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Lec 11 Monday Sept 28

Recall: Given vs U, U', V, V'

Linear map

$$\lambda : Hom(U, U') \otimes Hom(V, V') \to Hom(V \otimes U, U' \otimes V') \quad *$$

$$f \otimes g \to \lambda(f \otimes g)$$

$$\lambda(f \otimes g)(v \otimes u) = f(u) \otimes g(v)$$

Th 22 λ is an iso of vs, provided that at least one of the following holds

(i) U and V are fin dim'l

(ii) U and U' ..

(iii) V and V' ..

pf (i) (Cont)

After applying some isomorphisms, $*$ becomes

$$\lambda : \bigoplus_{i,j \in I \times J} U' \otimes V' \to \bigoplus_{i,j \in I \times J} U' \otimes V' \quad **$$

<u>claim</u> $**$ is identity map

Pick basis $\{u_i\}_{i \in I}$ for U,

" " $\{v_j\}_{j \in J}$ for V.

$V \otimes U$ has basis

$$\{ v_j \otimes u_i \}_{i,j \in I \times J}$$

We have vector space isomorphisms

$$\mathrm{Hom}(U, U') \longleftrightarrow \bigoplus_{i \in I} U'$$

$$f \longleftrightarrow \bigoplus_{i} f(u_i)$$

$$\mathrm{Hom}(V, V') \longleftrightarrow \bigoplus_{j \in J} V'$$

$$g \longleftrightarrow \bigoplus_{j} g(v_j)$$

$$\mathrm{Hom}(U, U') \otimes \mathrm{Hom}(V, V') \longleftrightarrow \bigoplus_{i,j \in I \times J} U' \otimes V'$$

$$f \otimes g \longleftrightarrow \bigoplus_{i,j} f(u_i) \otimes g(v_j)$$

$$\mathrm{Hom}(V \otimes U, U' \otimes V') \longleftrightarrow \bigoplus_{i,j \in I \times J} U' \otimes V'$$

$$h \longleftrightarrow \bigoplus_{i,j} h(v_j \otimes u_i)$$

View $*$: $$\mathrm{Hom}(U,U') \otimes \mathrm{Hom}(V,V') \xrightarrow{\lambda} \mathrm{Hom}(V\otimes U, U'\otimes V')$$

$$f \otimes g \longrightarrow h$$

vs iso $\updownarrow$ $\qquad \updownarrow \qquad \updownarrow \qquad \updownarrow$

$$\bigoplus_{i,j} f(u_i)\otimes g(v_j) \longrightarrow \bigoplus_{i,j} h(v_j \otimes u_i)$$

View $**$: $$\bigoplus_{i,j\in I\times J} U'\otimes V' \xrightarrow{\lambda} \bigoplus_{i,j\in I\times J} U'\otimes V'$$

$$h(v\otimes u) = f(u)\otimes g(v) \qquad \forall u\in U \ \forall v\in V$$

So $$\bigoplus_{i,j} h(v_j\otimes u_i) = \bigoplus_{i,j} f(u_i)\otimes g(v_j)$$

In other words,

$**$ is the identity map.

Therefore $*$ is an iso of vector spaces.

(ii), (iii) Similar

(detail)

(ii) Write

$$U \cong \bigoplus_{i \in I} U_i \qquad U_i \simeq k \qquad |I| < \infty$$

$$U' \simeq \bigoplus_{j \in J} U_j' \qquad U_j' \simeq k \qquad |J| < \infty$$

$$\mathrm{Hom}(U, U') \simeq \mathrm{Hom}\left(\bigoplus_i U_i, \bigoplus_j U_j'\right)$$

$$\simeq \bigoplus_{i,j \in I \times J} \mathrm{Hom}(U_i, U_j')$$

$$\simeq \bigoplus_{i,j \in I \times J} k$$

$$\mathrm{Hom}(U, U') \otimes \mathrm{Hom}(V, V') \simeq \left(\bigoplus_{i,j \in I \times J} k\right) \otimes \mathrm{Hom}(V, V')$$

