Lecture 18 Wednesday Oct 14

Bialgebras

Given a vector space A that is both an algebra and coalgebra

obs $A \otimes A$ is both ...

Coalg A	Extra Conditions	
$\Delta : A \to A \otimes A$	Δ is algebra morphism	I
$\varepsilon : A \to k$	ε is algebra morphism	II
Alg A		
$\mu : A \otimes A \to A$	μ is coalgebra morphism	III
$\eta \quad k \to A$	η is coalgebra morphism	IV

I requirement

$\forall\, a, b \in A$

$$\Delta(ab) = \Delta(a)\,\Delta(b)$$

$$= \left(\sum_{(a)} a' \otimes a''\right)\left(\sum_{(b)} b' \otimes b''\right)$$

$$= \sum_{(a)} \sum_{(b)} a'b' \otimes a''b''$$

Also

$$\Delta(1) = 1 \otimes 1$$

II requirement

$\forall\ a, b \in A$

$$\varepsilon(ab) = \varepsilon(a)\,\varepsilon(b)$$

Also

$$\varepsilon(1) = 1$$

III requirement:

these commute

$$\begin{tikzcd}
A\otimes A \arrow[r, "\mu"] \arrow[d, "\Delta_{A\otimes A}"'] & A \arrow[d, "\Delta_A"] \\
A\otimes A\otimes A\otimes A \arrow[r, "\mu\otimes\mu"'] & A\otimes A
\end{tikzcd}
\qquad
\begin{tikzcd}
A\otimes A \arrow[r, "\mu"] \arrow[d, "\varepsilon_{A\otimes A}"'] & A \arrow[d, "\varepsilon_A"] \\
k \arrow[r, "id"'] & k
\end{tikzcd}$$

$\forall\, a,b \in A$

$$\begin{tikzcd}
a\otimes b \arrow[r] \arrow[d, "\Delta_A\otimes\Delta_A"'] & ab \arrow[dd] \\
\left(\sum_{(a)} a'\otimes a''\right)\otimes\left(\sum_{(b)} b'\otimes b''\right) \arrow[d, "id\otimes\tau_{A,A}\otimes id"] & \\
\sum_{(a)}\sum_{(b)} a'\otimes b'\otimes a''\otimes b'' \arrow[r] & \Delta(ab) \overset{\text{require}}{=} \sum_{(a)}\sum_{(b)} (a'b')\otimes(a''b'')
\end{tikzcd}$$

Also

$$\begin{tikzcd}
a\otimes b \arrow[r] \arrow[d] & ab \arrow[d] \\
\varepsilon(a)\varepsilon(b) \arrow[r] & \varepsilon(ab) \overset{\text{require}}{=} \varepsilon(a)\varepsilon(b)
\end{tikzcd}$$

IV requirement :

these commute :

$$\begin{tikzcd} k \arrow[r, "\eta"] \arrow[d, "\Delta_k"'] & A \arrow[d, "\Delta_A"] \\ k \otimes k \arrow[r, "\eta \otimes \eta"'] & A \otimes A \end{tikzcd}$$

$$\begin{tikzcd} k \arrow[r, mapsto, "\eta"] \arrow[d, "\varepsilon_k"'] & A \arrow[d, "\varepsilon_A"] \\ k \arrow[r, "id"'] & k \end{tikzcd}$$

$$\begin{tikzcd} 1 \arrow[r] \arrow[dd] & 1 \arrow[d] \\ & \Delta(1) \arrow[d, equal, "\text{require}"] \\ 1 \otimes 1 \arrow[r] & 1 \otimes 1 \end{tikzcd}$$

$$\begin{tikzcd} 1 \arrow[r] \arrow[dd] & 1 \arrow[d] \\ & \varepsilon(1) \arrow[d, equal, "\text{require}"] \\ 1 \arrow[r, mapsto] & 1 \end{tikzcd}$$

The above calculations show:

thm. 14 Given a vector space A that is both an algebra and coalgebra

TFAE:

(i) I and II hold

(ii) III and IV hold

(iii) $\Delta(1) = 1 \otimes 1,$

$\varepsilon(1) = 1$

$$\Delta(ab) = \sum_{(a)} \sum_{(b)} a'b' \otimes a''b'' \qquad \forall\, a, b \in A$$

"

$$\varepsilon(ab) = \varepsilon(a)\,\varepsilon(b)$$

$\square$

— o —

DEF 15 Ref to thm 14, A is called a <u>bialgebra</u> whenever (i)—(iii) hold.

Ex 16 Recall $M(2)$ is both alg + coalg. 10/14/15

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By constr Δ + ε are algebra morphisms so $M(2)$ is a bialgebra.

Ex 17 Given a group G ident e

Consider the group algebra kG

So G is a basis for vector space kG.

Via Ex 7, kG becomes coalgebra.

$\Delta(x) = x \otimes x \qquad x \in G$

$\varepsilon(x) = 1 \qquad$ ..

Now kG is both alg + coalg.

