LEC 17 Monday Oct 12

Given vector spaces $V \neq 0$, $W \neq 0$

Given bases X for V, Y for W

Ex 7 yields

Coalgebras V, W — I

Prop 9 yields

Coalgebra $V \otimes W$ — II

Using the

basis $\{x \otimes y \mid x \in X, y \in Y\}$ for $V \otimes W$

Ex 7 yields

Coalgebra $V \otimes W$ — III

Compare II, III

LEM 10 the above coalgebras II, III are the same.

pf (ex)

(detail)

I

$\Delta_V \quad V \to V \otimes V$

$x \to x \otimes x$

$\Delta_W : W \to W \otimes W$

$y \to y \otimes y$

$\varepsilon_V : V \to k$

$x \to 1$

$\varepsilon_W : W \to k$

$y \to 1$

II

$$\Delta : \quad V \otimes W \longrightarrow V \otimes V \otimes W \otimes W \longrightarrow V \otimes W \otimes V \otimes W$$

$$x \otimes y \longrightarrow x \otimes x \otimes y \otimes y \longrightarrow (x \otimes y) \otimes (x \otimes y)$$

$$\varepsilon : \quad V \otimes W \longrightarrow k \otimes k \longrightarrow k$$

$$x \otimes y \longrightarrow 1 \otimes 1 \longrightarrow 1$$

III

$$\Delta : \quad V \otimes W \longrightarrow V \otimes W \otimes V \otimes W$$

$$x \otimes y \longrightarrow (x \otimes y) \otimes (x \otimes y)$$

$$\varepsilon : \quad V \otimes W \longrightarrow k$$

$$x \otimes y \longrightarrow 1$$

$\square$

Given fin dim'l algebras A, B

Get coalgebras A^*, B^*.

Given algebra morphism $\varphi : A \to B$.

LEM 11 The transpose

$$\varphi^* : B^* \to A^*$$

is a coalgebra morphism.

pf $\forall f \in A^*$ write

$$\Delta_{A^*}(f) = \sum_i f_i' \otimes f_i''$$

$\forall g \in B^*$ write

$$\Delta_{B^*}(g) = \sum_j g_j' \otimes g_j''$$

$\forall a_1, a_2 \in A$

$$f(a_1 a_2) = \sum_i f_i'(a_1)\, f_i''(a_2)$$

$\forall b_1, b_2 \in B$

$$g(b_1 b_2) = \sum_j g_j'(b_1)\, g_j''(b_2)$$

Take $b_1 = \varphi(a_1)$ $b_2 = \varphi(a_2)$

Also $\forall a \in A$

$$\langle g, \varphi(a) \rangle = \langle \varphi^*(g), a \rangle$$

check this diag commutes:

$$\begin{tikzcd} B^* \arrow[r, "\varphi^*"] \arrow[d, "\Delta_{B^*}"'] & A^* \arrow[d, "\Delta_{A^*}"] \\ B^* \otimes B^* \arrow[r, "\varphi^* \otimes \varphi^*"'] & A^* \otimes A^* \end{tikzcd}$$

$$\begin{tikzcd} g \arrow[r] \arrow[d] & \varphi^*(g) \arrow[d] \\ & \Delta_{A^*}(\varphi^*(g)) \overset{?}{=} \\ \sum_j g_j' \otimes g_j'' \arrow[r] & \sum_j \varphi^*(g_j') \otimes \varphi^*(g_j'') \end{tikzcd}$$

Require:

$$\varphi^*(g)(a_1 a_2) \overset{?}{=} \sum_j \underbrace{\varphi^*(g_j')(a_1)}\ \underbrace{\varphi^*(g_j'')(a_2)}$$

$$\varphi^*(g)(a_1 a_2) = \langle \varphi^*(g), a_1 a_2 \rangle = \langle g, \varphi(a_1 a_2) \rangle = \langle g, \underbrace{\varphi(a_1)}_{=b_1} \underbrace{\varphi(a_2)}_{=b_2} \rangle = g(b_1 b_2)$$

$$\varphi^*(g_j')(a_1) = \langle \varphi^*(g_j'), a_1 \rangle = \langle g_j', \underbrace{\varphi(a_1)}_{=b_1} \rangle$$

$$\varphi^*(g_j'')(a_2) = \langle \varphi^*(g_j''), a_2 \rangle = \langle g_j'', \underbrace{\varphi(a_2)}_{=b_2} \rangle$$

$$\sum_j \langle g_j', \varphi(a_1) \rangle \langle g_j'', \varphi(a_2) \rangle = \sum_j g_j'(b_1)\, g_j''(b_2)$$

$$// \checkmark$$

check this diag commutes:

$$\begin{tikzcd} B^* \arrow[r, "\varphi^*"] \arrow[d, "\varepsilon_{B^*}"'] & A^* \arrow[d, "\varepsilon_{A^*}"] \\ k \arrow[r, "id"'] & k \end{tikzcd}$$