$$\simeq \bigoplus_{i,j \in I \times J} k \otimes \mathrm{Hom}(V, V')$$

$$\simeq \bigoplus_{i,j \in I \times J} \mathrm{Hom}(V, V')$$

Also

$$\mathrm{Hom}(V \otimes U, U' \otimes V') \simeq \mathrm{Hom}\Big(\bigoplus_{i \in I} \underbrace{(V \otimes \overset{\overset{k}{\text{is}}}{U_i})}_{\overset{\text{is}}{V}}, \bigoplus_{j \in J} \underbrace{(\overset{\overset{k}{\text{is}}}{U'_j} \otimes V')}_{\overset{\text{is}}{V'}}\Big)$$

$$\simeq \mathrm{Hom}\Big(\bigoplus_{i \in I} V, \bigoplus_{j \in J} V'\Big)$$

$$\simeq \bigoplus_{i,j \in I \times J} \mathrm{Hom}(V, V')$$

$*$ becomes

$$\lambda : \bigoplus_{i,j \in I \times J} \mathrm{Hom}(V, V') \longrightarrow \bigoplus_{i,j \in I \times J} \mathrm{Hom}(V, V') \qquad **$$

Show $**$ is identity map.

Fix a basis $\{u_i\}_{i\in I}$ for U

-- $\{u_j'\}_{j\in J}$ for U'

$Hom(U, U')$ has basis $\{e_{ji}\}_{i,j\in I\times J}$

$$e_{ji}(u_r) = \delta_{ir} u_j'$$

VS iso

$$Hom(U, U') \otimes Hom(V, V') \longleftrightarrow \bigoplus_{i,j\in I\times J} Hom(V, V')$$

$$\sum_{i,j} e_{ji} \otimes f_{ij} \longleftrightarrow \bigoplus_{i,j} f_{ij}$$

Also

$$Hom(V\otimes U, U'\otimes V') \longleftrightarrow \bigoplus_{i,j\in I\times J} Hom(V, V')$$

$$h \longleftrightarrow \bigoplus h_{ij} :$$

where

$$h(v\otimes u_i) = \sum_{j\in J} u_j' \otimes h_{ij}(v) \qquad \forall i\in I \quad \forall v\in V$$

View * $\quad \mathrm{Hom}(U,U') \otimes \mathrm{Hom}(V,V') \xrightarrow{\lambda} \mathrm{Hom}(V\otimes U, U'\otimes V')$

$$\sum_{ij} e_{ji} \otimes f_{ij} \longrightarrow h$$

$$\updownarrow \qquad\qquad \updownarrow$$

$$\bigoplus_{ij} f_{ij} \longrightarrow \bigoplus_{ij} h_{ij}$$

View **: $\quad \bigoplus_{ij \in I\times J} \mathrm{Hom}(V,V') \xrightarrow{\lambda} \bigoplus_{ij\in I\times J} \mathrm{Hom}(V,V')$

Find h_{ij}

$\forall u \in U \quad \forall v \in V$

$$h(v\otimes u) = \sum_{ij} \lambda(e_{ji} \otimes f_{ij})(v\otimes u)$$

$$= \sum_{ij} e_{ji}(u) \otimes f_{ij}(v)$$

So for $i \in I$

$$h(v\otimes u_i) = \sum_j u_j' \otimes f_{ij}(v)$$

Also

$$h(v\otimes u_i) = \sum_j u_j' \otimes h_{ij}(v)$$

so $\quad f_{ij} = h_{ij} \qquad \forall i \in I \quad \forall j \in J$

** is the identity map. $\square$

Given a vectr space V recall the dual vectrspace

$$V^* = \mathrm{Hom}(V, k)$$

COR 23 For vectr spaces U, V

$\exists$ linear map

$$\lambda : U^* \otimes V^* \longrightarrow (V \otimes U)^*$$

$$f \otimes g \longrightarrow \lambda(f \otimes g)$$

s/t $\lambda(f \otimes g)(v \otimes u) = f(u)\, g(v) \quad \forall u \in U \;\; \forall v \in V.$

this λ is an isomorphism provided U or V is finite dim'l.

