Lec 16 Friday Oct 9

More on coalgebras.

Aside Given fin dim'l vector spaces U, V

Recall vector space iso

$$\mathrm{Hom}(U,V) \longrightarrow \mathrm{Hom}(V^*, U^*) \qquad ☆$$
$$\varphi \longrightarrow \varphi^*$$

s.t $\langle \varphi^*(g), u \rangle = \langle g, \varphi(u) \rangle \quad \forall u \in U \ \forall g \in V^*$

Also

$$U^* \otimes V^* \longrightarrow (V \otimes U)^* \qquad ☆☆$$
$$f \otimes g \longrightarrow H$$

s.t $H(v \otimes u) = f(u)\, g(v) \quad \forall u \in U \ \forall v \in V$

If we drop the requirement that U, V are fin dim'l then the maps ☆, ☆☆ still exist, but are no longer bijections in gen.

Ex 6 Given coalgebra C,

turn the dual space $A = C^*$ into an algebra.

<u>Define μ</u> :

$$\mu : \quad C^* \otimes C^* \longrightarrow (C \otimes C)^* \xrightarrow{\Delta^*} C^*$$

$\uparrow$

$U^* \otimes V^* \to (V \otimes U)^*$

$f \otimes g \to H$

$H(v \otimes u) = f(u)\, g(v)$

$\forall \; f, g \in C^*$ we have

$$\underbrace{\mu(f \otimes g)\;(c)}_{(fg)(c)} = \sum_i g(c_i')\, f(c_i'') \qquad c \in C$$

where

$$\Delta(c) = \sum_i c_i' \otimes c_i''$$

<u>Define η</u> :

$$\eta : \quad k \longrightarrow k^* \xrightarrow{\varepsilon^*} C^*$$

$h(1) \leftrightarrow h$

Get

$$\eta(1) = \varepsilon$$

check assoc:

For $f, g, h \in C^*$ require

$$(fg)h \stackrel{?}{=} f(gh)$$

$\forall c \in C$ require

$$((fg)h)(c) = (f(gh))(c) \qquad ☆$$

Write
$$\Delta(c) = \sum_i c_i' \otimes c_i''$$

$\forall\ d = c_i'$ write
$$\Delta(d) = \sum_j d_j' \otimes d_j''$$

$\forall\ e = c_i''$ write
$$\Delta(e) = \sum_l e_l' \otimes e_l''$$

Have
$$\sum_{i,j} (c_i')_j' \otimes (c_i')_j'' \otimes c_i'' = \sum_{i,l} c_i' \otimes (c_i'')_l' \otimes (c_i'')_l'' \qquad ☆☆$$

By constr

$$((fg)h)(c) = \sum_{i,l} h(c_i')\, g((c_i'')_l')\, f((c_i'')_l'') = h \otimes g \otimes f \text{ at } ☆☆$$

$$\shortparallel$$

$$(f(gh))(c) = \sum_{i,j} h((c_i')_j')\, g((c_i')_j'')\, f(c_i'') = h \otimes g \otimes f \text{ at } ☆☆$$

So ☆ holds.

check unit:

show $\eta(1) = \varepsilon$ is identity of C^*

$\forall g \in C^*$

$$\eta(1)\, g \overset{?}{=} g$$

$\forall c \in C$

$$(\eta(1) g)(c) \overset{?}{=} g(c)$$

$$\shortparallel$$

$$\underbrace{\sum_i g(c_i')\, \underset{\varepsilon}{\underset{\shortparallel}{\eta(1)}}(c_i'')}_{\shortparallel}$$

where $\Delta(c) = \sum_i c_i' \otimes c_i''$

$$\sum_i g(c_i')\, \varepsilon(c_i'')$$

$$\shortparallel$$

$$g\Big(\underbrace{\sum_i c_i'\, \varepsilon(c_i'')}_{\substack{\shortparallel \\ c}} \Big)$$ since ε is counit

We similarly have

$$g\, \eta(1) = g$$

We just turned C^* into an algebra.

We also have the opposite algebra, such that

$\forall\ f, g \in C^*$ and $\forall\ c \in C$

$$(fg)(c) = \sum_i f(c_i') \, g(c_i'') \qquad *$$

where

$$\Delta(c) = \sum_i c_i' \otimes c_i''$$

By <u>the algebra C^*</u> we mean $*$ $\square$

Note Given f.d alg A

$$A \longrightarrow A^* \longrightarrow (A^*)^*$$

alg coalg alg

Recall vs iso

$$A \longrightarrow (A^*)^* \qquad ☆$$

$$a \longrightarrow \hat{a} \qquad \hat{a}(f) = f(a)$$

the map ☆ is an algebra iso (ex)

pf detail: $\forall\, a, b$ show

$$\widehat{ab} = \hat{a}\,\hat{b}$$

$\forall\, f \in A^*$ show

$$\Delta(f) = \sum_i f_i' \otimes f_i''$$

$$\widehat{ab}(f) \overset{?}{=} (\hat{a}\,\hat{b})(f)$$

$$\widehat{ab}(f) = f(ab) = \sum_i f_i'(a)\, f_i''(b)$$

$$(\hat{a}\,\hat{b})(f) = \sum_i \underbrace{\hat{a}(f_i')}_{= f_i'(a)}\, \underbrace{\hat{b}(f_i'')}_{= f_i''(b)}$$