$$\begin{tikzcd} g \arrow[r] \arrow[d] & \varphi^*(g) \arrow[d] \\ g(1_B) \arrow[r, "?"'] & \varphi^*(g)(1_A) \end{tikzcd}$$

$$g(1_B) \stackrel{?}{=} \varphi^*(g)(1_A)$$

$$\begin{array}{ccc} g(1_B) & \stackrel{?}{=} & \varphi^*(g)(1_A) \\ \| & & \| \\ \langle g, 1_B \rangle & & \langle \varphi^*(g), 1_A \rangle \\ & & \| \\ & \stackrel{\checkmark}{=} & \langle g, \underset{\substack{\| \\ 1_B}}{\varphi(1_A)} \rangle \end{array}$$

$\square$

Given coalgebras A, B

Get algebras A^*, B^*

Given coalgebra morphism

$$\varphi : A \to B$$

LEM 12 the transpose

$$\varphi^* : B^* \to A^*$$

is an algebra morphism.

pf $\forall a \in A$ write

$$\Delta_A(a) = \sum_i a_i' \otimes a_i''$$

write

$$b = \varphi(a)$$

$$b_i' = \varphi(a_i') \qquad \forall i$$

$$b_i'' = \varphi(a_i'')$$

We have

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "\Delta_A"'] & B \arrow[d, "\Delta_B"] \\ A \otimes A \arrow[r, "\varphi \otimes \varphi"'] & B \otimes B \end{tikzcd}$$

commutes

Gives

$$\Delta_B(b) = \sum_i b_i' \otimes b_i''$$

By construction, $\forall\ f, g \in A^*$

$$(fg)(a) = \sum_i f(a_i')\, g(a_i'')$$

Also $\forall\ F, G \in B^*$

$$(FG)(b) = \sum_i F(b_i')\, G(b_i'')$$

Also $\forall\ H \in B^*$

$$\langle H, \varphi(a) \rangle = \langle \varphi^*(H), a \rangle$$

check $\quad \varphi^*(FG) \overset{?}{=} \varphi^*(F)\, \varphi^*(G)$

require $\quad \varphi^*(FG)(a) \overset{?}{=} \big(\varphi^*(F)\, \varphi^*(G)\big)(a)$

$$\begin{aligned}
\varphi^*(FG)(a) &= \langle \varphi^*(FG), a \rangle \\
&= \langle FG, \underbrace{\varphi(a)}_{= b} \rangle \\
&= (FG)(b) \\
&= \sum_i F(b_i')\, G(b_i'')
\end{aligned}$$

$$\begin{aligned}
\big(\varphi^*(F)\, \varphi^*(G)\big)(a) &= \sum_i \underbrace{\varphi^*(F)(a_i')}\ \underbrace{\varphi^*(G)(a_i'')} \\
\varphi^*(F)(a_i') &= \langle \varphi^*(F), a_i' \rangle = \langle F, \underbrace{\varphi(a_i')}_{= b_i'} \rangle = F(b_i') \\
\varphi^*(G)(a_i'') &= \langle \varphi^*(G), a_i'' \rangle = \langle G, \underbrace{\varphi(a_i'')}_{= b_i''} \rangle = G(b_i'') \\
\big(\varphi^*(F)\, \varphi^*(G)\big)(a) &= \sum_i F(b_i')\, G(b_i'')
\end{aligned}$$

$$\sum_i F(b_i')\, G(b_i'') \overset{\checkmark}{=} \sum_i F(b_i')\, G(b_i'')$$

check
$$\varphi^*(1_{B^*}) = 1_{A^*}$$

Recall
$$1_{B^*} = \varepsilon_B \qquad 1_{A^*} = \varepsilon_A$$

We have
$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "\varepsilon_A"'] & B \arrow[d, "\varepsilon_B"] \\ k \arrow[r, "id"'] & k \end{tikzcd}$$
commutes

$$\varepsilon_A(a) = \varepsilon_B(\varphi(a))$$

$$\varphi^*(\varepsilon_B) \overset{?}{=} \varepsilon_A$$

$$\varphi^*(\varepsilon_B)(a) \overset{?}{=} \varepsilon_A(a)$$
$$\shortparallel$$
$$\langle \varphi^*(\varepsilon_B), a \rangle$$
$$\shortparallel$$
$$\langle \varepsilon_B, \varphi(a) \rangle$$
$$\shortparallel$$
$$\varepsilon_B(\varphi(a))$$
$$\shortparallel$$
$$\varepsilon_A(a) \checkmark$$

$\square$

Given coalgebras A, B

Given coalgebra morphism

$$\varphi : A \to B$$

Describe $\quad \underset{\overset{\shortparallel}{C}}{\mathrm{Image}(\varphi)}, \qquad \underset{\overset{\shortparallel}{I}}{\ker(\varphi)}$

these commute:

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ \Delta_A \downarrow & & \downarrow \Delta_B \\ A \otimes A & \xrightarrow{\varphi \otimes \varphi} & B \otimes B \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ \varepsilon_A \downarrow & & \downarrow \varepsilon_B \\ k & \xrightarrow{id} & k \end{array}$$

consider C

obs

$$\Delta_B(C) \subseteq C \otimes C$$

C becomes coalgebra with

$$\Delta_C = \Delta_B \big|_C$$

$$\varepsilon_C = \varepsilon_B \big|_C$$

Also

$$\varphi : A \to C$$

is coalg morphism.