This kG is a bialg.

pf Require

$\Delta(e) \overset{?}{=} e \otimes e$ yes since $e \in G$

$\varepsilon(e) = 1$ yes ..

$\forall\, x, y \in G$

$\Delta(xy) \overset{?}{=} xy \otimes xy$ yes since $xy \in G$

$$\begin{array}{ccc} \varepsilon(xy) = & \varepsilon(x) & \varepsilon(y) \\ \| & \| & \| \\ 1 & 1 & 1 \end{array}$$

ok $\square$

Note 18 In Ex 17, never used the fact that x^{-1} exists $\forall x \in G$.

Need only the mult

$$\begin{array}{ccc} G \times G & \longrightarrow & G \\ x \quad y & \longrightarrow & xy \end{array}$$

s.t $(xy)z = x(yz)$ $\forall x, y, z \in G$ and $\exists\, e \in G$ s.t. $ex = xe = x$ $\forall x \in G$. "G is monoid"

Ex 19 Given set X

Consider free alg $k\{X\}$

Via Ex 7 $k\{X\}$ has coalg str s.t.

$\Delta(w) = w \otimes w$ $\forall$ words w on X

$\varepsilon(w) = 1$

this turns $k\{X\}$ into a bialgebra.

pf Apply Note 18 to

G = set of all words on X

mult is concatenation.

$e = \phi$.

$\square$

Given fin dim'l bialgebra A

A		A^*
alg	$\Rightarrow$	coalg
coalg	$\Rightarrow$	alg

Prop 20 the above A^* is a bialgebra

pf Given

$$\Delta(1_A) = 1_A \otimes 1_A \tag{1}$$

$$\varepsilon(1_A) = 1 \tag{2}$$

$\forall a, b \in A$

$$\Delta(ab) = \sum_{(a)} \sum_{(b)} a'b' \otimes a''b'' \tag{3}$$

$$\varepsilon(ab) = \varepsilon(a)\,\varepsilon(b) \tag{4}$$

Given from constr :

$\forall\, f, g \in A^*\quad \forall a \in A,$

$$(fg)(a) = \sum_{a} f(a')\, g(a'') \qquad (5)$$

$\forall\, a, b \in A\quad \forall f \in A^*,$

$$f(ab) = \sum_{(f)} f'(a)\, f''(b) \qquad (6)$$

where $\Delta^*(f) = \sum_{f} f' \otimes f''$

$$1_{A^*} = \varepsilon \qquad (7)$$

$\forall f \in A^*$

$$\varepsilon^*(f) = f(1_A) \qquad (8)$$

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Need:

$$\Delta^*(1_{A^*}) = 1_{A^*} \otimes 1_{A^*}$$ I

$$\varepsilon^*(1_{A^*}) = 1$$ II

$\forall\, f, g \in A^*$

$$\Delta^*(fg) = \sum_{(f)} \sum_{(g)} f'g' \otimes f''g''$$ III

$$\varepsilon^*(fg) = \varepsilon^*(f)\,\varepsilon^*(g)$$ IV

I :

$$\Delta^*(\varepsilon) \overset{?}{\underset{(7)}{=}} \varepsilon \otimes \varepsilon$$

$\forall\, a, b \in A$

$$\varepsilon(ab) \overset{?}{\underset{(4),(6)}{=}} \varepsilon(a)\,\varepsilon(b) \qquad \checkmark$$

II

$$\begin{array}{l} \mathcal{E}^*(\varepsilon) \overset{?}{=} 1 \\ \parallel \ (8) \\ \mathcal{E}(1_A) \\ \parallel \ (2) \\ 1 \quad \checkmark \end{array}$$

III $\forall\, f, g \in A^*$

$$\Delta^*(fg) \overset{?}{=} \sum_{(f)} \sum_{(g)} f'g' \otimes f''g''$$

$\forall\, a, b \in A$

$$(fg)(ab) \underset{(6)}{\overset{?}{=}} \sum_{(f)} \sum_{(g)} \underbrace{(f'g')(a)}_{\| (5)} \underbrace{(f''g'')(b)}_{\| (5)}$$

$$\| (3), (5) \qquad \sum_{(a)} f'(a')g'(a'') \quad \sum_{(b)} f''(b')g''(b'')$$

$$\sum_{(a)} \sum_{(b)} \underbrace{f(a'b')}_{\| (6)} \underbrace{g(a''b'')}_{\| (6)}$$

$$\sum_{(g)} g'(a'')g''(b'')$$

$$\sum_{(f)} f'(a')f''(b')$$

ok.

