

trace map, cont.
claim

the following diagram commutes:

$$\begin{array}{ccc}
 V \otimes V^* & \xrightarrow{\lambda_{V,V}} & \text{End}(V) \\
 \downarrow \tau_{V,V^*} & & \downarrow \text{tr} \\
 V^* \otimes V & \xrightarrow{\text{ev}} & k
 \end{array}$$

check: Pick basis $\{v_i\}_i$ of V

$$\begin{array}{ccc}
 v_i \otimes v^j & \rightarrow & e_{ij} \\
 \downarrow & & \downarrow \delta_{ij} \\
 v^j \otimes v_i & \rightarrow & \langle v^j, v_i \rangle
 \end{array}$$

OK

Motivated by the above claim, we officially define the trace map in the following basis free way:

DEF 39 For a fin dim'l vector space V ,

$$\text{tr}: \text{End}(V) \xrightarrow{\lambda_{V,V}} V \otimes V^* \xrightarrow{\tau_{V,V^*}} V^* \otimes V \xrightarrow{\text{ev}} k$$

We prove a few basic facts about tr using the above definition.

LEM 40 Given a fin dim vectn space V .

$F, F' \in \text{End}(V)$,

$$\text{tr}(F \circ F') = \text{tr}(F' \circ F)$$

pf Consider vectn space U

$$V \otimes V^* \xleftrightarrow{\text{isom}} \text{End}(V)$$

$$v \otimes f \leftrightarrow F$$

$$F(w) = f(w)v \quad \forall w \in V$$

$$v' \otimes f' \leftrightarrow F'$$

$$F'(w) = f'(w)v'$$

$$f(v) v \otimes f' \leftrightarrow F \circ F'$$

$$f'(v') v' \otimes f \leftrightarrow F' \circ F$$

$$\text{tr}: F \circ F' \xrightarrow{\text{isom}} f(v) v \otimes f' \xrightarrow{\text{isom}} f(v) f'(v') \xrightarrow{\text{ev}} f(v) f'(v')$$

Similarly

$$\text{tr} F' \circ F \rightarrow f'(v') f(v')$$

Result follows. □

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LEM 41 Given a fin dim'l vector space V .

For $f \in \text{End}(V)$,

$$\text{tr}(f) = \text{tr}(f^*).$$

pf Represent f by a matrix m with respect to a basis $\{v_i\}$ of V .

By LEM 38, the usual transpose m^t represents f^* with respect to the dual basis $\{v^i\}$.

Sum the diagonal entries of m, m^t to get result. \square

LEM 42 Given a fin dim'l vector space $V \neq 0$,

For $f \in \text{End}(V)$, $\text{tr}(f)$ is equal to the composition

$$k \rightarrow V \otimes V^* \xrightarrow{\text{Tr} \circ f} V^* \otimes V \xrightarrow{\text{id} \otimes f} V^* \otimes V \xrightarrow{\text{ev}} k$$

pf Pick a basis $\{v_i\}_i$ for V .

$$1 \rightarrow \sum_i v_i \otimes v_i^* \rightarrow \sum_i v_i \otimes v_i \rightarrow \sum_i v_i \otimes f(v_i) \rightarrow \underbrace{\sum_i \langle v_i, f(v_i) \rangle}_{\text{tr}(f)}$$

□

the partial transpose (differs from ~~ext~~)

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Given f vs V, W, X, Y

Given $f \in \text{Hom}(V \otimes W, X \otimes Y)$

View $f^* \in \text{Hom}(Y^* \otimes X^*, W^* \otimes V^*)$

We define the partial transposes

$$f^+ \in \text{Hom}(V \otimes X^*, W^* \otimes Y)$$

$$f^x \in \text{Hom}(Y^* \otimes W, X \otimes V^*)$$

as follows.