$= \checkmark$

__Note__ Given f.d. coalg C

$$\underset{\text{coalg}}{C} \to \underset{\text{alg}}{C^*} \to \underset{\text{coalg}}{(C^*)^*}$$

the map

$$C \longrightarrow (C^*)^* \qquad \hat{c}(f) = f(c)$$
$$c \longrightarrow \hat{c}$$

is coalg iso (ex)

Ex 7 Given vector space $V \neq 0$

Given basis X for V

We have bijection of sets

algebra with mult

$(\varphi \phi)(x) = \varphi(x)\, \phi(x)$
$\forall x \in X$

$$V^* \longleftrightarrow \{ \varphi \mid \varphi : X \to k \}$$

$$f \longrightarrow f|_X$$

☆

Define linear maps

$$\Delta : \quad V \to V \otimes V \qquad x \in X$$
$$x \longmapsto x \otimes x$$

$$\varepsilon : \quad V \longrightarrow k \qquad x \in X$$
$$x \longmapsto 1$$

Then Δ, ε turn V into a coalgebra (ex)

the corresponding algebra V^* is the one inherited via bijection ☆

(ex)

□

Ex 8 Given fin dim'l vs $V \neq 0$

Consider alg $A = End(V)$

Via Ex 5, A^* becomes coalgebra.

Describe coalg A^*.

Pick basis $\{v_i\}$ for V

dual basis $\{v^i\}$ for V^*

basis $\{e_{ij}\}$ for A $\qquad e_{ij}(v_r) = \delta_{jr} v_i$

dual basis $\{x_{ij}\}$ for A^*

We have

$$\Delta : \begin{array}{ccc} A^* & \longrightarrow & A^* \otimes A^* \\ x_{ij} & \longrightarrow & \sum_{l} x_{il} \otimes x_{lj} \end{array} \qquad \forall i, j$$

$$\varepsilon : \begin{array}{ccc} A^* & \longrightarrow & k \\ x_{ij} & \longrightarrow & \delta_{ij} \end{array} \qquad \forall i, j.$$

check Δ:

Require: $\forall\, i,j$

$$x_{ij}(ab) \overset{?}{=} \sum_{\ell} x_{i\ell}(a)\, x_{\ell j}(b) \qquad \forall\, a,b \in A$$

take $a = e_{rs}$, $b = e_{xy}$

$$\underbrace{x_{ij}(\underbrace{e_{rs}\, e_{xy}}_{=\,\delta_{sx}\, e_{ry}})}_{\delta_{sx}\,\delta_{ir}\,\delta_{jy}} \overset{?}{=} \sum_{\ell} \underbrace{x_{i\ell}(e_{rs})}_{=\,\delta_{ir}\,\delta_{\ell s}} \; \underbrace{x_{\ell j}(e_{xy})}_{=\,\delta_{\ell x}\,\delta_{jy}}$$

$= \checkmark$

Check ε:

Require: $\forall\, i,j$

$$x_{ij} \overset{?}{=} \sum_{\ell} \underbrace{\varepsilon(x_{i\ell})}_{=\,\delta_{i\ell}}\, x_{\ell j}\,, \qquad x_{ij} \overset{?}{=} \sum_{\ell} x_{i\ell}\, \underbrace{\varepsilon(x_{\ell j})}_{=\,\delta_{\ell j}}$$

$\checkmark$ $\qquad$ $\checkmark$

Ex 8, cont.

Recall vs iso

$$A \simeq V \otimes V^*$$

$$A^* \simeq (V \otimes V^*)^* \simeq (V^*)^* \otimes V^* \simeq V \otimes V^* \simeq A$$

Gives vs iso

$$A \longrightarrow A^*$$
$$f \longrightarrow f^\vee$$

Eval map satisfies

$$\langle , \rangle \quad \begin{array}{ccc} A^* & \times & A & \longrightarrow & k \\ f^\vee & & g & \longrightarrow & tr(fg) \end{array} \qquad \forall f, g \in A$$

(ex)

So $\quad \langle e_{ij}^\vee, e_{rs} \rangle = tr(e_{ij}\, e_{rs}) \qquad \forall i,j,r,s$

$$= \delta_{jr} \delta_{is}$$

ALSO $\quad \langle x_{ij}, e_{rs} \rangle = \delta_{ir} \delta_{js} \qquad \forall i,j,r,s$

So $\quad e_{ij}^\vee = x_{ji} \qquad \forall i,j$

Via vs iso $\begin{array}{l} A \to A^* \\ f \to f^\vee \end{array}$

alg str on A induces
alg str on A^* $\qquad *$

Via inverse iso $\begin{array}{l} A^* \to A \\ f^\vee \to f \end{array}$

coalg str on A^* induces
coalg str on A $\qquad **$

Describe $*$, $**$.

