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trace map, cont.

claim: the following diagram commutes:

$$\begin{tikzcd} V \otimes V^* \arrow[r, "\lambda_{V,V}"] \arrow[d, "\tau_{V,V^*}"'] & \mathrm{End}(V) \arrow[d, "tr"] \\ V^* \otimes V \arrow[r, "ev"'] & k \end{tikzcd}$$

check: Pick basis $\{v_i\}_i$ of V

$$\begin{tikzcd} v_i \otimes v^j \arrow[r] \arrow[d] & e_{ij} \arrow[d] \\ v^j \otimes v_i \arrow[r] & \delta_{ij} = \langle v^j, v_i \rangle \end{tikzcd}$$

ok

Motivated by the above claim, we officially define the trace map in the following basis free way:

DEF 39 For a fin dim'l vector space V,

$$tr: \mathrm{End}(V) \xrightarrow[\overleftarrow{\lambda_{V,V}}]{} V \otimes V^* \xrightarrow[\tau_{V,V^*}]{} V^* \otimes V \xrightarrow[ev]{} k$$

We prove a few basic facts about tr using the above definition.

LEM 40 Given a fin dim vectr space V.

For $F, F' \in End(V)$,

$$tr(F \circ F') = tr(F' \circ F)$$

pf Consider vectr space iso

$$V \otimes V^* \xleftrightarrow{\lambda_{V,V}} End(V)$$

$$v \otimes f \longleftrightarrow F \qquad F(w) = f(w)v \qquad \forall w \in V$$

$$v' \otimes f' \longleftrightarrow F' \qquad F'(w) = f'(w)v'$$

$$f(v')\, v \otimes f' \longleftrightarrow F \circ F'$$

$$f'(v)\, v' \otimes f \longleftrightarrow F' \circ F$$

$$tr: \quad F \circ F' \xrightarrow{\lambda_{V,V}^{-1}} f(v')\, v \otimes f' \xrightarrow{\tau_{V,V^*}} f(v')\, f' \otimes v \xrightarrow{ev} f(v')\, f'(v)$$

Similarly

$$tr \quad F' \circ F \longrightarrow f'(v) f(v')$$

Result follows. $\square$

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LEM 41 Given a fin dim'l vector space V.

For $f \in End(V)$,

$tr(f) = tr(f^*)$.

pf Represent f by a matrix m with respect to a basis $\{v_i\}_i$ of V.

By LEM 38, the usual transpose m^t represents f^* with respect to the dual basis $\{v^i\}_i$

Sum the diagonal entries of m, m^t to get result. $\square$

LEM 42 Given a fin dim'l vector space $V \neq 0$.

For $f \in \mathrm{End}(V)$, $\mathrm{tr}(f)$ is equal to the composition

$$k \xrightarrow{\delta} V \otimes V^* \xrightarrow{\tau_{V,V^*}} V^* \otimes V \xrightarrow{id \otimes f} V^* \otimes V \xrightarrow{ev} k$$

pf Pick a basis $\{v_i\}_i$ for V.

$$1 \to \sum_i v_i \otimes v^i \to \sum_i v^i \otimes v_i \to \sum_i v^i \otimes f(v_i) \to \underbrace{\sum_i \underbrace{\langle v^i, f(v_i) \rangle}_{= f_i^i}}_{= \mathrm{tr}(f)}$$

$\square$

the partial transpose (differs from text)

Given fd vs V, W, X, Y

Given
$$f \in Hom(V \otimes W, X \otimes Y)$$

View
$$f^* \in Hom(Y^* \otimes X^*, W^* \otimes V^*)$$

We define the partial transposes
$$f^+ \in Hom(V \otimes X^*, W^* \otimes Y),$$
$$f^\times \in Hom(Y^* \otimes W, X \otimes V^*)$$

as follows.

We have vector space isomorphisms

$$\mathrm{Hom}(V\otimes W, X\otimes Y) \overset{\lambda}{\cong} \mathrm{Hom}(W,X)\otimes \mathrm{Hom}(V,Y) \cong X\otimes W^*\otimes Y\otimes V^* \qquad ☆$$

Also

$$\mathrm{Hom}(Y^*\otimes X^*, W^*\otimes V^*) \overset{\lambda}{\cong} \mathrm{Hom}(X^*\otimes W^*, Y^*\otimes V^*) \cong W^*\otimes X\otimes V^*\otimes Y \qquad ☆$$

From the ☆ pt of view, the transpose map $f \to f^*$ becomes

$$X\otimes W^*\otimes Y\otimes V^* \xrightarrow[\tau_{X,W^*}\otimes \tau_{Y,V^*}]{} W^*\otimes X\otimes V^*\otimes Y \qquad ☆☆$$