pf In thm 22 take

$U' = k, \quad V' = k.$ $\square$

Cor 24 For vectr spaces U, V

$\exists$ lin map

$$\lambda_{U,V} : V \otimes U^* \to Hom(U, V)$$

$$v \otimes f \to \lambda_{U,V}(v \otimes f)$$

s.t. $\lambda_{U,V}(v \otimes f)(u) = f(u)v$

$\forall u \in U \ \forall v \in V$

Moreover $\lambda_{U,V}$ is a bijection provided that U or V is fin dim'l.

pf Apply Th 22 with notation

Th22	U	V	U'	V'
new	K	U	V	K

(pf detail) Recall Th 2k: $\lambda : \mathrm{Hom}(U,U') \otimes \mathrm{Hom}(V,V') \to \mathrm{Hom}(V\otimes U, U'\otimes V')$ $\quad \ast$

In Th 2k take $U = k, \quad V' = k$

vector space isomorphisms:

$$\mathrm{Hom}(k, U') \leftrightarrow U'$$
$$f \leftrightarrow f(1)$$

$$\mathrm{Hom}(V, k) = V^*$$

$$V \otimes k \leftrightarrow V$$
$$v \otimes 1 \leftrightarrow v$$

$$U' \otimes k \leftrightarrow U'$$
$$u' \otimes 1 \leftrightarrow u'$$

$$\mathrm{Hom}(V\otimes k, U'\otimes k) \leftrightarrow \mathrm{Hom}(V, U')$$
$$h \leftrightarrow H$$

where $h(v\otimes 1) = H(v) \otimes 1 \quad \forall v \in V$

$\ast$ becomes

$$\lambda : U' \otimes V^* \longrightarrow \mathrm{Hom}(V, U') \qquad \ast\ast$$

Desc $\ast\ast$

(details, cont)

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View *

$$\mathrm{Hom}(k, U') \otimes \mathrm{Hom}(V, k) \xrightarrow{\lambda} \mathrm{Hom}(V \otimes k, U' \otimes k)$$

$$\begin{array}{ccc} f \otimes g & \longrightarrow & h \\ \updownarrow & & \updownarrow \\ f(1) \otimes g & \longrightarrow & H \end{array}$$

View **

$$U' \otimes V^* \xrightarrow[\lambda]{} \mathrm{Hom}(V, U')$$

$$h(v \otimes 1) \overset{m22}{=} f(1) \otimes \underbrace{g(v)}_{\in k}$$

$$= g(v) \, f(1) \otimes 1$$

Also

$$h(v \otimes 1) = H(v) \otimes 1$$

so

$$H(v) = g(v) \, f(1) \qquad \forall v \in V.$$

COR 25 For a finite dim'l vs V

$$\lambda_{V,V} : V \otimes V^* \longrightarrow End(V)$$

is an iso of vector spaces.

Given fd vector spaces U, U', V, V'

Recall vs iso

$$\lambda: \quad Hom(U,U') \otimes Hom(V,V') \longrightarrow Hom(V \otimes U, U' \otimes V') \qquad *$$

We saw vs isomorphisms

$$Hom(U,U') \simeq U' \otimes U^*$$

$$Hom(V,V') \simeq V' \otimes V^*$$

$$Hom(V \otimes U, U' \otimes V') \simeq U' \otimes V' \otimes (V \otimes U)^*$$

$$\simeq U' \otimes V' \otimes U^* \otimes V^*$$

So $*$ becomes an iso

$$U' \otimes U^* \otimes V' \otimes V^* \longrightarrow U' \otimes V' \otimes U^* \otimes V^* \qquad **$$

Natural to guess that $**$ is

$$id \otimes \tau_{U^*,V'} \otimes id: \quad u \otimes f \otimes v \otimes g \longrightarrow u \otimes v \otimes f \otimes g$$

This is correct, as we now show.