Consider I

$$\begin{tikzcd} I \arrow[r, "\varphi"] \arrow[dr, "\Delta_A"', "\varphi\otimes\varphi"] & 0 \arrow[d, "\Delta_B"] \\ & 0 \end{tikzcd}$$

$$\Delta_A(I) \subseteq \ker(\varphi\otimes\varphi) = I\otimes A + A\otimes I$$

$$\begin{tikzcd} I \arrow[r, "\varphi"] \arrow[d, dash, "\varepsilon_A"'] & 0 \arrow[d, "\varepsilon_B"] \\ 0 \arrow[r, "id"'] & 0 \end{tikzcd}$$

$$\varepsilon_A(I) = 0.$$

DEF 13 Given coalg A

A <u>coideal</u> of A is a subspace I of A

s.t.

$$\Delta(I) \subseteq I\otimes A + A\otimes I,$$

$$\varepsilon(I) = 0.$$

Given coalgebra A

Given coideal I of A

Consider quotient vector space $\tilde{A} = A/I$

We now turn $\tilde{A}$ into a coalgebra.

Consider linear map

$$\mathrm{can}: \begin{array}{ccc} A & \longrightarrow & \tilde{A} \\ a & \longmapsto & a+I \end{array}$$

$\exists$ unique linear map

$$\tilde{\Delta}: \tilde{A} \longrightarrow \tilde{A} \otimes \tilde{A}$$

s.t. this diag commutes:

$$\begin{tikzcd} A \arrow[r, "\mathrm{can}"] \arrow[d, "\Delta"'] & \tilde{A} \arrow[d, "\tilde{\Delta}"] \\ A \otimes A \arrow[r, "\mathrm{can} \otimes \mathrm{can}"'] & \tilde{A} \otimes \tilde{A} \end{tikzcd}$$

Indeed $\forall\, a \in A$

$$\tilde{\Delta}(a) = \sum_i (a_i' + I) \otimes (a_i'' + I)$$

where $\Delta(a) = \sum_i a_i' \otimes a_i''$

Also, $\exists$ unique linear map

$$\tilde{\varepsilon} : \tilde{A} \to k$$

s.t

$$\begin{tikzcd} A \arrow[r, "can"] \arrow[d, "\varepsilon"'] & \tilde{A} \arrow[d, "\tilde{\varepsilon}"] \\ k \arrow[r, "id"'] & k \end{tikzcd}$$

commutes

The maps $\tilde{\Delta}$, $\tilde{\varepsilon}$ turn $\tilde{A}$ into a coalgebra

By Constr $can : A \to \tilde{A}$ is coalgebra morphism.

Sweedler notation

Given coalgebra C

For $c \in C$ write

$$\Delta(c) = \sum_i c_i' \otimes c_i'' \qquad *$$

Counit property:

$$c = \sum_i \varepsilon(c_i')\, c_i'' = \sum_i c_i'\, \varepsilon(c_i'') \qquad **$$

In, suppress index i and write

$$\Delta(c) = \sum_{(c)} c' \otimes c''$$

$**$ becomes

$$c = \sum_{(c)} \varepsilon(c')\, c'' = \sum_{(c)} c'\, \varepsilon(c'')$$

Coassoc property becomes

$$\sum_{(c)} \Delta(c') \otimes c'' = \sum_{(c)} c' \otimes \Delta(c'')$$

We abbreviate this common value by

$$\sum_{(c)} c' \otimes c'' \otimes c''' \qquad \text{(Long form)}$$

or

$$\Delta^{(2)}(c) \qquad \text{(short form)}$$

Similarly

$$\sum_{(c)} \Delta(c') \otimes c'' \otimes c''' = \sum_{(c)} c' \otimes \Delta(c'') \otimes c''' = \sum_{(c)} c' \otimes c'' \otimes \Delta(c''')$$

is abbreviated by

$$\sum_{(c)} c' \otimes c'' \otimes c''' \otimes c^{iv} \qquad \text{(Long form)}$$

or

$$\Delta^{(3)}(c) \qquad \text{(short form)}$$

etc

Consider algebra C^*.

For $f, g \in C^*$ and $c \in C$,

$$(fg)(c) = \sum_{(c)} f(c')\, g(c'')$$

Given algebra A, consider coalgebra A^*.

For $f \in A^*$ and $a, b \in A$,

$$f(ab) = \sum_{(f)} f'(a)\, f''(b)$$