Factor ☆☆ as follows:

$$\begin{tikzcd}
 & W^*\otimes X\otimes Y\otimes V^* \arrow[dr, "id\otimes \tau_{Y,V^*}"] & \\
X\otimes W^*\otimes Y\otimes V^* \arrow[ur, "\tau_{X,W^*}\otimes id"] \arrow[dr, "id\otimes\tau_{Y,V^*}"'] & & W^*\otimes X\otimes V^*\otimes Y \\
 & X\otimes W^*\otimes V^*\otimes Y \arrow[ur, "\tau_{X,W^*}\otimes id"'] &
\end{tikzcd} \qquad *$$

We have vs isomorphisms

$$\mathrm{Hom}(V\otimes X^*, W^*\otimes Y) \overset{\lambda}{\simeq} \mathrm{Hom}(X^*, W^*)\otimes \mathrm{Hom}(V, Y)$$
$$\simeq W^*\otimes X\otimes Y\otimes V^*$$

and

$$\mathrm{Hom}(Y^*\otimes W, X\otimes V^*) \overset{\lambda}{\simeq} \mathrm{Hom}(W, X)\otimes \mathrm{Hom}(Y^*, V^*)$$
$$\simeq X\otimes W^*\otimes V^*\otimes Y$$

At orig "Hom" level, $*$ becomes

$$\begin{tikzcd}
& \mathrm{Hom}(V\otimes X^*, W^*\otimes Y) \arrow[dr] & \\
\mathrm{Hom}(V\otimes W, X\otimes Y) \arrow[ur] \arrow[dr] & & \mathrm{Hom}(Y^*\otimes X^*, W^*\otimes V^*) \\
& \mathrm{Hom}(Y^*\otimes W, X\otimes V^*) \arrow[ur] &
\end{tikzcd}$$

$$\begin{tikzcd}
& f^+ \arrow[dr] & \\
f \arrow[ur] \arrow[dr] & & f^* \\
& f^\times \arrow[ur] &
\end{tikzcd}$$

One checks

$$\langle y^* \otimes w,\ f^+(v \otimes x^*) \rangle = \langle y^* \otimes x^*,\ f(v \otimes w) \rangle,$$

$$\langle v \otimes x^*,\ f^\times(y^* \otimes w) \rangle = \langle y^* \otimes x^*,\ f(v \otimes w) \rangle$$

$\forall$ $v \in V$, $w \in W$, $x^* \in X^*$, $y^* \in Y^*$

Pick bases for V, W, X, Y

vs	basis
V	v_i
W	w_j
X	x_r
Y	y_s

So for $f \in \mathrm{Hom}(V \otimes W, X \otimes Y)$,

$$f(v_i \otimes w_j) = \sum_{r,s} \langle y^s \otimes x^r, f(v_i \otimes w_j) \rangle \, x_r \otimes y_s$$

$\forall i, j$

the transpose $f^* \in \mathrm{Hom}(Y^* \otimes X^*, W^* \otimes V^*)$

satisfies

$w^j \in$

$$f^*(y^s \otimes x^r) = \sum_{i,j} \underbrace{\langle f^*(y^s \otimes x^r), v_i \otimes w_j \rangle}_{\begin{gathered} \parallel \\ \langle y^s \otimes x^r, f(v_i \otimes w_j) \rangle \end{gathered}}$$

the partial transpose
$f^+ \in \mathrm{Hom}(V \otimes X^*, W^* \otimes Y)$

satisfies

$$f^+(v_i \otimes x^r) = \sum_{j,s} \underbrace{\langle y^s \otimes w_j, f^+(v_i \otimes x^r) \rangle}_{\begin{gathered} \parallel \\ \langle y^s \otimes x^r, f(v_i \otimes w_j) \rangle \end{gathered}} \, w^j \otimes y_s$$

$\forall i, r$

The partial transpose

$$f^X \in \mathrm{Hom}\left(Y^* \otimes W,\ X \otimes V^* \right)$$

satisfies

$$f^X(y^a \otimes w_j) = \sum_{i,r} \underbrace{\left\langle v_i \otimes x^r,\ f^X(y^a \otimes w_j) \right\rangle}_{\substack{\parallel \\ \left\langle y^a \otimes x^r,\ f(v_i \otimes w_j) \right\rangle}} \quad x_r \otimes v^i$$

$\forall j, a$

Tensor products of algebras

LEM 43 Given algebras A, B.
Then the vector space $A \otimes B$ becomes an algebra with mult

$$\mu : \begin{array}{ccccc} A \otimes B & \times & A \otimes B & \to & A \otimes B \\ a \otimes b & & a' \otimes b' & \to & (aa') \otimes (bb') \end{array}$$

The mult identity is $1 \otimes 1$.

