

LEC 17 Monday Oct 12

10/12/15

1

Given vector spaces  $V \neq 0, W \neq 0$

Given bases  $X$  for  $V, Y$  for  $W$

Ex 7 yields

Coalgebras  $V, W$

I

Prop 9 yields

Coalgebra  $V \otimes W$

II

Using the

basis  $\{x \otimes y \mid x \in X, y \in Y\}$  for  $V \otimes W$

Ex 7 yields

Coalgebra  $V \otimes W$

III

Compare II, III

LEM 10 the above coalgebras II, III are the same.

pt (ex)

(detail)

I

$$\Delta_V: V \rightarrow V \otimes V$$

$$x \rightarrow x \otimes x$$

$$\Delta_W: W \rightarrow W \otimes W$$

$$y \rightarrow y \otimes y$$

$$\epsilon_V: V \rightarrow K$$

$$x \rightarrow 1$$

$$\epsilon_W: W \rightarrow K$$

$$y \rightarrow 1$$

II

$$\Delta: V \otimes W \rightarrow V \otimes V \otimes W \otimes W \rightarrow V \otimes W \otimes V \otimes W$$

$$x \otimes y \rightarrow x \otimes x \otimes y \otimes y \rightarrow (x \otimes y) \otimes (x \otimes y)$$

$$\epsilon: V \otimes W \rightarrow K \otimes K \rightarrow K$$

$$x \otimes y \rightarrow 1 \otimes 1 \rightarrow 1$$

III

$$\Delta: V \otimes W \rightarrow V \otimes W \otimes V \otimes W$$

$$x \otimes y \rightarrow (x \otimes y) \otimes (x \otimes y)$$

$$\epsilon: V \otimes W \rightarrow K$$

$$x \otimes y \rightarrow 1$$



Given fin dim'l algebras  $A, B$

Get coalgebras  $A^*, B^*$

Given algebra morphism  $\varphi: A \rightarrow B$ .

LEM 11 The transpose

$$\varphi^*: B^* \rightarrow A^*$$

is a coalgebra morphism.

pf  $\forall f \in A^*$  write

$$\Delta_{A^*}(f) = \sum_i f_i' \otimes f_i''$$

$\forall g \in B^*$  write

$$\Delta_{B^*}(g) = \sum_j g_j' \otimes g_j''$$

$\forall a_1, a_2 \in A$

$$f(a_1, a_2) = \sum_i f_i'(a_1) f_i''(a_2)$$

$\forall b_1, b_2 \in B$

$$g(b_1, b_2) = \sum_j g_j'(b_1) g_j''(b_2)$$

Take

$$b_1 = \varphi(a_1) \quad b_2 = \varphi(a_2)$$

Also  $\forall a \in A$

$$\langle g, \varphi(a) \rangle = \langle \varphi^*(g), a \rangle$$

check this diag commutes:

$$\begin{array}{ccc}
 B^* & \xrightarrow{\varphi^*} & A^* \\
 \Delta_{B^*} \downarrow & & \downarrow \Delta_{A^*} \\
 B^* \otimes B^* & \xrightarrow{\varphi^* \otimes \varphi^*} & A^* \otimes A^*
 \end{array}$$

$$\begin{array}{ccc}
 g & \longrightarrow & \varphi^*(g) \\
 \downarrow & & \downarrow \Delta_{A^*}(\varphi^*(g)) \\
 \sum_j g_j^i \otimes g_j^{ii} & \longrightarrow & \sum_j \varphi^*(g_j^i) \otimes \varphi^*(g_j^{ii})
 \end{array}$$

Require:

$$\begin{array}{l}
 \varphi^*(g)(a_1, a_2) \stackrel{?}{=} \sum_j \underbrace{\varphi^*(g_j^i)(a_1)}_{\langle \varphi^*(g_j^i), a_1 \rangle} \underbrace{\varphi^*(g_j^{ii})(a_2)}_{\langle \varphi^*(g_j^{ii}), a_2 \rangle} \\
 \parallel \\
 \langle \varphi^*(g), a_1, a_2 \rangle \\
 \parallel \\
 \langle g, \varphi(a_1), \varphi(a_2) \rangle \\
 \parallel \\
 \underbrace{\langle g, \varphi(a_1), \varphi(a_2) \rangle}_{\langle g, b_1, b_2 \rangle} \\
 \parallel \\
 \sum_j \underbrace{g_j^i(b_1)}_{b_1} \underbrace{g_j^{ii}(b_2)}_{b_2}
 \end{array}$$

