Lec 12 Wednesday Sept 30 9/30/15

Given fin. dim'l vector spaces U, V.

Recall vs iso

$$\lambda_{U,V} : V \otimes U^* \longrightarrow Hom(U, V)$$

$$v \otimes f \longrightarrow \lambda_{U,V}(v \otimes f)$$

s.t. $\lambda_{U,V}(v \otimes f)(u) = f(u)v$

Given bases $\{v_i\}_{i \in I}$ for V

$\{u_j\}_{j \in J}$ for U

$V \otimes U^*$ has basis

$$v_i \otimes u^j \qquad i \in I, \quad j \in J \qquad \star$$

Describe image of $\star$ under $\lambda_{U,V}$

LEM 28. With above notation,

$$\lambda_{U,V}(v_i \otimes u^j) = e_{ij}$$

where $e_{ij}(u_r) = \delta_{jr} v_i \qquad \forall i, j, r$

pf $\lambda_{U,V}(v_i \otimes u^j)(u_r) = u^j(u_r) v_i$

$= \delta_{jr} v_i$

$\square$

Matrix representations

Given f.d v.s. U, V as above.

Given $f \in Hom(U, V)$

For $j \in J$

$$f(u_j) = \sum_{i \in I} f_j^i v_i \qquad f_j^i \in k$$

So $(f_j^i)_{i,j \in I \times J}$ is matrix representing f relative the bases $\{v_i\}_{i \in I}$, $\{u_j\}_{j \in J}$.

Obs

$$f_j^i = \langle v^i, f(u_j) \rangle \qquad \forall i \in I \ \forall j \in J$$

By constr $\quad$ ☆☆

$$f = \sum_{i,j \in I \times J} f_j^i e_{ij}$$

where the $\{e_{ij}\}$ are from LEM 28.

LEM 29 With the above notation, the map

$$\lambda_{U,V} : V \otimes U^* \to Hom(U, V)$$

sends

$$\sum_{i,j \in I \times J} f_j^i v_i \otimes u^j \to f$$

pf By LEM 28 and ☆☆. $\square$

The identity map $I \in End(V)$

LEM 30 Given a fin dim'l vector space $V \neq 0$,

under the map

$$\lambda_{V,V} : V \otimes V^* \to End(V)$$

the preimage of I is

$$\sum_i v_i \otimes v^i$$

where $\{v_i\}_i$ is any basis for V.

pf. Use LEM 29. $\square$

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DEF 31 Given a fin dim'l vector space $V \neq 0$

Consider the linear maps

$$\begin{array}{ccccc} k & \longrightarrow & \mathrm{End}(V) & \xleftarrow{\lambda_{V,V}} & V \otimes V^* \\ 1 & \longmapsto & I & & \end{array}$$

The coevaluation map is the composition

$$\delta : \begin{array}{ccccc} k & \longrightarrow & \mathrm{End}(V) & \xrightarrow{\overrightarrow{\lambda_{V,V}}} & V \otimes V^* \\ 1 & \longmapsto & I & & \end{array}$$

LEM 32 With the above notation,

$$\delta : \begin{array}{ccc} k & \longrightarrow & V \otimes V^* \\ 1 & \longmapsto & \sum_i v_i \otimes v^i \end{array}$$

where $\{v_i\}_i$ is any basis for V.

pf By LEM 30

$$\begin{array}{ccc} \mathrm{End}(V) & \xleftarrow{\lambda_{V,V}} & V \otimes V^* \\ I & \longleftarrow & \sum v_i \otimes v^i \end{array}$$

□

LEM 33 Given a fin. dim'l vector space $V \neq 0$,

(i) the composition

$$V \longrightarrow k \otimes V \xrightarrow{\delta \otimes id} V \otimes V^* \otimes V \xrightarrow{id \otimes ev} V \otimes k \longrightarrow V$$

$$v \to 1 \otimes v \qquad\qquad v \otimes 1 \to v$$

is the identity map.

