Since I is homogeneous,

$$S(V) = \sum_{n \in \mathbb{N}} S^n(V) \quad \text{(dir sum)}$$

$$\cong \bigoplus_{n \in \mathbb{N}} S^n(V)$$

Observe

$$T^n(V) \cap I = 0 \quad \text{for } n \leq 1$$

So

$$S^0(V) \cong T^0(V) = k$$

$$S^1(V) \cong T^1(V) = V$$

The composition

$$V \xrightarrow[\text{incl}]{} T(V) \xrightarrow[\text{quot}]{} S(V) \qquad *$$

is injective.

Identify V with its image under $*$.

Then algebra $S(V)$ is generated by V.

LEM 53 Given a vector space V and an algebra A. Given a linear map

$$f : V \to A$$

such that

$$f(u) f(v) = f(v) f(u) \qquad \forall\, u, v \in V.$$

Then $\exists$ unique algebra morphism

$$\bar{f} : S(V) \to A$$

such that

$$\begin{tikzcd} V \arrow[r, "\text{incl}"] \arrow[dr, "f"'] & S(V) \arrow[d, "\bar{f}"] \\ & A \end{tikzcd}$$

commutes. $*$

pf The map

$$f : V \to A$$

induces an algebra morphism

$$\bar{f} : T(V) \to A.$$

$\forall u, v \in V,$

$$\bar{f}(u \otimes v - v \otimes u) = f(u) f(v) - f(v) f(u)$$
$$= 0$$

So $$I \subseteq \ker(\bar{f})$$

So $\bar{f}$ induces an alg morphism

$$\bar{f} : \underbrace{T(V)/I}_{\substack{\shortparallel \\ S(V)}} \longrightarrow A$$

By constr $*$ commutes.

The uniqueness claim is clear. $\square$

Given a vector space V.

Given basis X for V.

Recall polynomial algebra

$$k[X]$$

$\uparrow$ view as commuting indets

LEM 54 With the above notation, $\exists$ algebra iso

$$S(V) \to k[X]$$

that sends $x \to x$ $\forall x \in X$.

pf $\exists$ linear map

$$f : V \to k[X]$$
$$x \to x \quad \forall x \in X$$

The map f induces an algebra morphism

$$\bar{f} : S(V) \to k[X] \qquad *$$

Also, the map

$$\theta : X \xrightarrow{\text{incl}} V \xrightarrow{\text{incl}} S(V)$$

θ induces an algebra morphism

$$\bar{\theta}: \quad k[x] \longrightarrow S(V) \qquad **$$
$$x \longrightarrow x$$

$\square$

The maps $*$, $**$ are inverses, hence bijections.

LEM 55 Given vector spaces U, V.

$\exists$ algebra isomorphism

$$S(U \oplus V) \to S(U) \otimes S(V)$$

that sends

$$(u,v) \to u \otimes 1 + 1 \otimes v \qquad \forall u \in U \ \forall v \in V.$$

pf The algebras

$$S(U \oplus V), \qquad S(U) \otimes S(V)$$

are commutative.

The linear map

$$f: \begin{array}{ccc} U \oplus V & \to & S(U) \otimes S(V) \\ (u,v) & \to & u \otimes 1 + 1 \otimes v \end{array}$$

induces the alg morphism

$$\bar{f}: S(U \oplus V) \to S(U) \otimes S(V) \qquad *$$

Also, the linear map

$$\begin{array}{ccccc} U & \to & U \oplus V & \xrightarrow{\text{incl}} & S(U \oplus V) \\ u & \to & (u, 0) & & \end{array}$$

induces an algebra morphism

$$S(U) \longrightarrow S(U \oplus V) \qquad \star$$

Similarly we obtain an algebra morphism

$$S(V) \longrightarrow S(U \oplus V) \qquad \star\star$$

By Prop 46, $\star$ and $\star\star$ induce an algebra morphism

$$S(U) \otimes S(V) \longrightarrow S(U \oplus V) \qquad \ast\ast$$

The maps $\ast$, $\ast\ast$ are inverses, hence bijections. $\square$

III Hopf algebras

In terms of commuting diagrams, an algebra is a nonzero vector space A together with linear maps

$$\mu : A \otimes A \to A, \qquad \eta : k \to A$$

that make the following diagrams commute:

(Assoc)

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "\mu \otimes id"] \arrow[d, "id \otimes \mu"'] & A \otimes A \arrow[d, "\mu"] \\ A \otimes A \arrow[r, "\mu"'] & A \end{tikzcd}$$