We have vector space isomorphisms

$$\begin{aligned} \text{Hom}(V \otimes W, X \otimes Y) &\stackrel{\lambda}{\cong} \text{Hom}(W, X) \otimes \text{Hom}(V \otimes Y) \\ &\cong X \otimes W^* \otimes Y \otimes V^* \end{aligned}$$

★

Also

$$\begin{aligned} \text{Hom}(Y^* \otimes X^*, W^* \otimes V^*) &\stackrel{\lambda}{\cong} \text{Hom}(X^* \otimes W^*, Y^* \otimes V^*) \\ &\cong W^* \otimes X \otimes V^* \otimes Y \end{aligned}$$

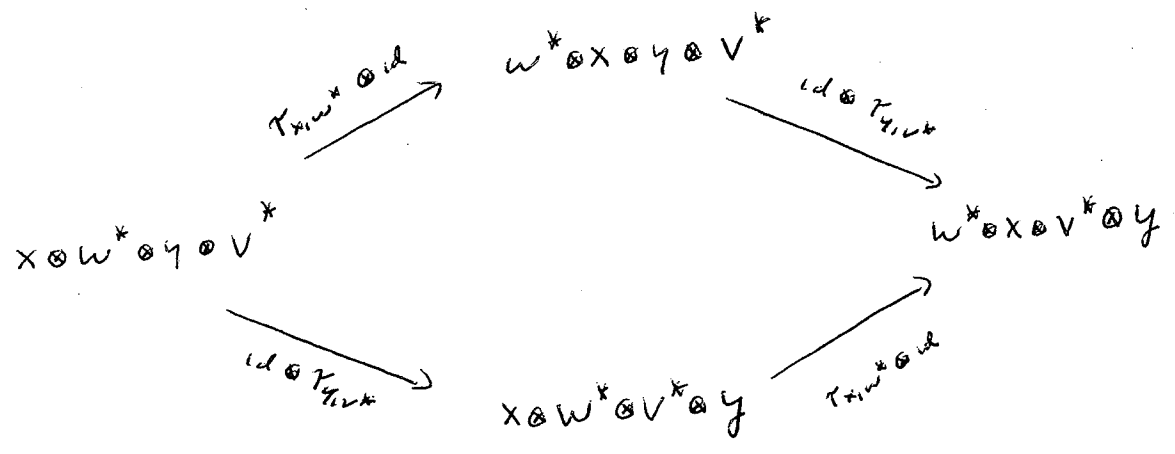
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From the ★ pt of view, the transpose map $f \rightarrow f^*$ becomes

$$X \otimes W^* \otimes Y \otimes V^* \xrightarrow{\tau_{X, W^*} \otimes \tau_{Y, V^*}} W^* \otimes X \otimes V^* \otimes Y$$

★★

Factor ★★ as follows:



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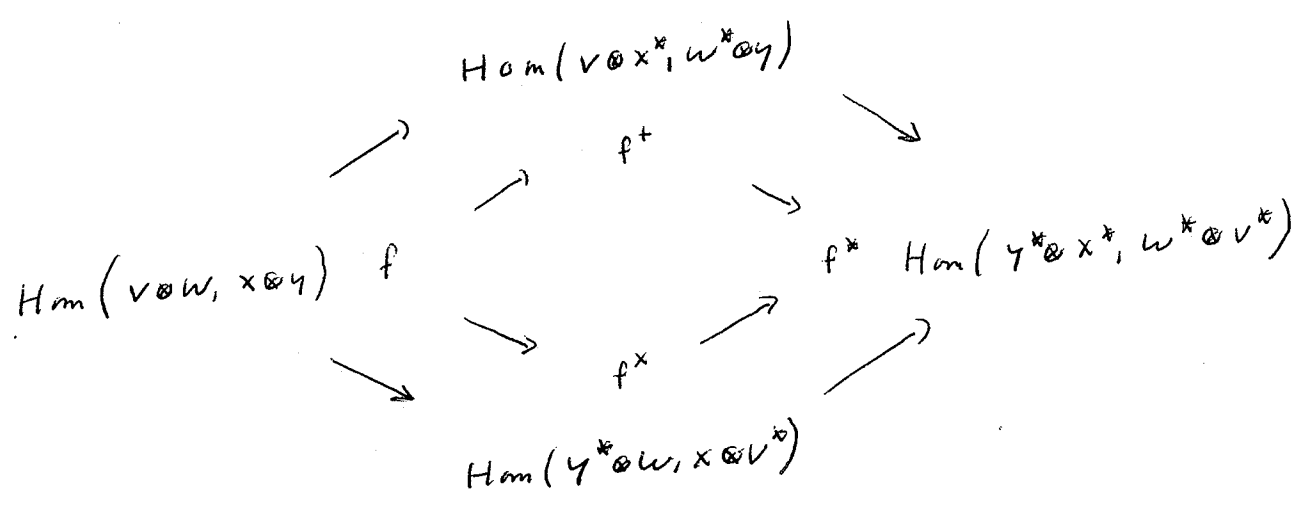
We have vs isomorphisms

$$\begin{aligned} \text{Hom}(V \otimes X^*, W^* \otimes Y) &\cong^{\lambda} \text{Hom}(X^*, W^*) \otimes \text{Hom}(V, Y) \\ &\cong W^* \otimes X \otimes Y \otimes V^* \end{aligned}$$

and

$$\begin{aligned} \text{Hom}(Y^* \otimes W, X \otimes V^*) &\cong^{\lambda} \text{Hom}(W, X) \otimes \text{Hom}(Y^*, V^*) \\ &\cong X \otimes W^* \otimes V \otimes Y \end{aligned}$$

