The inclusion maps

$$X \overset{}{\underset{\text{incl}}{\hookrightarrow}} X \cup Y \overset{}{\underset{\text{incl}}{\hookrightarrow}} k\{X \cup Y\}$$

induce alg morphism

incl: $k\{X\} \to k\{X \cup Y\}$

$x \mapsto x$

Sim $\exists$ alg morphism

$k\{Y\} \to k\{X \cup Y\}$

$y \mapsto y$

Define alg morphisms φ, ϕ by

φ : $k\{X\} \xrightarrow[\text{incl}]{} k\{X \cup Y\} \longrightarrow k\{X \cup Y\}/L$

$r \mapsto r + L$

ϕ : $k\{Y\} \xrightarrow[\text{incl}]{} k\{X \cup Y\} \longrightarrow k\{X \cup Y\}/L$

$r \mapsto r + L$

Obs $\varphi(x)\,\phi(y) = \phi(y)\,\varphi(x)$ $\quad \forall x \in X,\ \forall y \in Y.$

By Prop 46 $\exists$ alg morphism

$$\varphi \otimes \phi : \quad k\{X\} \otimes k\{Y\} \longrightarrow k\{X \cup Y\}/L$$

$$x \otimes 1 \longrightarrow x + L \qquad \forall x \in X$$

$$1 \otimes y \longrightarrow y + L \qquad \forall y \in Y$$

The maps χ and $\varphi \otimes \phi$ are inverses, hence bijections.

Result follows. $\square$

Aside about vector spaces

Given vs U, V.

Given subspaces $I \subseteq U$, $J \subseteq V$

Get lin maps

$$\text{can}: \begin{array}{l} U \to U/I \\ u \mapsto u+I \end{array} \qquad \text{can}: \begin{array}{l} V \to V/J \\ v \mapsto v+J \end{array}$$

Consider lin map

$$U \otimes V \xrightarrow[\text{can} \otimes \text{can}]{} (U/I) \otimes (V/J) \qquad *$$

Find $\ker(*)$

Obs
$$I \otimes V + U \otimes J \subseteq \ker(*) \qquad **$$

LEM 50 $I \otimes V + U \otimes J = \ker(*)$

pf By $**$, the map $*$ factors into the composition of lin maps

$$U \otimes V \xrightarrow{\text{can}} \frac{U \otimes V}{I \otimes V + U \otimes J} \xrightarrow{\varphi} \left(\frac{U}{I}\right) \otimes \left(\frac{V}{J}\right)$$

$$u \otimes v + I \otimes V + U \otimes J \longrightarrow (u+I) \otimes (v+J)$$

Show φ is bijection. Display inverse of φ.

Start with bilin map

$$\theta : \quad U/I \times V/J \longrightarrow \frac{U \otimes V}{I \otimes V + U \otimes J}$$

$$u+I, \quad v+J \longrightarrow u \otimes v + I \otimes V + U \otimes J$$

θ induces lin map

$$\bar{\theta} : \quad (U/I) \otimes (V/J) \longrightarrow \frac{U \otimes V}{I \otimes V + U \otimes J}$$

$$(u+I) \otimes (v+J) \longrightarrow u \otimes v + I \otimes V + U \otimes J$$

the maps φ and $\bar{\theta}$ are inverses, hence bijections.

Result follows. $\square$

COR 51 Given vs U, V, U', V'

Given lin maps

$$f : U \to U' \qquad g : V \to V'$$

Write

$$I = \ker(f) \qquad J = \ker(g)$$

then the linear map

$$U \otimes V \xrightarrow[f \otimes g]{} U' \otimes V'$$

has kernel

$$I \otimes V + U \otimes J.$$

pf Use LEM 50 $\square$

Given disjoint sets

$X, \quad Y$

Free algebras

$$k\{X\}, \quad k\{Y\}, \quad k\{X \cup Y\}$$

Given

I = 2-sided ideal of $k\{X\}$ s.t. $I \neq k\{X\}$

J = ... $k\{Y\}$ s.t. $J \neq k\{Y\}$

Quotient algebras

$$A = k\{X\}/I,$$

$$B = k\{Y\}/J$$

the composition of linear maps

$$k\{X \cup Y\} \longrightarrow k\{X\} \otimes k\{Y\} \xrightarrow[\text{can} \otimes \text{can}]{} A \otimes B \qquad \ast$$

$$x \longmapsto x \otimes 1$$
$$y \longmapsto 1 \otimes y$$

is surj.