pf. Start with mult maps

$$\mu_A : \begin{array}{ccccc} A & \times & A & \to & A \\ a & & a' & \to & aa' \end{array}$$

$$\mu_B : \begin{array}{ccccc} B & \times & B & \to & B \\ b & & b' & & bb' \end{array}$$

these induce linear maps

$$\bar{\mu}_A : A \otimes A \to A$$

$$\bar{\mu}_B : B \otimes B \to B$$

Define a linear map

$$\bar{\mu}: A \otimes B \otimes A \otimes B \xrightarrow{id \otimes T_{B,A} \otimes id} A \otimes A \otimes B \otimes B \xrightarrow{\bar{\mu}_A \otimes \bar{\mu}_B} A \otimes B$$

$\bar{\mu}$ induces a bilinear map

$$\mu: \begin{array}{ccccc} A \otimes B & \times & A \otimes B & \longrightarrow & A \otimes B \\ r & & s & \longrightarrow & \bar{\mu}(r \otimes s) \end{array}$$

One checks that μ satisfies the requirements. $\square$

The last assertion is clear.

LEM 44 Given algebras A, B.

The following are injective algebra morphisms:

$$i_A : \begin{array}{l} A \to A\otimes B \\ a \to a\otimes 1 \end{array} \qquad i_B : \begin{array}{l} B \to A\otimes B \\ b \to 1\otimes b \end{array}$$

Moreover

$$i_A(a)\, i_B(b) = i_B(b)\, i_A(a) = a\otimes b \qquad \forall a\in A \ \forall b \in B$$

pf i_A is alg morphism ✓

check inj. suppose $a\otimes 1 = 0$ show $a = 0$.

Suppose $a \neq 0$ $\exists$ lin trans $\varphi : A \to k$ that sends $a \to 1$.

$\exists$ lin trans $\phi : B \to k$ that sends $1 \to 1$.

Define bilin map $\theta : \begin{array}{l} A\times B \to k \\ r, s \to \varphi(r)\phi(s) \end{array}$

$\exists$ lin map $\bar{\theta} : A\otimes B \to k$ s.t.

$$\begin{tikzcd} A\times B \arrow[r, "\varphi_0"] \arrow[dr, "\theta"'] & A\otimes B \arrow[d, "\bar{\theta}"] \\ & k \end{tikzcd}$$

com.

But

$$\begin{tikzcd} a, 1 \arrow[r] \arrow[dr] & a\otimes 1 = 0 \arrow[d] \\ & 1 \end{tikzcd}$$

gives a contr. □

Given algebras A, B.

Recall bilinear map

$$\varphi_0 : \begin{array}{ccc} A \times B & \longrightarrow & A \otimes B \\ a \quad b & \longrightarrow & a \otimes b \end{array}$$

Given algebra C

Given any algebra morphism

$$\psi : A \otimes B \to C$$

Consider composition

$$\theta : \quad A \times B \xrightarrow{\varphi_0} A \otimes B \xrightarrow{\psi} C \qquad \text{(bilinear)}$$

By construction the diagram

$$\begin{tikzcd} A \times B \arrow[r, "\varphi_0"] \arrow[dr, "\theta"'] & A \otimes B \arrow[d, "\psi = \bar{\theta}"] \\ & C \end{tikzcd}$$

commutes.

Describe θ :

LEM 45 With the above notation,

(i) The map

$$\varphi : \begin{array}{l} A \longrightarrow C \\ a \longrightarrow \theta(a,1) \end{array}$$ is an algebra morphism

(ii) The map

$$\phi : \begin{array}{l} B \longrightarrow C \\ b \longrightarrow \theta(1,b) \end{array}$$ is an algebra morphism

(iii) $\forall a \in A, \ \forall b \in B,$

$$\varphi(a)\,\phi(b) = \phi(b)\,\varphi(a) \qquad *$$

pf (i), (ii) Routine

(iii) Each side of $*$ is equal to $\theta(a,b)$. $\square$

We now reverse direction.

Prop 46 Given algebras A, B, C.

Given algebra morphisms

$$\varphi : A \to C, \qquad \phi : B \to C$$

such that

$$\varphi(a)\,\phi(b) = \phi(b)\,\varphi(a) \qquad \forall a \in A \quad \forall b \in B.$$

Then

(i) the map

$$\theta : \begin{array}{ccc} A \times B & \longrightarrow & C \\ a \quad b & \longrightarrow & \varphi(a)\,\phi(b) \end{array}$$

is bilinear.

(ii) the induced map

$$\bar{\theta} : \begin{array}{ccc} A \otimes B & \longrightarrow & C \\ a \otimes b & \longrightarrow & \theta(a, b) \end{array}$$

is an algebra morphism.