Start with alg str on A^* with mult $\qquad$ ☆

$$x_{ij}\, x_{rs} = \delta_{jr}\, x_{is} \qquad \forall\, i,j,r,s$$

Via Ex 6 the alg A^* gives coalg A s.t.

$$\Delta \quad \begin{array}{l} A \to A \otimes A \\ e_{ij} \to \sum_l e_{il} \otimes e_{lj} \end{array} \qquad \forall\, i,j \qquad ☆☆$$

$$\varepsilon \quad \begin{array}{l} A \to k \\ e_{ij} \to \delta_{ij} \end{array} \qquad \forall\, i,j.$$

then $*$ is the opposite of ☆
$**$ is the opposite of ☆☆

Ex 8 Detail

$A = \mathrm{End}(V)$

$\{v_i\}$ basis for V

$\{v^i\}$ dual basis for V^*

$\{e_{ij}\}$ basis for A

$e_{ij}(v_r) = \delta_{jr} v_i$

$$A \overset{\mathrm{I}}{\simeq} V \otimes V^* \overset{\mathrm{II}}{\simeq} (V^*)^* \otimes V^* \overset{\mathrm{III}}{\simeq} (V \otimes V^*)^* \overset{\mathrm{IV}}{\simeq} A^*$$

$$f \longrightarrow f^\vee$$

Describe $f^\vee$

Show

$$\begin{array}{lccccl} & A^* & \times & A & \longrightarrow & k \\ \langle\, , \,\rangle & f^\vee & & h & \longrightarrow & tr(fh) \end{array} \qquad \forall\, f, h \in A$$

I detail

$$V \otimes V^* \longrightarrow A$$

$$v \otimes f \longrightarrow H$$

$$H(w) = f(w)v \qquad \forall w \in V$$

We have

$$v_i \otimes v^j \longrightarrow e_{ij}$$

check

$$v^j(v_k)v_i \overset{?}{=} e_{ij}(v_k)$$

$$\| \qquad\qquad \|$$

$$\delta_{jk} v_i \quad \checkmark \quad \delta_{jk} v_i$$

II detail

$$V \longrightarrow (V^*)^*$$
$$v \longrightarrow \hat{v}$$

$$\hat{v}(g) = g(v) \qquad \forall g \in V^*$$

III detail

Recall in general

$$V^* \otimes U^* \longrightarrow (U \otimes V)^*$$

$$g \otimes f \longrightarrow H$$

$$H(u \otimes v) = f(u)\, g(v) \qquad \forall u \in U \quad \forall v \in V$$

So

$$(V^*)^* \otimes V^* \longrightarrow (V \otimes V^*)^*$$

$$\hat{v} \otimes f \longrightarrow H$$

$$\begin{aligned} H(w \otimes g) &= f(w)\, \hat{v}(g) \\ &= f(w)\, g(v) \end{aligned}$$

In particular

$$(V^*)^* \otimes V^* \longrightarrow (V \otimes V^*)^*$$

$$\hat{v}_i \otimes v^j \longrightarrow E_{ij}$$

$$\begin{aligned} E_{ij}(v_r \otimes v^s) &= v^j(v_r)\, v^s(v_i) \\ &= \delta_{jr}\, \delta_{si} \end{aligned}$$

IV detail

$$\begin{array}{ccc} (V\otimes V^*)^* & & V\otimes V^* \\ \varphi^* \uparrow & & \downarrow \varphi \\ A^* & & A \end{array}$$

$\forall v \in V \quad \forall f \in V^* \quad \forall g \in A^*$

$$\langle g, \varphi(v\otimes f) \rangle = \langle \varphi^*(g), v\otimes f \rangle$$

so

$$\langle g, e_{ij} \rangle = \langle \varphi^*(g), v_i \otimes v^j \rangle$$

conclusion:

$$\overset{\mathrm{I}}{A} \simeq \overset{\mathrm{II}}{V \otimes V^*} \simeq (V^*)^* \otimes V^* \simeq \overset{\mathrm{III}}{(V \otimes V^*)^*} \simeq \overset{\mathrm{IV}}{A^*}$$

$$e_{ij} \leftrightarrow v_i \otimes v^j \leftrightarrow \hat{v}_i \otimes v^j \leftrightarrow E_{ij} \leftrightarrow e_{ij}^\vee$$