Next goal: describe $\ker(\ast)$

Recall incl maps

$$k\{X\} \to k\{X \cup Y\}, \qquad k\{Y\} \to k\{X \cup Y\}$$
$$x \to x \qquad\qquad y \to y$$

Identify $k\{X\}$, $k\{Y\}$ with their images.

Define

$\mathbb{I}$ = 2-sided ideal of $k\{X \cup Y\}$ gen by I

$\mathbb{J}$ = ... J

L = ... $\{xy - yx \mid x \in X, y \in Y\}$

Prop 52 $\ker(*) = \mathbb{I} + \mathbb{J} + L$

pf For lin map

$$k\{X\} \otimes k\{Y\} \xrightarrow[\text{can} \otimes \text{can}]{} A \otimes B$$

the kernel is

$$I \otimes k\{Y\} + k\{X\} \otimes J$$

by Cor 51

Under lin map

$$k\{x \cup y\} \longrightarrow k\{x\} \otimes k\{y\} \qquad ☆$$

$$x \longrightarrow x \otimes 1$$

$$y \longrightarrow 1 \otimes y$$

- $\ker(☆) = L$ by Prop 49
- ☆ sends $\mathbb{I}$ onto $I \otimes k\{y\}$
- ☆ sends $\mathbb{J}$ onto $k\{x\} \otimes J$

So far

$$\ker(*) \supseteq \mathbb{I} + \mathbb{J} + L$$

show $\subseteq$:

Given $w \in \ker(*)$

show $w \in \mathbb{I} + \mathbb{J} + L$

Under ☆,

$$\underbrace{\text{image of } w}_{\shortparallel} \text{ contained in } I \otimes k\{y\} + k\{x\} \otimes J$$

$$w' = w_1 + w_2, \quad w_1 \in I \otimes k\{y\}, \quad w_2 \in k\{x\} \otimes J$$

$\exists\ i \in \mathbb{I}$ that ☆ sends to w_1

$\exists\ j \in \mathbb{J}$ that ☆ sends to w_2

Obs

$$w' - i - j \in \ker(☆) = L$$

So

$$w' \in \mathbb{I} + \mathbb{J} + L \qquad \square$$

Recall from Chapter I the algebra

$$M(2) = K[x_{11}, x_{12}, x_{21}, x_{22}] \qquad x_{ij} \text{ commute}$$

$$M(2)^{\otimes 2} = K[x'_{ij}, x''_{ij} \mid 1 \le i,j \le 2] \qquad x'_{ij}, x''_{ij} \text{ commute}$$

Recall Alg morphism

$$\Delta : \begin{array}{ccc} M(2) & \to & M(2)^{\otimes 2} \\ x_{ij} & \to & \sum_{\ell=1}^{2} x'_{i\ell} x''_{\ell j} \end{array} \qquad 1 \le i,j \le 2$$

We have alg iso

$$\begin{array}{ccc} M(2)^{\otimes 2} & \leftrightarrow & M(2) \otimes M(2) \\ x'_{ij} & \leftrightarrow & x_{ij} \otimes 1 \\ x''_{ij} & \leftrightarrow & 1 \otimes x_{ij} \end{array}$$

Replacing $M(2)^{\otimes 2}$ by $M(2) \otimes M(2)$

Δ becomes

$$\Delta : \begin{array}{ccc} M(2) & \to & M(2) \otimes M(2) \\ x_{ij} & \to & \sum_{\ell=1}^{2} x_{i\ell} \otimes x_{\ell j} \end{array} \qquad 1 \le i,j \le 2$$

Write $d = x_{11}x_{22} - x_{12}x_{21} \in M(2)$

We have $\Delta(d) = d \otimes d.$

Similar comments for $GL(2)$, $SL(2)$.