IV $\quad \forall f, g \in A^*$

$$
\begin{array}{ccc}
\varepsilon^*(fg) & \overset{?}{=} & \varepsilon^*(f) \quad \varepsilon^*(g) \\
\|^{(8)} & & \|^{(8)} \qquad \|^{(8)} \\
(fg)(1_A) & & f(1_A) \qquad g(1_A) \\
\|^{(1),(5)} & & \\
f(1_A)\, g(1_A) & & \text{ok}
\end{array}
$$

$\square$

LEM 21
Given bialgebra A

(i) If we replace the algebra A by its opposite, then the result is a bialgebra (called A^{op})

(ii) If we replace the coalgebra A by its opposite, then the result is a bialgebra (called A^{cop})

pf Given

$\Delta(1_A) = 1_A \otimes 1_A$

$\varepsilon(1_A) = 1$

$\forall a, b \in A$

$\Delta(ab) = \sum_{(a)} \sum_{(b)} a'b' \otimes a''b''$ *

$\varepsilon(ab) = \varepsilon(a)\,\varepsilon(b)$ **

(i)
In A^{op}

$1_{A^{op}} = 1_A$

Δ, ε unchanged

Need $\Delta(a \underset{\text{opp}}{\uparrow} b) \overset{?}{=} \sum_{(a)} \sum_{(b)} a' \underset{\text{op}}{\uparrow} b' \otimes a'' \underset{\text{op}}{\uparrow} b''$

$\Delta(ba) \overset{?}{=} \sum_{(a)} \sum_{(b)} b'a' \otimes b''a''$ in orig bialg A

This is * with $a \leftrightarrow b$

$$\varepsilon(ab) \overset{?}{=} \varepsilon(a)\,\varepsilon(b)$$

$\uparrow$ op

$$\varepsilon(ba) \overset{?}{=} \varepsilon(a)\,\varepsilon(b) \qquad \text{in orig bialg } A$$

this is ** with $a \leftrightarrow b$

(iii) In A^{cop}

$$1_{A^{cop}} = 1_A$$

ε unchanged

$$\Delta^{op}(1_A) = 1_A \otimes 1_A \qquad \checkmark$$

$\forall\, a, b \in A$

need

$$\Delta^{op}(ab) \overset{?}{=} \sum_{(a)} \sum_{(b)} a''b'' \otimes a'b' \qquad \text{yes by constr.}$$

where

$$\Delta^{op}(a) = \sum_{(a)} a'' \otimes a'$$

$$\Delta^{op}(b) = \sum_{(b)} b'' \otimes b'$$

$\square$

Given monoid G as in Note 18,

assume $|G| < \infty$.

Recall $V = kG$ is bialgebra.

Describe the bialgebra V^*

<u>Algebra V^*</u>: We saw

$\forall f, g \in V^*$

$(fg)(x) = f(x)\, g(x) \qquad \forall x \in G$

<u>Coalg V^*</u>:

let $G^* = \{x^* / x \in G\}$ denote the basis for V^*

dual to the basis G for V.

Get

$$\Delta_{V^*} : \begin{array}{ccc} V^* & \longrightarrow & V^* \otimes V^* \\ z^* & \longrightarrow & \displaystyle\sum_{\substack{x, y \in G \\ xy = z}} x^* \otimes y^* \end{array} \qquad \forall z \in G$$

check: $\forall a, b \in G$

$$\underset{\substack{\| \\ \delta_{z,ab}}}{z^*(ab)} \overset{?}{=} \underbrace{\sum_{\substack{x, y \in G \\ xy = z}} \underbrace{x^*(a)}_{\delta_{x,a}} \underbrace{y^*(b)}_{\delta_{y,b}}}_{\delta_{z,ab}}$$

$$\underset{ok}{=}$$

To clarify Δ_{V^*}, consider the linear map

$$V^* \xrightarrow{\Delta_{V^*}} V^* \otimes V^* \longrightarrow (V \otimes V)^* \qquad ✩$$

$g \otimes f \longrightarrow H$

$H(u \otimes v) = f(u) g(v)$

Describe ✩:

For $z \in G$, ✩ sends

$$z^* \longrightarrow \sum_{\substack{x, y \in G \\ xy = z}} x^* \otimes y^* \longrightarrow Z$$

Find Z:

For $a, b \in G$,

$$Z(a \otimes b) = \sum_{\substack{x, y \in G \\ xy = z}} \underbrace{y^*(a)}_{\delta_{y, a}} \underbrace{x^*(b)}_{\delta_{x, b}}$$

$$= \begin{cases} 1 & \text{if } ba = z \\ 0 & \text{if } ba \neq z \end{cases}$$

$$= z^*(ba)$$

So for $f \in V^*$ ✩ sends

$$f \to F$$

where

$$F(a \otimes b) = f(ba) \qquad \forall\, a, b \in G$$

Given vector space V. Consider tensor algebra $T(V)$.

We now turn $T(V)$ into a bialgebra.

thm 22 Given vector space V.

(i) $\exists$ alg morphism

$$\Delta : T(V) \longrightarrow T(V) \otimes T(V)$$

that sends

$$v \longrightarrow v \otimes 1 + 1 \otimes v \qquad \forall v \in V.$$

(ii) $\exists$ alg morphism

$$\varepsilon : T(V) \longrightarrow k$$

that sends

$$v \longrightarrow 0 \qquad \forall v \in V$$

(iii) Δ, ε turn $T(V)$ into a coalgebra

(iv) the coalgebra $T(V)$ is co-commutative.

(v) the above algebra and coalgebra structures turn $T(V)$ into a bialgebra.