(Unit)

$$\begin{tikzcd} k \otimes A \arrow[r, "\eta \otimes id"] \arrow[dr] & A \otimes A \arrow[d, "\mu"] & A \otimes k \arrow[l, "id \otimes \eta"'] \arrow[dl, "\mu"] \\ & A & \end{tikzcd}$$

$$\begin{tikzcd} 1 \otimes a \arrow[r, mapsto] & a \end{tikzcd}$$

The above algebra A is commutative whenever this diagram commutes:

$$\begin{tikzcd} A\otimes A \arrow[rr, "\tau_{A,A}"] \arrow[dr, "\mu"'] & & A\otimes A \arrow[dl, "\mu"] \\ & A & \end{tikzcd}$$

Given algebras A, A', an algebra morphism is a linear map $f: A \to A'$ s.t. these diagrams commute:

$$\begin{tikzcd} & A \arrow[d, "f"] \\ k \arrow[ur, "\eta"] \arrow[dr, "\eta'"'] & \\ & A' \end{tikzcd} \qquad \begin{tikzcd} A\otimes A \arrow[r, "f\otimes f"] \arrow[d, "\mu"'] & A'\otimes A' \arrow[d, "\mu'"] \\ A \arrow[r, "f"'] & A' \end{tikzcd}$$

DEF 1 A <u>coalgebra</u> is a non0 vector space C together with linear maps

$$\Delta : C \longrightarrow C \otimes C, \qquad \varepsilon : C \to k$$

s.t. the following diagrams commutes

(Coassoc)

$$\begin{tikzcd}
C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes C \arrow[d, "id \otimes \Delta"] \\
C \otimes C \arrow[r, "\Delta \otimes id"'] & C \otimes C \otimes C
\end{tikzcd}$$

(Counit)

$$\begin{tikzcd}
k \otimes C & C \otimes C \arrow[l, "\varepsilon \otimes id"'] \arrow[r, "id \otimes \varepsilon"] & C \otimes k \\
& C \arrow[u, "\Delta"] \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

$1 \otimes c \mapsfrom c$ $\qquad$ $c \mapsto c \otimes 1$

Coalg C is cocommutative whenever this diag commutes:

$$
\begin{tikzcd}
 & C \arrow[dl, "\Delta"'] \arrow[dr, "\Delta"] & \\
C \otimes C \arrow[rr, "\tau_{C,C}"'] & & C \otimes C
\end{tikzcd}
$$

For coalgebras C, C' a co-algebra morphism $C \to C'$ is a linear map $f: C \to C'$ that makes these diagrams commute:

$$
\begin{tikzcd}
 & C \arrow[dl, "\varepsilon"'] \arrow[d, "f"] \\
k & C' \arrow[l, "\varepsilon'"]
\end{tikzcd}
\qquad
\begin{tikzcd}
C \arrow[r, "f"] \arrow[d, "\Delta"'] & C' \arrow[d, "\Delta'"] \\
C \otimes C \arrow[r, "f \otimes f"'] & C' \otimes C'
\end{tikzcd}
$$

Ex 2 Recall algebra $M(2)$ with gens x_{ij}, $1 \le i,j \le 2$

Recall algebra morphisms

$$\Delta : M(2) \longrightarrow M(2) \otimes M(2)$$

$$x_{ij} \longrightarrow \sum_{\ell} x_{i\ell} \otimes x_{\ell j} \qquad 1 \le i,j \le 2$$

$$\varepsilon : M(2) \longrightarrow k$$

$$x_{ij} \longrightarrow \delta_{ij}$$

Then Δ, ε turn $M(2)$ into a coalgebra.

Check:

(Coassoc)

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes C \arrow[d, "\Delta \otimes id"] \\ C \otimes C \arrow[r, "id \otimes \Delta"'] & C \otimes C \otimes C \end{tikzcd}
\qquad
\begin{tikzcd} x_{ij} \arrow[r] \arrow[d] & \sum_{\ell} x_{i\ell} \otimes x_{\ell j} \arrow[d] \\ \sum_{k} x_{ik} \otimes x_{kj} \arrow[r] & \sum_{\ell, k} x_{ik} \otimes x_{k\ell} \otimes x_{\ell j} \end{tikzcd} \checkmark$$

(counit L)

$$\begin{tikzcd} k \otimes C & C \otimes C \arrow[l, "\varepsilon \otimes id"'] \\ & C \arrow[u, "\Delta"'] \arrow[ul, "1 \otimes c \,\leftarrow\, c"] \end{tikzcd}
\qquad
\begin{tikzcd} \sum_{\ell} \delta_{i\ell} \otimes x_{\ell j} \arrow[d, phantom, "\shortparallel"] & \sum_{\ell} x_{i\ell} \otimes x_{\ell j} \arrow[l] \\ 1 \otimes x_{ij} & x_{ij} \arrow[u] \arrow[l] \end{tikzcd}$$

(counit R)

sim.