Tensor algebras

(Motivation)

Given set X

Recall free alg $k\{X\}$

Given alg A and function $f : X \to A$

$\exists$ unique alg morphism $\bar{f} : k\{X\} \to A$ s.t

$$\begin{tikzcd} X \arrow[r, "\text{incl}"] \arrow[dr, "f"'] & k\{X\} \arrow[d, "\bar{f}"] \\ & A \end{tikzcd}$$

commutes.

The map

$$\mathrm{Hom}_{set}(X, A) \leftrightarrow \mathrm{Hom}_{alg}(k\{X\}, A)$$

$$f \to \bar{f}$$

is a bijection.

We now describe a "basis free" analog of $k\{X\}$

Given vectr space V

Define tensor algebra $T(V)$ as follows

Define vector spaces

$$T^0(V) = k$$

$$T^1(V) = V$$

$$T^2(V) = V \otimes V$$

$$\cdots$$

$$T(V) = \bigoplus_{n \in \mathbb{N}} T^n(V) \qquad *$$

$*$ becomes a graded algebra with mult

$$\begin{aligned} T^r(V) \times T^s(V) &\to T^{r+s}(V) \\ u \cdot v &\to u \otimes v \end{aligned}$$

The incl map

$$\begin{aligned} i_V \quad V &\to T^1(V) \subseteq T(V) \\ v &\to v \end{aligned}$$

is inj.

Given alg A

Given lin map

$$f : V \to A$$

$\exists$ unique alg morphism

$$\bar{f} : T(V) \to A$$

s.t.

$$\begin{tikzcd} V \arrow[r, "i_V"] \arrow[dr, "f"'] & T(V) \arrow[d, "\bar{f}"] \\ & A \end{tikzcd}$$

commutes

For $n \in \mathbb{N}$ and $v_1, \dots, v_n \in V$

$$\bar{f}(v_1 \otimes \cdots \otimes v_n) = f(v_1) \cdots f(v_n)$$

The map

$$\begin{aligned} \mathrm{Hom}(V, A) &\longleftrightarrow \mathrm{Hom}_{alg}(T(V), A) \\ f &\longrightarrow \bar{f} \end{aligned}$$

is a bijection.

Let X denote a basis for V

Then $\exists$ algebra iso

$$k\{X\} \longrightarrow T(V)$$

that sends

$$x_1 x_2 \cdots x_n \longrightarrow x_1 \otimes x_2 \otimes \cdots \otimes x_n \qquad \forall n \in \mathbb{N}$$
$$\forall x_1, \ldots, x_n \in X.$$

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Symmetric algebras

Given a vector space V

Recall tensor algebra $T(V)$

Define

I = 2-sided ideal of $T(V)$ gen by

$u \otimes v - v \otimes u$ $\quad u, v \in V$

Ideal I is homogeneous.

Define quotient algebra

$S(V) = T(V)/I$ "symmetric algebra on V"

$S(V)$ is commutative.

$S(V)$ inherits the grading of $T(V)$:

For $n \in \mathbb{N}$ define

$$S^n(V) = \text{image of } T^n(V) \text{ under quot map } T(V) \to S(V)$$

$$= \frac{T^n(V) + I}{I}$$

$$\simeq \frac{T^n(V)}{T^n(V) \cap I}$$

Since I is homogeneous,

$$S(V) = \sum_{n \in \mathbb{N}} S^n(V) \qquad \text{(dir sum)}$$

$$\cong \bigoplus_{n \in \mathbb{N}} S^n(V)$$

Observe

$$T^n(V) \cap I = 0 \qquad \text{for } n \leq 1$$

So

$$S^0(V) \cong T^0(V) = k$$

$$S^1(V) \cong T^1(V) = V$$

The composition

$$V \xrightarrow[\text{incl}]{} T(V) \xrightarrow[\text{quot}]{} S(V) \qquad *$$

is injective.

Identify V with its image under $*$.

Then algebra $S(V)$ is generated by V.