

Since I is homogeneous,

$$S(V) = \sum_{n \in \mathbb{N}} S^n(V) \quad (\text{dir sum})$$

$$\cong \bigoplus_{n \in \mathbb{N}} S^n(V)$$

Observe

$$T^n(V) \cap I = 0 \quad \text{for } n \geq 1$$

So

$$S^0(V) \cong T^0(V) = k$$

$$S^1(V) \cong T^1(V) = V$$

The composition

$$V \xrightarrow{\text{incl}} T(V) \xrightarrow{\text{quot}} S(V) \quad *$$

is injective.

Identify V with its image under $*$.

Then algebra $S(V)$ is generated by V .

LEM 53 Given a vector space V and an algebra A . Given a linear map

$$f: V \rightarrow A$$

such that

$$f(u)f(v) = f(uv) \quad \forall u, v \in V.$$

then \exists unique algebra morphism

$$\bar{f}: S(V) \rightarrow A$$

such that

$$\begin{array}{ccc} V & \xrightarrow{\text{incl}} & S(V) \\ f \searrow & & \swarrow \bar{f} \\ & A & \end{array}$$

commutes.

*

pf The map

$$f: V \rightarrow A$$

induces an algebra morphism

$$\bar{f}: T(V) \rightarrow A.$$

$\forall u, v \in V,$

$$\begin{aligned} \bar{f}(u \otimes v - v \otimes u) &= f(u)f(v) - f(v)f(u) \\ &= 0 \end{aligned}$$

So $I \subseteq \ker(\bar{f})$

So \bar{f} induces an alg morphism

$$\bar{f}: \underbrace{T(V)/I}_S \rightarrow A$$

S(V)

By const * commutes.

The uniqueness claim is clear.



Given a vector space V .

Given basis X for V .

Recall polynomial algebra

$$k[X]$$

↑ view as commuting indets

LEM 54 With the above notation, \exists algebra iso

$$S(V) \rightarrow k[X]$$

that sends

$$x \rightarrow x$$

$$\forall x \in X$$

pf \exists linear map

$$f: V \rightarrow k[X]$$

$$x \rightarrow x$$

$$\forall x \in X$$

the map f induces an algebra morphism

$$\bar{f}: S(V) \rightarrow k[X]$$

Also, the map

$$\theta: X \rightarrow V \rightarrow S(V)$$

$$\text{incl} \quad \text{incl}$$

*

θ induces an algebra morphism

$$\begin{aligned} \bar{\theta} : k[x] &\rightarrow S(V) \\ x &\rightarrow x \end{aligned}$$

$\ast \ast$

The maps $\ast, \ast \ast$ are inverses, hence bijections.

□

LEM 55 Given vector spaces U, V .

\exists algebra isomorphism

$$S(U \oplus V) \rightarrow S(U) \otimes S(V)$$

that sends

$$(u, v) \rightarrow u \otimes 1 + 1 \otimes v$$

$\forall u \in U$
 $\forall v \in V$.

pf the algebras

$$S(U \oplus V), \quad S(U) \otimes S(V)$$

are commutative.

The linear map

$$f: \begin{array}{l} U \oplus V \rightarrow S(U) \otimes S(V) \\ (u, v) \rightarrow u \otimes 1 + 1 \otimes v \end{array}$$

induces the alg morphism

$$\bar{f}: S(U \oplus V) \rightarrow S(U) \otimes S(V)$$

Also, the linear map

$$\begin{array}{l} U \rightarrow U \oplus V \rightarrow S(U \oplus V) \\ u \rightarrow (u, 0) \quad \text{incl} \end{array}$$

induces an algebra morphism

$$S(u) \rightarrow S(u \otimes v)$$

10/7/15
★⁷

Similarly we obtain an algebra morphism

$$S(v) \rightarrow S(u \otimes v)$$

★ ★

By Prop 46, ★ and ★★ induce an algebra morphism

$$S(u) \otimes S(v) \rightarrow S(u \otimes v)$$

★★

The maps ★, ★★ are inverses, hence bijectims.