At orig "Hom" level, * becomes



One checks

$$\langle y^* \otimes w, f^+(v \otimes x^*) \rangle = \langle y^* \otimes x^*, f(v \otimes w) \rangle,$$

$$\langle v \otimes x^*, f^*(y^* \otimes w) \rangle = \langle y^* \otimes x^*, f(v \otimes w) \rangle$$

$\forall v \in V, w \in W, x^* \in X^*, y^* \in Y^*$

Pick bases for V, W, X, Y

v_i	basis
v	v_i
w	w_j
x	x_r
y	y_s

So for $f \in \text{Hom}(V \otimes W, X \otimes Y)$,

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$$f(v_i \otimes w_j) = \sum_{r,s} \langle y^s \otimes x^r, f(v_i \otimes w_j) \rangle x^r \otimes y^s \quad \forall i, j$$

the transpose $f^* \in \text{Hom}(Y^* \otimes X^*, W^* \otimes V^*)$

satisfies

$$f^*(y^s \otimes x^r) = \sum_{i,j} \underbrace{\langle f^*(y^s \otimes x^r), v_i \otimes w_j \rangle}_{\langle y^s \otimes x^r, f(v_i \otimes w_j) \rangle} w^i \otimes v^j$$

the partial transpose

$f^+ \in \text{Hom}(V \otimes X^*, W^* \otimes Y)$

satisfies

$$f^+(v_i \otimes x^r) = \sum_{j,s} \underbrace{\langle v_i \otimes x^r, f^+(v_i \otimes x^r) \rangle}_{\langle (v_i \otimes x^r), f(v_i \otimes w_j) \rangle} w^j \otimes y^s \quad \forall i, r$$

The partial transpose

$$f^x \in \text{Hom}(Y^* \otimes W, X \otimes V^*)$$

satisfies

$$f^x(y^i \otimes w_j) = \sum_{i,r} \underbrace{\langle v_i \otimes x^r, f^x(y^i \otimes w_j) \rangle}_{\parallel} x_r \otimes v^i$$

$$\langle y^i \otimes x^r, f(v_i \otimes w_j) \rangle$$

v_j, 2

Tensor products of algebras

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LEM 43 Given algebras A, B .

then the vector space $A \otimes B$ becomes an algebra with mult

$$\begin{array}{rcll} & A \otimes B & \times & A \otimes B & \rightarrow & A \otimes B \\ \mu: & a \otimes b & & a' \otimes b' & \rightarrow & (aa') \otimes (bb') \end{array}$$

The mult identity is $1 \otimes 1$.

pf. Start with mult maps

$$\begin{array}{rcll} \mu_A: & A & \times & A & \rightarrow & A \\ & a & & a' & \rightarrow & aa' \end{array}$$

$$\begin{array}{rcll} \mu_B: & B & \times & B & \rightarrow & B \\ & b & & b' & \rightarrow & bb' \end{array}$$

these induce linear maps

$$\bar{\mu}_A : A \otimes A \rightarrow A$$

$$\bar{\mu}_B : B \otimes B \rightarrow B$$

Define a linear map

$$\bar{\mu}: A \otimes B \otimes A \otimes B \xrightarrow{\text{id} \otimes T_{B,A} \otimes \text{id}} A \otimes A \otimes B \otimes B \xrightarrow{\bar{\mu}_A \otimes \bar{\mu}_B} A \otimes B$$

$\bar{\mu}$ induces a bilinear map

$$\mu: A \otimes B \times A \otimes B \rightarrow A \otimes B$$

$$r \quad a \quad \rightarrow \quad \bar{\mu}(ra)$$

One checks that μ satisfies the requirements.

The last assertion is clear.

□

LEM 44 Given algebras A, B .

