^*$$

$$(fg)(x) = f(x)g(x) \quad \forall x \in G$$

Coalg V^* :

Let $G^* = \{x^* \mid x \in G\}$ denote the basis for V^*
dual to the basis G for V .

Get

$$\Delta_{V^*} : \begin{array}{ccc} V^* & \longrightarrow & V^* \otimes V^* \\ z^* & \longrightarrow & \sum_{\substack{x, y \in G \\ xy = z}} x^* \otimes y^* \end{array} \quad \forall z \in G$$

check: $\forall a, b \in G$

$$\begin{aligned} z^*(ab) & \stackrel{?}{=} \sum_{\substack{x, y \in G \\ xy = z}} \underbrace{x^*(a)}_{\delta_{x,a}} \underbrace{y^*(b)}_{\delta_{y,b}} \\ \parallel & \\ \delta_{z,ab} & \underbrace{\hspace{10em}}_{\parallel} \\ & \delta_{z,ab} \\ & \text{ok} \end{aligned}$$

To clarify Δ_{V^*} , consider the linear map

$$\begin{array}{ccc}
 V^* & \longrightarrow & V^* \otimes V^* & \longrightarrow & (V \otimes V)^* & \star \\
 & & \Delta_{V^*} & & g \otimes f & \rightarrow H \\
 & & & & H(u \otimes v) = f(u)g(v) &
 \end{array}$$

Describe \star :

For $z \in G$, \star sends

$$z^* \longrightarrow \sum_{\substack{x, y \in G \\ xy = z}} x^* \otimes y^* \longrightarrow z$$

Find Z :

For $a, b \in G$,

$$\begin{aligned}
 Z(a \otimes b) &= \sum_{\substack{x, y \in G \\ xy = z}} \underbrace{y^*(a)}_{\delta_{y,a}} \underbrace{x^*(b)}_{\delta_{x,b}} \\
 &= \begin{cases} 1 & \text{if } ba = z \\ 0 & \text{if } ba \neq z \end{cases} \\
 &= z^*(ba)
 \end{aligned}$$

So for $f \in V^*$ \star sends

$$f \rightarrow F$$

where

$$F(a \otimes b) = f(ba)$$

$\forall a, b \in G$

Given vector space V . Consider tensor algebra $T(V)$.

We now turn $T(V)$ into a bialgebra.

Thm 22 Given vector space V .

(i) \exists alg morphism

$$\Delta: T(V) \rightarrow T(V) \otimes T(V)$$

that sends

$$v \rightarrow v \otimes 1 + 1 \otimes v \quad \forall v \in V.$$

(ii) \exists alg morphism

$$\varepsilon: T(V) \rightarrow k$$

that sends

$$v \rightarrow 0 \quad \forall v \in V$$

(iii) Δ, ε turn $T(V)$ into a coalgebra

(iv) the coalgebra $T(V)$ is co-commutative.

(v) the above algebra and coalgebra structures turn $T(V)$ into a bialgebra.