□

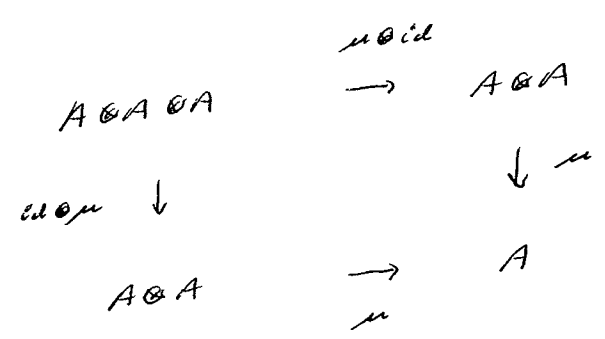
III Hopf algebras

In terms of commuting diagrams, an algebra is a non 0 vector space A together with linear maps

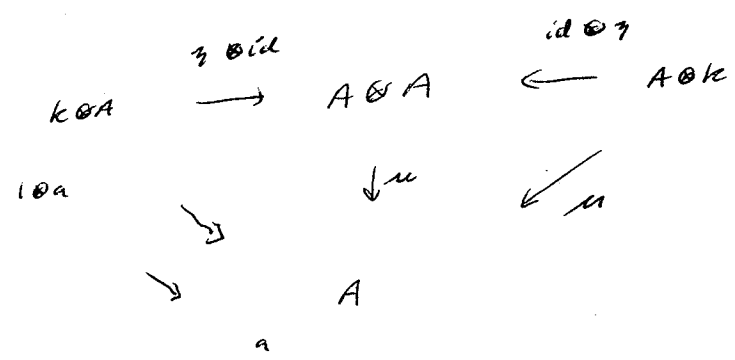
$$\mu: A \otimes A \rightarrow A, \quad \eta: k \rightarrow A$$

that make the following diagrams commute:

(Assoc)

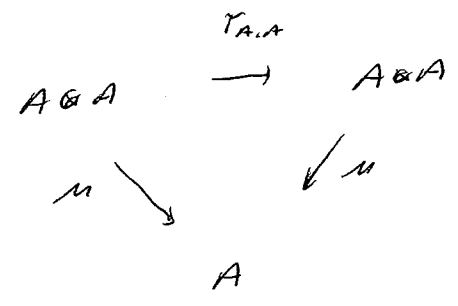


(Unit)

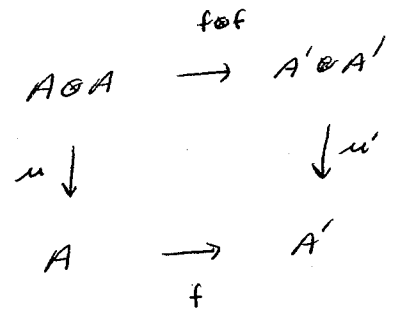
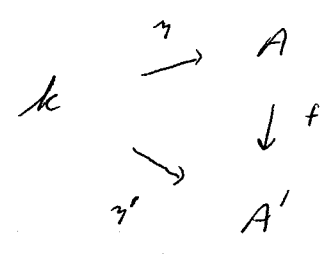


The above algebra A is commutative whenever

this diagram commutes:



Given algebras A, A' , an algebra morphism is a linear map $f: A \rightarrow A'$ s.t. these diagrams commute:



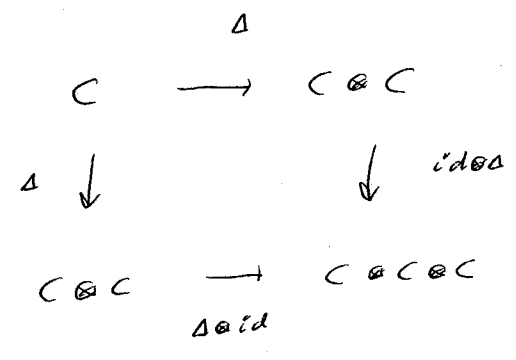
DEF 1 A coalgebra is a non

vector space C together with linear maps

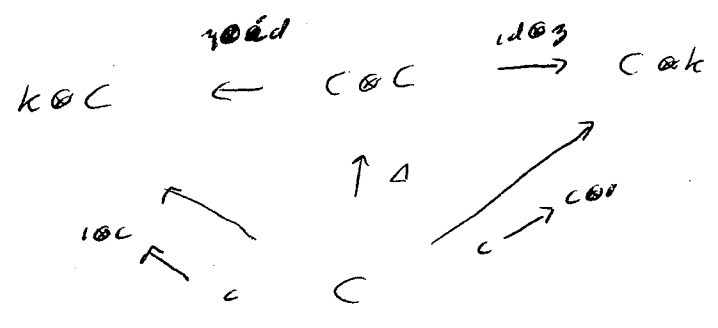
$$\Delta: C \rightarrow C \otimes C, \quad \epsilon: C \rightarrow k$$

s.t. the following diagrams commute

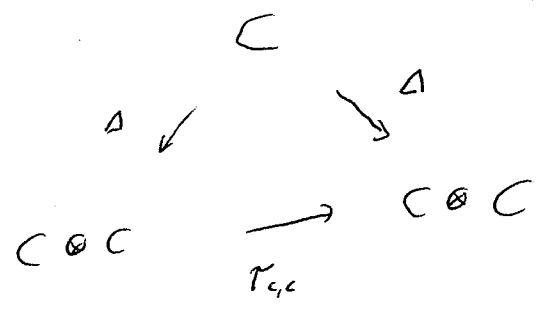
(Coassoc)