The following are injective algebra morphisms:

$$i_A: \begin{array}{l} A \rightarrow A \otimes B \\ a \rightarrow a \otimes 1 \end{array} \quad i_B: \begin{array}{l} B \rightarrow A \otimes B \\ b \rightarrow 1 \otimes b \end{array}$$

Moreover

$$i_A(a) i_B(b) = i_B(b) i_A(a) = a \otimes b \quad \forall a \in A \quad \forall b \in B$$

pt i_A is alg morphism

check inj. suppose $a \otimes 1 = 0$ show $a = 0$.

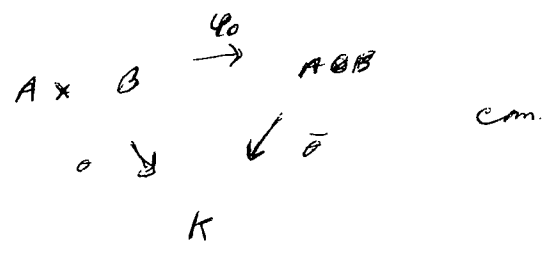
Suppose $a \neq 0$ \exists lin trans $\varphi: A \rightarrow K$ that sends $a \rightarrow 1$.

\exists lin trans $\phi: B \rightarrow K$ that sends $1 \rightarrow 1$.

Define bilin map

$$\theta: \begin{array}{l} A \times B \rightarrow K \\ r, s \rightarrow \varphi(r) \phi(s) \end{array}$$

\exists lin map $\bar{\theta}: A \otimes B \rightarrow K$ s.t.



But $a, 1 \rightarrow a \otimes 1 = 0$



□

Given algebras A, B .

Recall bilinear map

$$\begin{aligned} \varphi_0: A \times B &\longrightarrow A \otimes B \\ a \quad b &\longrightarrow a \otimes b \end{aligned}$$

Given algebra C

Given any algebra morphism

$$\psi: A \otimes B \longrightarrow C$$

Consider composition

$$\theta: A \times B \xrightarrow{\varphi_0} A \otimes B \xrightarrow{\psi} C \quad (\text{bilinear})$$

By construction the diagram

$$\begin{array}{ccc} A \times B & \xrightarrow{\varphi_0} & A \otimes B \\ & \searrow \theta & \swarrow \psi \\ & & C \end{array} \quad \psi \circ \varphi_0 = \theta$$

commutes.

Describe θ :

LEM 45 With the above notation,

(i) The map

$$\varphi: \begin{array}{l} A \rightarrow C \\ a \rightarrow \theta(a, 1) \end{array} \quad \text{is an algebra morphism}$$

(ii) The map

$$\phi: \begin{array}{l} B \rightarrow C \\ b \rightarrow \theta(1, b) \end{array} \quad \text{is an algebra morphism}$$

(iii) $\forall a \in A, \forall b \in B,$

$$\varphi(a)\phi(b) = \phi(b)\varphi(a) \quad *$$

pf (i), (ii) Routine

(iii) Each side of * is equal to $\theta(a, b)$. □

We now reverse direction.

Prop 46 Given algebras A, B, C .

Given algebra morphisms

$$\psi: A \rightarrow C,$$

$$\phi: B \rightarrow C$$

such that

$$\psi(a)\phi(b) = \phi(b)\psi(a) \quad \forall a \in A \quad \forall b \in B.$$

Then

(i) the map

$$\theta: \begin{array}{ccc} A \times B & \longrightarrow & C \\ a \quad b & \longrightarrow & \psi(a)\phi(b) \end{array}$$

is bilinear.

(ii) the induced map

$$\bar{\theta}: \begin{array}{ccc} A \otimes B & \longrightarrow & C \\ a \otimes b & \longrightarrow & \theta(a, b) \end{array}$$

is an algebra morphism.

(iii) the following diagrams commute:



pf (i) ✓

(ii) Check $\bar{\theta}$ respects mult. ?