$\forall\, r, s$

$$\langle e_{ij}^\vee, e_{rs} \rangle = \langle \varphi^*(e_{ij}^\vee), v_r \otimes v^s \rangle$$

$$= \langle E_{ij}, v_r \otimes v^s \rangle$$

$$= \delta_{jr} \delta_{is}$$

$$= \mathrm{tr}(e_{ij} e_{rs})$$

Given

fin dim'l coalgebras A, B I

Get

algebras A^*, B^* II

algebra $B^* \otimes A^*$ III

vs iso $B^* \otimes A^* \simeq (A \otimes B)^*$ IV

coalgebra $A \otimes B$ V

$\uparrow$
describe

<u>Guess</u>:

$$\Delta : \quad A \otimes B \xrightarrow[\Delta_A \otimes \Delta_B]{} A \otimes A \otimes B \otimes B \xrightarrow[id \otimes \tau_{A,B} \otimes id]{} A \otimes B \otimes A \otimes B$$

$$\varepsilon : \quad A \otimes B \xrightarrow[\varepsilon_A \otimes \varepsilon_B]{} k \otimes k \longrightarrow k$$

$$1 \otimes 1 \longrightarrow 1$$

Check Δ

I $\quad \forall a \in A \qquad \forall b \in B$

write

$$\Delta_A(a) = \sum_i a_i' \otimes a_i'' \qquad \Delta_B(b) = \sum_j b_j' \otimes b_j''$$

II $\quad \forall f, g \in A^*, \qquad \forall F, G \in B^*,$

$$(fg)(a) = \sum_i f(a_i')\, g(a_i'') \qquad (FG)(b) = \sum_j F(b_j')\, G(b_j'')$$

III

$$(F \otimes f)(G \otimes g) = (FG) \otimes (fg)$$

IV

$$F \otimes f \leftrightarrow \varphi \qquad \varphi(a \otimes b) = f(a)\, F(b)$$

$$G \otimes g \leftrightarrow \phi \qquad \phi(a \otimes b) = g(a)\, G(b)$$

$$FG \otimes fg \leftrightarrow \varphi\phi \qquad (\varphi\phi)(a \otimes b) = (fg)(a)\, (FG)(b)$$

V

$$\Delta : a \otimes b \rightarrow \Delta_A(a) \otimes \Delta_B(b) = \left(\sum_i a_i' \otimes a_i''\right) \otimes \left(\sum_j b_j' \otimes b_j''\right)$$

$$= \sum_{i,j} a_i' \otimes a_i'' \otimes b_j' \otimes b_j'' \qquad ?$$

$$\Rightarrow \sum_{i,j} (a_i' \otimes b_j') \otimes (a_i'' \otimes b_j'')$$

Require

$$(\varphi\phi)(a\otimes b) \overset{?}{=} \sum_{i,j} \underbrace{\varphi(a_i'\otimes b_j')}_{= f(a_i')\,F(b_j')} \;\underbrace{\phi(a_i''\otimes b_j'')}_{= g(a_i'')\,G(b_j'')}$$

Have

$$(\varphi\phi)(a\otimes b) = \underbrace{(fg)(a)}_{= \sum_i f(a_i')\,g(a_i'')} \;\underbrace{(FG)(b)}_{= \sum_j F(b_j')\,G(b_j'')}$$

$$= \sum_{i,j} f(a_i')\,F(b_j')\,g(a_i'')\,G(b_j'')$$

✓

check ε

I $\quad a = \sum_i \varepsilon_A(a_i') a_i'' \qquad a = \sum_i a_i' \varepsilon_A(a_i'')$

$\quad b = \sum_j \varepsilon_B(b_j') b_j'' \qquad b = \sum_j b_j' \varepsilon_B(b_j'')$

II $\quad \varepsilon_A =$ mult ident of A^*

$\quad \varepsilon_B =$ " " B^*

III $\quad \varepsilon_B \otimes \varepsilon_A =$ mult ident of $B^* \otimes A^*$

IV $\quad \varepsilon_B \otimes \varepsilon_A \leftrightarrow \mathbb{1} \qquad \mathbb{1}(a \otimes b) = \varepsilon_A(a)\, \varepsilon_B(b)$

$\mathbb{1}$ is mult ident of algebra $(A \otimes B)^*$

V $\quad$ Require: ε is mult ident of $(A \otimes B)^*$

show $\varepsilon = \mathbb{1}$

$$\varepsilon : \quad a \otimes b \longrightarrow \varepsilon_A(a) \otimes \varepsilon_B(b) \longrightarrow \varepsilon_A(a)\, \varepsilon_B(b)$$
$$= \mathbb{1}(a \otimes b)$$

So $\varepsilon = \mathbb{1}$ ✓

Guess is correct.