// ✓

check this diag commutes:

$$\begin{array}{ccc}
 B^* & \xrightarrow{\varphi^*} & A^* \\
 \varepsilon_{B^*} \downarrow & & \downarrow \varepsilon_{A^*} \\
 K & \xrightarrow{id} & K
 \end{array}$$

$$\begin{array}{ccc}
 g & \longrightarrow & \varphi^*(g) \\
 \downarrow & & \downarrow \\
 g(1_B) & \xrightarrow{?} & \varphi^*(g)(1_A)
 \end{array}$$

$$\begin{array}{ccc}
 g(1_B) & \stackrel{?}{=} & \varphi^*(g)(1_A) \\
 \parallel & & \parallel \\
 \langle g, 1_B \rangle & & \langle \varphi^*(g), 1_A \rangle \\
 \stackrel{=}{\sim} & & \parallel \\
 \checkmark & & \langle g, \varphi(1_A) \rangle \\
 & & \parallel \\
 & & 1_B
 \end{array}$$

□

Given coalgebras  $A, B$

Get algebras  $A^*, B^*$

Given coalgebra morphism

$$\varphi: A \rightarrow B$$

LEM 12 the transpose

$$\varphi^*: B^* \rightarrow A^*$$

is an algebra morphism.

Pf  $\forall a \in A$  write

$$\Delta_A(a) = \sum_i a_i' \otimes a_i''$$

write

$$b = \varphi(a)$$

$$b_i' = \varphi(a_i') \quad \forall i$$

$$b_i'' = \varphi(a_i'')$$

We have

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ \Delta_A \downarrow & & \downarrow \Delta_B \\ A \otimes A & \xrightarrow{\varphi \otimes \varphi} & B \otimes B \end{array}$$

Commutative

Given

$$A_B(b) = \sum_i b_i' \otimes b_i''$$

By construction.  $\forall f, g \in A^*$

$$(fg)(a) = \sum_i f(a_i') g(a_i'')$$

Also  $\forall F, G \in B^*$

$$(FG)(b) = \sum_i F(b_i') G(b_i'')$$

Also  $\forall H \in B^*$

$$\langle H, \varphi(a) \rangle = \langle \varphi^*(H), a \rangle$$

check  $\varphi^*(FG) \stackrel{?}{=} \varphi^*(F) \varphi^*(G)$

require  $\varphi^*(FG)(a) \stackrel{?}{=} (\varphi^*(F) \varphi^*(G))(a)$

"		
$\langle \varphi^*(FG), a \rangle$	"	$\sum_i \underbrace{\varphi^*(F)(a_i')} \underbrace{\varphi^*(G)(a_i'')}$
"		$\langle \varphi^*(F), a_i' \rangle \langle \varphi^*(G), a_i'' \rangle$
$\langle FG, \varphi(a) \rangle$	$\langle F, \varphi(a_i') \rangle$	$\langle G, \varphi(a_i'') \rangle$
"	$\langle F, \varphi(a_i') \rangle$	$\langle G, \varphi(a_i'') \rangle$
$(FG)(b)$	$F(b_i')$	$G(b_i'')$
"	"	
$\sum_i F(b_i') G(b_i'')$	$\sum_i F(b_i') G(b_i'')$	
	"	

check

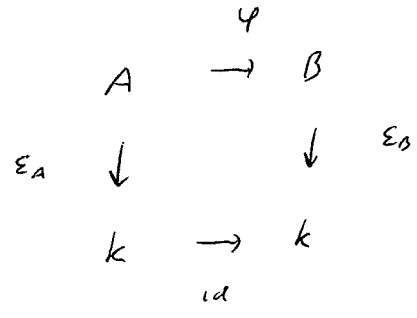
$$\varphi^*(1_{B^*}) = 1_{A^*}$$

Recall

$$1_{B^*} = \varepsilon_B$$

$$1_{A^*} = \varepsilon_A$$

We have



Comutes

$$\varepsilon_A(a) = \varepsilon_B(\varphi(a))$$

$$\varphi^*(\varepsilon_B) \stackrel{?}{=} \varepsilon_A$$

$$\varphi^*(\varepsilon_B)(a) \stackrel{?}{=} \varepsilon_A(a)$$

$$\llcorner \varphi^*(\varepsilon_B), a \rceil$$

$$\llcorner \varepsilon_B, \varphi(a) \rceil$$

$$\varepsilon_B(\varphi(a))$$

$$\varepsilon_A(a) \checkmark$$





Given coalgebras  $A, B$

Given coalgebra morphism

$$\varphi: A \rightarrow B$$

Describe

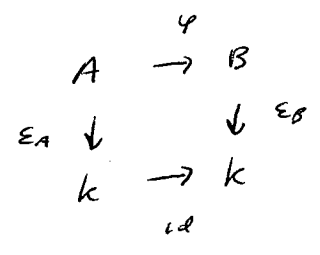
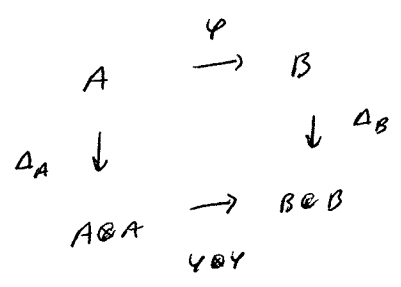
Image( $\varphi$ ),

ker( $\varphi$ )

"  
C

"  
I

these commute:



consider C

obs

$$\Delta_B(C) \subseteq C \otimes C$$

C becomes coalgebra with

$$\Delta_C = \Delta_B|_C$$

$$\varepsilon_C = \varepsilon_B|_C$$

Also

$$\varphi: A \rightarrow C$$

is coalg morphism.