$$\bar{\theta}(uv) = \bar{\theta}(u) \bar{\theta}(v)$$

$\forall u, v \in A \otimes B$

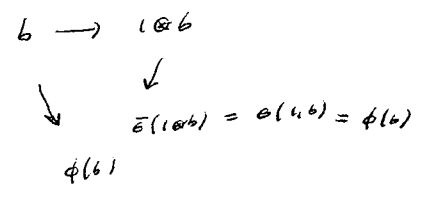
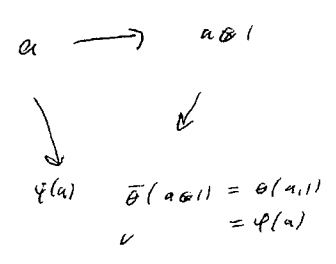
wlog $u = a \otimes b \quad v = a' \otimes b'$

$$\begin{aligned} \bar{\theta}(u) &= \bar{\theta}(a \otimes b) \\ &= \varphi(a) \phi(b) \end{aligned}$$

$$\begin{aligned} \bar{\theta}(v) &= \bar{\theta}(a' \otimes b') \\ &= \varphi(a') \phi(b') \end{aligned}$$

$$\begin{aligned} \bar{\theta}(uv) &= \bar{\theta}(aa' \otimes bb') \\ &= \varphi(aa') \phi(bb') \\ &= \underbrace{\varphi(a) \varphi(a')}_{\varphi(b) \varphi(a')} \phi(b) \phi(b') \\ &= \bar{\theta}(u) \bar{\theta}(v) \quad \checkmark \end{aligned}$$

(iii) obs.

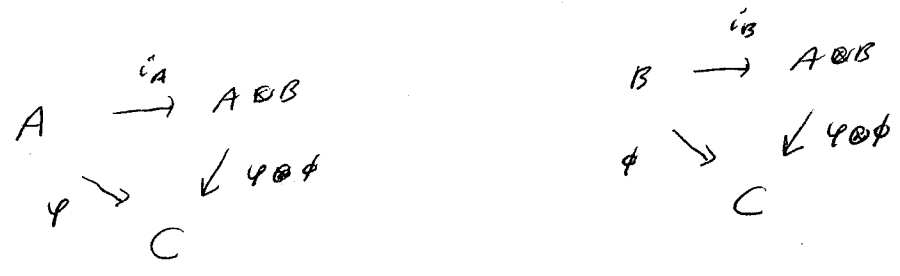


Def 47 Referring to Prop 46, define

$$\psi \otimes \phi = \bar{\theta}$$

Cor 48 With the notation/assumptions of Prop 46,

$\psi \otimes \phi$ is the unique algebra morphism $A \otimes B \rightarrow C$ that makes these diagrams commute:



Given algebras A, B, C .

The following is a bijection

$$\psi \otimes \phi \quad \text{Hom}_{\text{alg}}(A \otimes B, C)$$



$$(\psi, \phi) \quad \left\{ (\psi, \phi) \in \text{Hom}_{\text{alg}}(A, C) \times \text{Hom}_{\text{alg}}(B, C) \mid \begin{array}{l} \psi(a)\phi(b) = \phi(b)\psi(a) \\ \forall a \in A, \forall b \in B \end{array} \right\}$$

For C commutative, this bijection becomes

$$\text{Hom}_{\text{alg}}(A \otimes B, C) \iff \text{Hom}_{\text{alg}}(A, C) \times \text{Hom}_{\text{alg}}(B, C)$$

Given X, Y disjoint sets

Free algebras

$$k\{x\}, k\{y\}.$$

Compare algebras

$$k\{x \cup y\}, k\{x\} \otimes k\{y\}$$

\exists alg morphism

$$\theta : k\{x \cup y\} \rightarrow k\{x\} \otimes k\{y\}$$

that sends

$$x \rightarrow x \otimes 1$$

$\forall x \in X$

$$y \rightarrow 1 \otimes y$$

$\forall y \in Y$

θ is surj

Consider $\ker(\theta)$.

$$\forall x \in X \quad \forall y \in Y$$

$$\begin{aligned} \theta(xy - yx) &= x \otimes 1 \otimes y - 1 \otimes y \otimes x \\ &= x \otimes y - x \otimes y \\ &= 0 \end{aligned}$$

Define

$$L = \text{2-sided ideal of } k\{x \cup y\} \text{ gen by}$$

$$xy - yx \quad x \in X \quad y \in Y$$

So

$$L \subseteq \ker(\theta)$$

Prop 49

$$L = \ker(\theta)$$

pf

θ induces surj alg morphism

$$\frac{k\{x \cup y\}}{L} \longrightarrow k\{x\} \otimes k\{y\} \quad *$$

$$\begin{array}{lcl} x + L & \longrightarrow & x \otimes 1 \quad \forall x \in X \\ y + L & \longrightarrow & 1 \otimes y \quad \forall y \in Y \end{array}$$

show $*$ is bijectm.

Display inverse of $*$