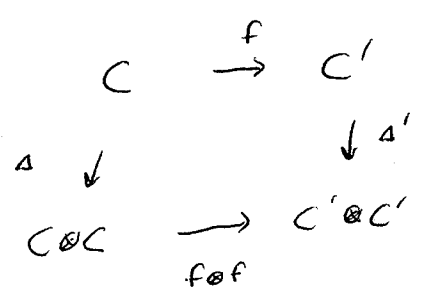
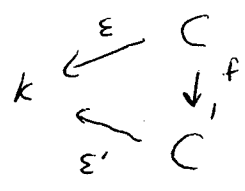
(Counit)



Coalg C is co commutative whenever
 this diag commutes:



For coalgebras C, C' a co-algebra morphism
 $C \rightarrow C'$ is a linear map $f: C \rightarrow C'$ that makes these
 diagrams commute:



EX 2 Recall algebra $M(2)$ with gens x_{ij} $1 \leq i, j \leq 2$

Recall algebra morphisms

$$\Delta: M(2) \rightarrow M(2) \otimes M(2)$$

$$x_{ij} \rightarrow \sum_l x_{il} \otimes x_{lj} \quad 1 \leq i, j \leq 2$$

$$\epsilon: M(2) \rightarrow k$$

$$x_{ij} \rightarrow \delta_{ij}$$

Then Δ, ϵ turn $M(2)$ into a coalgebra.

Check:
(Coassoc)

$$C \xrightarrow{\Delta} C \otimes C$$

$$\Delta \downarrow \qquad \qquad \downarrow \Delta \otimes id$$

$$C \otimes C \xrightarrow{id \otimes \Delta} C \otimes C \otimes C$$

$$x_{ij} \rightarrow \sum_l x_{il} \otimes x_{lj}$$

$$\downarrow \qquad \qquad \qquad \downarrow$$

$$\sum_k x_{ik} \otimes x_{kj} \rightarrow \sum_{l,k} x_{il} \otimes x_{kl} \otimes x_{lj}$$

(Co-unit L)

$$k \otimes C \xleftarrow{\epsilon \otimes id} C \otimes C$$

$$id_C \swarrow \qquad \uparrow \Delta$$

$$R_C \quad C$$

$$\sum_l \delta_{il} \otimes x_{lj} \xleftarrow{=} \sum_l x_{il} \otimes x_{lj}$$

$$1 \otimes k_j \xleftarrow{=} x_{ij}$$

(Co-unit R)

sim

Ex 3

K has coalg structure with

$$\Delta: \begin{aligned} k &\rightarrow k \otimes k \\ 1 &\rightarrow 1 \otimes 1 \end{aligned}$$

$$\varepsilon: \begin{aligned} k &\rightarrow k \\ 1 &\rightarrow 1 \end{aligned}$$

Ex 4 Given coalg C, Δ, ε define

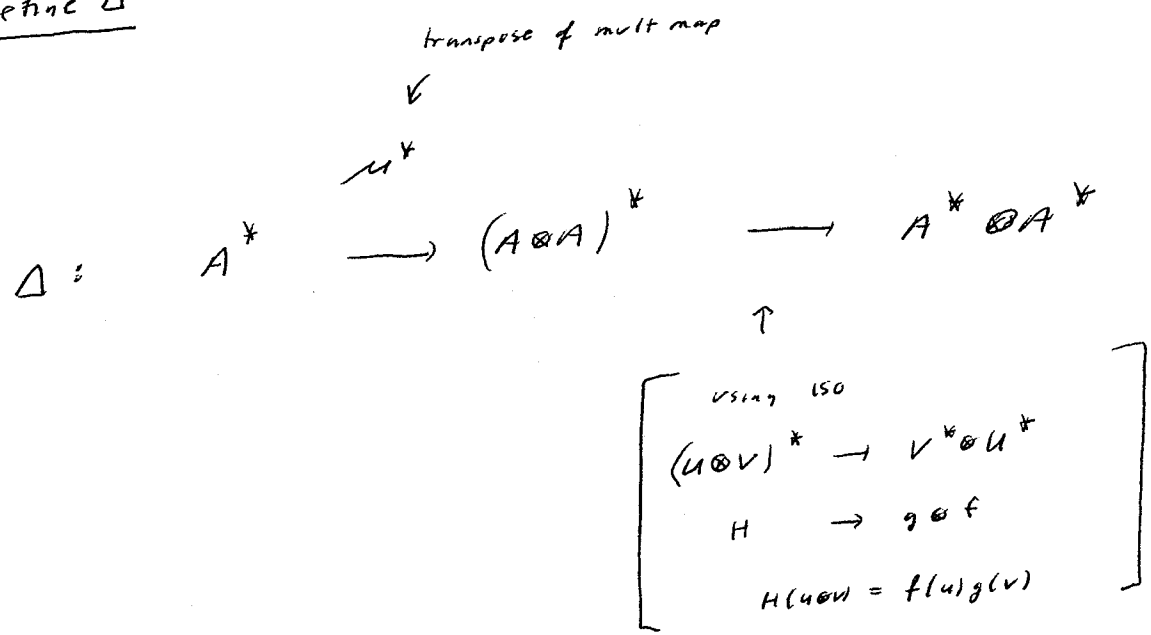
$$\Delta^{op}: C \rightarrow C \otimes C \xrightarrow{\tau_{C,C}} C \otimes C$$