Consider  $I$

$$\begin{array}{ccc}
 I & \xrightarrow{\varphi} & 0 \\
 \Delta_A \downarrow & & \downarrow \Delta_B \\
 & \xrightarrow{\varphi \otimes \varphi} & 0
 \end{array}$$

$$\Delta_A(I) \subseteq \ker(\varphi \otimes \varphi) = I \otimes A + A \otimes I$$

$$\begin{array}{ccc}
 I & \xrightarrow{\varphi} & 0 \\
 \varepsilon_A \downarrow & & \downarrow \varepsilon_B \\
 0 & \xrightarrow{\text{id}} & 0
 \end{array}$$

$$\varepsilon_A(I) = 0.$$

DEF 13 Given coalg  $A$

A coideal of  $A$  is a subspace  $I$  of  $A$

s.t.  $\Delta(I) \subseteq I \otimes A + A \otimes I,$

$$\varepsilon(I) = 0.$$

Given coalgebra  $A$

Given coideal  $I$  of  $A$

Consider quotient vector space

$$\tilde{A} = A/I$$

We now turn  $\tilde{A}$  into a coalgebra.

Consider linear map

$$\begin{array}{l} \text{can:} \\ A \rightarrow \tilde{A} \\ a \rightarrow a+I \end{array}$$

$\exists$  unique linear map

$$\tilde{\Delta}: \tilde{A} \rightarrow \tilde{A} \otimes \tilde{A}$$

set this diag commutes:

$$\begin{array}{ccc} A & \xrightarrow{\text{can}} & \tilde{A} \\ \Delta \downarrow & & \downarrow \tilde{\Delta} \\ A \otimes A & \xrightarrow{\text{can} \otimes \text{can}} & \tilde{A} \otimes \tilde{A} \end{array}$$

Indeed  $\forall a \in A$

$$\tilde{\Delta}(a) = \sum_i (a_i' + I) \otimes (a_i'' + I)$$

where

$$\Delta(a) = \sum_i a_i' \otimes a_i''$$

Also,  $\exists$  unique linear map

$$\tilde{\varepsilon}: \tilde{A} \rightarrow K$$

s.t

$$\begin{array}{ccc}
 A & \xrightarrow{\text{can}} & \tilde{A} \\
 \varepsilon \downarrow & & \downarrow \tilde{\varepsilon} \\
 K & \xrightarrow{\text{id}} & K
 \end{array}
 \quad \text{commutes}$$

The maps  $\tilde{\Delta}, \tilde{\varepsilon}$  turn  $\tilde{A}$  into a coalgebra

By constr  $\text{can}: A \rightarrow \tilde{A}$  is coalgebra morphism.

Sweedler notation

Given coalgebra  $C$

For  $c \in C$  write

$$\Delta(c) = \sum_i c_i' \otimes c_i''$$

Counit property:

$$c = \sum_i \varepsilon(c_i') c_i'' = \sum_i c_i' \varepsilon(c_i'')$$

In, suppress index  $i$  and write

$$\Delta(c) = \sum_{(c)} c' \otimes c''$$

\* becomes

$$c = \sum_{(c)} \varepsilon(c') c'' = \sum_{(c)} c' \varepsilon(c'')$$

Coassoc property becomes

$$\sum_{(c)} \Delta(c') \otimes c'' = \sum_{(c)} c' \otimes \Delta(c'')$$

We abbreviate this commutative by

$$\sum_{(c)} c' \otimes c'' \otimes c'''$$

(Long form)

or

$$\Delta^{(2)}(c)$$

(short form)

Similarly

$$\sum_{(c)} \Delta(c') \otimes c'' \otimes c''' = \sum_{(c)} c' \otimes \Delta(c'') \otimes c''' = \sum_{(c)} c' \otimes c'' \otimes \Delta(c''')$$

is abbreviated by

$$\sum_{(c)} c' \otimes c'' \otimes c''' \otimes c^{iv} \quad (\text{Long form})$$

or

$$\Delta^{(3)}(c)$$

(short form)

etc

Consider algebra  $C^*$ .

For  $f, g \in C^*$  and  $c \in C$ ,

$$(fg)(c) = \sum_{(c)} f(c') g(c'')$$

Given algebra  $A$ , consider coalgebra  $A^*$ .

For  $f \in A^*$  and  $a, b \in A$ ,

$$f(ab) = \sum_{(f)} f'(a) f''(b)$$