(ii) the composition

$$V^* \longrightarrow V^* \otimes k \xrightarrow{id \otimes \delta} V^* \otimes V \otimes V^* \xrightarrow{ev \otimes id} k \otimes V^* \longrightarrow V^*$$

$$f \to f \otimes 1 \qquad\qquad 1 \otimes f \to f$$

is the identity map.

pf Pick a basis $\{v_i\}_i$ for V

(i)
$$v \to 1 \otimes v \to \sum_i v_i \otimes v^i \otimes v \to \sum_i v_i \otimes \langle v^i, v \rangle \to \sum_i \langle v^i, v \rangle v_i = v$$

(ii)
$$f \to f \otimes 1 \to \sum_i f \otimes v_i \otimes v^i \to \sum_i \langle f, v_i \rangle \otimes v^i \to \sum_i \langle f, v_i \rangle v^i = f$$

$\square$

LEM 34 Given a vector space V $\exists$ linear map

$$\varphi : \begin{array}{l} V \to (V^*)^* \\ v \to \varphi(v) \end{array}$$

s.t. $\forall f \in V^*$

$$\varphi(v)(f) = f(v) \qquad \forall v \in V$$

The map φ is injective

Suppose $\dim(V) < \infty$. Then φ is a vector space isomorphism.

pf $\forall v \in V$ the map

$$\varphi(v) : \begin{array}{l} V^* \to K \\ f \to f(v) \end{array}$$

is linear and hence in $(V^*)^*$

This gives a map

$$\varphi : \begin{array}{l} V \to (V^*)^* \\ v \to \varphi(v) \end{array}$$

One checks φ is linear.

Show φ is injective: Given $v \in V$ s.t. $\varphi(v) = 0$

show $v = 0$. Suppose $v \neq 0$.

Since $\varphi(v) = 0$,

$$f(v) = 0 \quad \forall f \in V^*$$

Let U = complement of Kv in V ie $V = Kv + U$ ds.

$\exists f \in V^*$ s.t $f(v) = 1$ and $f(U) = 0$. This is a contradiction since $f(v) \neq 0$.

Now assume $\dim(V) < \infty$.

We have
$$\dim(V) = \dim(V^*)$$
and φ is inj so φ is surj, hence bij. $\square$

(Aside)

LEM For a vector space V the following diag commutes:

$$\begin{tikzcd} V \otimes V^* \arrow[r, "\varphi \otimes id"] \arrow[d, "T_{V,V^*}"'] & (V^*)^* \otimes V^* \arrow[d, "ev_{V^*}"] \\ V^* \otimes V \arrow[r, "ev_V"'] & K \end{tikzcd}$$

pf $\forall v \in V$ $\forall f \in V^*$

$$\begin{tikzcd} v \otimes f \arrow[r] \arrow[dd] & \varphi(v) \otimes f \arrow[d] \\ & \varphi(v)(f) \arrow[d, phantom, "\shortparallel"] \\ f \otimes v \arrow[r] & f(v) \end{tikzcd}$$

$\square$

LEM 35 Given fin dim'l vectr spaces U, V

$\exists$ vectr space iso

$$\mathrm{Hom}(U,V) \to \mathrm{Hom}(V^*, U^*)$$
$$\varphi \to \varphi^*$$

such that

$$\langle h, \varphi(u)\rangle = \langle \varphi^*(h), u\rangle \quad \forall h \in V^* \ \forall u \in U \ (\star)$$

pf We have vector space isomorphisms

$$\mathrm{Hom}(U,V) \overset{\lambda_{U,V}}{\cong} V \otimes U^* \overset{\tau_{V,U^*}}{\cong} U^* \otimes V$$

$$\mathrm{Hom}(V^*, U^*) \underset{\lambda_{V^*,U^*}}{\cong} U^* \otimes (V^*)^* \underset{\text{LEM 34}}{\cong} U^* \otimes V$$