$$\Delta \quad \quad \quad \tau_{C,C}$$

then $C, \Delta^{op}, \varepsilon$ is coalg (ex)

Ex 5 Given finite dim'l algebra A ,
turn the dual space $C = A^*$ into a coalgebra.

Define Δ



For $f \in A^*$, write

$$\Delta(f) = \sum_i f_i' \otimes f_i''$$

We have

$$f(ab) = \sum_i f_i'(b) f_i''(a) \quad \forall a, b \in A$$

Define ε

$$\varepsilon: \begin{array}{ccc} A^* & \rightarrow & k^* & \longrightarrow & k \\ \exists^* & & h & \longrightarrow & h(\cdot) \end{array}$$

For $f \in A^*$ we have

$$\varepsilon(f) = f(1)$$

check coassoc :

For $f \in A^*$ write

$$\Delta(f) = \sum_i f_i' \otimes f_i''$$

For $g = f_i'$ write

$$\Delta(g) = \sum_j g_j' \otimes g_j''$$

For $h = f_i''$ write

$$\Delta(h) = \sum_l h_l' \otimes h_l''$$

require

$$\sum_{i,j} (f_i')_j' \otimes (f_i')_j'' \otimes f_i'' = \sum_{i,j,l} f_i' \otimes (f_i'')_j' \otimes (f_i'')_j'' \quad *$$

By cmstr, $\forall a,b,c \in A$

$$f(abc) = \sum_{i,j} (f_i')_j'(c) (f_i')_j''(b) f_i''(a) \quad **$$

and

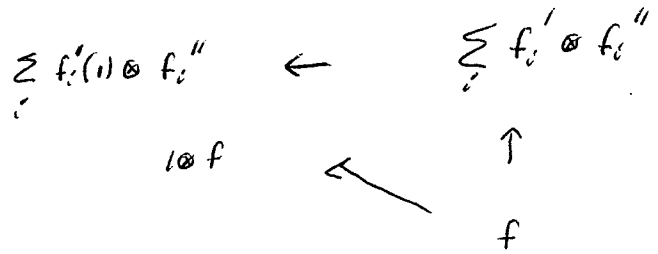
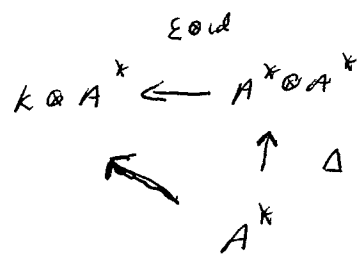
$$f(abc) = \sum_{i,j,l} f_i'(c) (f_i'')_j'(b) (f_i'')_j''(a)$$

By ** and since

$$(A \otimes A \otimes A)^* \cong A^* \otimes A^* \otimes A^*$$

we obtain *

check count:



Require

$$f = \sum_i f_i'(a) f_i''$$

$$f = \sum_i f_i' f_i''(a) \quad *$$

Have

$$f(ab) = \sum_i f_i'(b) f_i''(a)$$

$$\forall a, b \in A$$

Set $a=1$:

$$f(b) = \sum_i f_i'(b) f_i''(1)$$

$$\forall b \in A$$

Set $b=1$:

$$f(a) = \sum_i f_i'(1) f_i''(a)$$

$$\forall a \in A$$

Line * follows.

We just turned A^* into a coalgebra.

We also have the opposite coalgebra

s.t. $f \in A^*$

$$\Delta(f) = \sum_i f_i' \otimes f_i''$$

☆☆

where

$$f(ab) = \sum_i f_i'(a) f_i''(b)$$

$$\forall a, b \in A.$$

By the coalgebra A^* we mean

☆☆

□