$\square$

Ex 3 $\quad k$ has coalg structure with

$$\Delta : k \to k \otimes k$$
$$1 \to 1 \otimes 1$$

$$\varepsilon : k \to k$$
$$1 \to 1$$

Ex 4 Given Coalg C, Δ, ε define

$$\Delta^{op} : C \xrightarrow{\Delta} C \otimes C \xrightarrow{\tau_{C,C}} C \otimes C$$

then $C, \Delta^{op}, \varepsilon$ is coalg (ex)

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Ex 5 Given finite dim'l algebra A, turn the dual space $C = A^*$ into a coalgebra.

Define Δ

$$\Delta : \quad A^* \xrightarrow{\mu^*} (A \otimes A)^* \longrightarrow A^* \otimes A^*$$

transpose of mult map

using iso

$$\left[\begin{array}{l} (U \otimes V)^* \to V^* \otimes U^* \\ H \to g \otimes f \\ H(u \otimes v) = f(u) g(v) \end{array} \right]$$

For $f \in A^*$, write

$$\Delta(f) = \sum_i f_i' \otimes f_i''$$

We have

$$f(ab) = \sum_i f_i'(b) \, f_i''(a) \qquad \forall a, b \in A$$

Define ε

$$\varepsilon: \quad A^* \longrightarrow k^* \longrightarrow k$$

$$\mathfrak{Z}^* \qquad h \longrightarrow h(1)$$

For $f \in A^*$ we have

$$\varepsilon(f) = f(1)$$

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check coassoc:

for $f \in A^*$ write

$$\Delta(f) = \sum_i f_i' \otimes f_i''$$

For $g = f_i'$ write

$$\Delta(g) = \sum_j g_j' \otimes g_j''$$

For $h = f_i''$ write

$$\Delta(h) = \sum_l h_l' \otimes h_l''$$

require

$$\sum_{i,j} (f_i')_j' \otimes (f_i')_j'' \otimes f_i'' \overset{?}{=} \sum_{i,l} f_i' \otimes (f_i'')_l' \otimes (f_i'')_l'' \qquad *$$

By constr, $\forall a,b,c \in A$

$$f(abc) = \sum_{i,j} (f_i')_j'(c)\, (f_i')_j''(b)\, f_i''(a) \qquad **$$

and

$$f(abc) = \sum_{i,l} f_i'(c)\, (f_i'')_l'(b)\, (f_i'')_l''(a)$$

By $**$ and since

$$(A \otimes A \otimes A)^* = A^* \otimes A^* \otimes A^*$$

we obtain $*$.

check counit:

$$\begin{tikzcd} k \otimes A^* & A^* \otimes A^* \arrow[l, "\varepsilon \otimes id"'] \\ & A^* \arrow[u, "\Delta"'] \arrow[ul] \end{tikzcd} \qquad \begin{tikzcd} \sum_i f_i'(1) \otimes f_i'' & \sum_i f_i' \otimes f_i'' \arrow[l] \\ & f \arrow[u] \arrow[ul, "1 \otimes f"] \end{tikzcd}$$

Require

$$f = \sum_i f_i'(1)\, f_i'', \qquad f = \sum_i f_i'\, f_i''(1) \qquad ☆$$

Have

$$f(ab) = \sum_i f_i'(b)\, f_i''(a) \qquad \forall\, a, b \in A$$

Set $a = 1$:

$$f(b) = \sum_i f_i'(b)\, f_i''(1) \qquad \forall\, b \in A$$

Set $b = 1$:

$$f(a) = \sum_i f_i'(1)\, f_i''(a) \qquad \forall\, a \in A$$

Line ☆ follows.

We just turned A^* into a coalgebra.

We also have the opposite coalgebra

s.t. for $f \in A^*$

$$\Delta(f) = \sum_i f_i' \otimes f_i'' \qquad ☆☆$$

where

$$f(ab) = \sum_i f_i'(a)\, f_i''(b) \qquad \forall\, a, b \in A.$$

By <u>the coalgebra A^*</u> we mean ☆☆ $\square$