This gives vectr space iso

$$\mathrm{Hom}(U,V) \longrightarrow \mathrm{Hom}(V^*, U^*)$$
$$\varphi \to \varphi^*$$

Show this satisfies $\star$:

We have:

$$\mathrm{Hom}(U,V) \simeq V \otimes U^* \simeq U^* \otimes V$$

$$F \leftrightarrow v \otimes f \leftrightarrow f \otimes v$$

$$F(u) = f(u)v \qquad \forall u \in U$$

$$H(V^*, U^*) \simeq U^* \otimes (V^*)^* \simeq U^* \otimes V$$

$$G \leftrightarrow f \otimes g \leftrightarrow f \otimes v$$

$$G(h) = \underset{\overset{\parallel}{h(v)}}{g(h)} f = h(v) f \qquad \forall h \in V^*$$

Also $G = F^*$

Check

$$\langle h, F(u) \rangle \stackrel{?}{=} \langle F^*(h), u \rangle$$

$$\langle h, F(u)\rangle = f(u)\langle h, v\rangle = f(u)h(v)$$

$$\langle F^*(h), u\rangle = \langle G(h), u\rangle = h(v)\langle f, u\rangle = h(v) f(u)$$

$$f(u)h(v) \stackrel{\checkmark}{=} h(v) f(u)$$

$\square$

DEF 36. Referring to LEM 35, we call the map

$$\mathrm{Hom}(U,V) \to \mathrm{Hom}(V^*,U^*)$$
$$\varphi \to \varphi^*$$

the transpose map.

LEM 37 Given fin dim vs $U \neq 0$, $V \neq 0$.

Fn $f \in \mathrm{Hom}(U,V)$,

f^* is equal to the composition

$$V^* \to V^* \otimes k \xrightarrow{id \otimes \delta} V^* \otimes U \otimes U^* \xrightarrow{id \otimes f \otimes id} V^* \otimes V \otimes U^* \xrightarrow{ev \otimes id} k \otimes U^* \to U^*$$
$$h \to h \otimes 1 \qquad\qquad 1 \otimes f \to f$$

pf Pick a basis $\{u_i\}_i$ of U.

$\forall\, h \in V^*$

$$h \to h \otimes 1 \to \sum_i h \otimes u_i \otimes u^i \to \sum_i h \otimes f(u_i) \otimes u^i \to \sum_i \underbrace{\langle h, f(u_i)\rangle}_{\langle f^*(h),\, u_i\rangle} \otimes u^i$$

$$= \sum_i \langle f^*(h), u_i\rangle \otimes u^i \to \sum_i \langle f^*(h), u_i\rangle\, u^i = f^*(h)$$

$\square$

Given f.d. vector spaces U, V

Given bases $\{v_i\}_i$ for V

$\{u_j\}_j$ for U

Given $f \in \mathrm{Hom}(U, V)$

Recall $f(u_j) = \sum_i f^i_j v_i$ $\forall j$

LEM 38 With the above notation

$$f^*(v^i) = \sum_j f^i_j u^j \quad \forall i$$

pf Recall $\forall h \in U^*$

$$h = \sum_j \langle h, u_j \rangle u^j$$

So $f^*(v^i) = \sum_j \underbrace{\langle f^*(v^i), u_j \rangle}_{= \langle v^i, f(u_j) \rangle = f^i_j} u^j$

$\square$

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The trace map

Motivation

Given fin dim'l vectr space V

Given basis $\{v_i\}_i$ of V

For $f \in End(V)$ write

$$f(v_j) = \sum_i f^i_j v_i \qquad \forall j$$

Recall $tr(f) = \sum_i f^i_i$

Given linear map

$$tr : End(V) \to k$$

Consider

$$\begin{tikzcd} V \otimes V^* \arrow[r, "\lambda_{V,V}"] \arrow[dr, dashed, "?"'] & End(V) \arrow[d, "tr"] \\ & k \end{tikzcd}$$