(iii) the following diagrams commute:

$$\begin{tikzcd} A \arrow[r, "i_A"] \arrow[dr, "\varphi"'] & A \otimes B \arrow[d, "\bar{\theta}"] \\ & C \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "i_B"] \arrow[dr, "\phi"'] & A \otimes B \arrow[d, "\bar{\theta}"] \\ & C \end{tikzcd}$$

pf (i) ✓

(ii) Check $\bar{\theta}$ respects mult.

$$\bar{\theta}(uv) \overset{?}{=} \bar{\theta}(u)\,\bar{\theta}(v) \qquad \forall\, u,v \in A\otimes B$$

WLOG $u = a\otimes b$ $\quad v = a'\otimes b'$

$$\bar{\theta}(u) = \bar{\theta}(a\otimes b) = \varphi(a)\,\phi(b)$$

$$\bar{\theta}(v) = \bar{\theta}(a'\otimes b') = \varphi(a')\,\phi(b')$$

$$\begin{aligned}
\bar{\theta}(uv) &= \bar{\theta}(aa'\otimes bb') \\
&= \varphi(aa')\,\phi(bb') \\
&= \varphi(a)\,\underbrace{\varphi(a')\,\phi(b)}_{\phi(b)\,\varphi(a')}\,\phi(b') \\
&= \bar{\theta}(u)\,\bar{\theta}(v) \quad ✓
\end{aligned}$$

(iii) Obs

$$a \longrightarrow a\otimes 1 \qquad a \searrow \varphi(a) \qquad a\otimes 1 \swarrow \bar{\theta}(a\otimes 1) = \theta(a,1) = \varphi(a) \quad ✓$$

$$b \longrightarrow 1\otimes b \qquad b \searrow \phi(b) \qquad 1\otimes b \swarrow \bar{\theta}(1\otimes b) = \theta(1,b) = \phi(b)$$

$\square$

Def 47 Referring to Prop 46, define

$$\varphi \otimes \phi = \bar{\theta}$$

Cor 48 With the notation/assumptions of Prop 46, $\varphi \otimes \phi$ is the unique algebra morphism $A \otimes B \to C$ that makes these diagrams commute:

$$\begin{tikzcd} A \arrow[r, "i_A"] \arrow[dr, "\varphi"'] & A \otimes B \arrow[d, "\varphi \otimes \phi"] \\ & C \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "i_B"] \arrow[dr, "\phi"'] & A \otimes B \arrow[d, "\varphi \otimes \phi"] \\ & C \end{tikzcd}$$

Given algebras A, B, C,

The following is a bijection

$$\begin{array}{ccc} \varphi \otimes \phi & & Hom_{alg}(A \otimes B, C) \\ \updownarrow & & \updownarrow \\ (\varphi, \phi) & & \left\{ (\varphi, \phi) \in Hom_{alg}(A, C) \times Hom_{alg}(B, C) \;\middle/\; \begin{array}{l} \varphi(a)\phi(b) = \phi(b)\varphi(a) \\ \forall a \in A,\ \forall b \in B \end{array} \right\} \end{array}$$

For C commutative, this bijection becomes

$$Hom_{alg}(A \otimes B, C) \longleftrightarrow Hom_{alg}(A, C) \times Hom_{alg}(B, C)$$

Given X, Y disjoint sets

Free algebras

$$k\{X\}, \quad k\{Y\}.$$

Compare algebras

$$k\{X \cup Y\}, \quad k\{X\} \otimes k\{Y\}$$

$\exists$ alg morphism

$$\theta : \quad k\{X \cup Y\} \longrightarrow k\{X\} \otimes k\{Y\}$$

that sends

$$x \longrightarrow x \otimes 1 \qquad \forall x \in X$$

$$y \longrightarrow 1 \otimes y \qquad \forall y \in Y$$

θ is surj

Consider $\ker(\theta)$.

$\forall x \in X \quad \forall y \in Y$

$$\theta(xy - yx) = x \otimes 1 \; 1 \otimes y - 1 \otimes y \; x \otimes 1$$

$$= x \otimes y - x \otimes y$$

$$= 0$$

Define

L = 2-sided ideal of $k\{X \cup Y\}$ gen by

$xy - yx \qquad x \in X \qquad y \in Y$

So $\quad L \subseteq \ker(\theta)$

<u>Prop</u> 49 $\quad L = \ker(\theta)$

<u>pf</u> $\quad \theta$ induces surj alg morphism

$$\frac{k\{X \cup Y\}}{L} \longrightarrow k\{X\} \otimes k\{Y\} \qquad *$$

$$x + L \longrightarrow x \otimes 1 \qquad \forall x \in X$$

$$y + L \longrightarrow 1 \otimes y \qquad \forall y \in Y$$

show $*$ is bijection.

Display inverse of $*$