LEM 26 For vector spaces U, V, U', V' the following diagram commutes:

$$\begin{array}{ccc}
U' \otimes U^* \otimes V' \otimes V^* & \xrightarrow{\lambda_{U,U'} \otimes \lambda_{V,V'}} & \mathrm{Hom}(U,U') \otimes \mathrm{Hom}(V,V') \\
\downarrow {\scriptstyle id \otimes \tau_{U^*,V'} \otimes id} & & \\
U' \otimes V' \otimes U^* \otimes V^* & & \downarrow {\scriptstyle \lambda} \\
\downarrow {\scriptstyle id \otimes id \otimes \lambda} & & \\
U' \otimes V' \otimes (V \otimes U)^* & \xrightarrow[\lambda_{V\otimes U,\, U' \otimes V'}]{} & \mathrm{Hom}(V \otimes U, U' \otimes V')
\end{array}$$

pf. chase around diagram:

$$\begin{array}{ccc}
u' \otimes f \otimes v' \otimes g & \longrightarrow & F \otimes G \\
\downarrow & & \downarrow \\
u' \otimes v' \otimes f \otimes g & & \phi \\
\downarrow & & \overset{?}{\cong} \\
u' \otimes v' \otimes h & \longrightarrow & \varphi
\end{array}$$

$F(u) = f(u)\, u'$

$G(v) = g(v)\, v'$

$\phi(v \otimes u) = F(u) \otimes G(v)$

$= f(u)\, g(v)\, u' \otimes v'$

//

$h(v \otimes u) = f(u)\, g(v)$

$\varphi(v \otimes u) = h(v \otimes u)\, u' \otimes v'$

$= f(u)\, g(v)\, u' \otimes v'$

$\varphi = \phi$ ✓

□

Given vector spaces U, V, W

Composition

$f \circ g \quad U \xrightarrow{g} V \xrightarrow{f} W$

Gives bilin map

$$\mathrm{Hom}(V, W) \times \mathrm{Hom}(U, V) \to \mathrm{Hom}(U, W)$$

$$f \qquad g \qquad\qquad f \circ g$$

Induces lin map

$$\mathrm{Hom}(V, W) \otimes \mathrm{Hom}(U, V) \to \mathrm{Hom}(U, W)$$

comp: $f \otimes g \to f \circ g$

Recall lin maps

$\lambda_{V,W}$: $W \otimes V^* \to \mathrm{Hom}(V, W)$

$\lambda_{U,V}$: $V \otimes U^* \to \mathrm{Hom}(U, V)$

$\lambda_{U,W}$: $W \otimes U^* \to \mathrm{Hom}(U, W)$

$$\begin{array}{ccc} W \otimes V^* \otimes V \otimes U^* & \overset{?}{\longrightarrow} & W \otimes U^* \\ \downarrow \lambda_{V,W} \otimes \lambda_{U,V} & & \downarrow \lambda_{U,W} \\ \mathrm{Hom}(V,W) \otimes \mathrm{Hom}(U,V) & \underset{\text{comp}}{\longrightarrow} & \mathrm{Hom}(U,W) \end{array} \quad \star$$

Find map ? that makes $\star$ commute.

Consider bil map

$$\langle , \rangle \quad \begin{array}{ccccc} V^* & \times & V & \to & k \\ f & & v & \to & f(v) \end{array}$$

This induces lin map

$$ev_V \quad \begin{array}{ccc} V^* \otimes V & \to & k \\ f \otimes v & \to & f(v) = \langle f, v \rangle \end{array}$$

"evaluation map"

LEM 2[illegible] Fa vs U, V, W the linear map

$$id \otimes ev_V \otimes id : \; W \otimes V^* \otimes V \otimes U^* \longrightarrow W \otimes U^*$$
$$w \otimes f \otimes v \otimes g \longrightarrow w \otimes g \; f(v)$$

makes ☆ commute.

pf

$$\begin{array}{ccc} w \otimes f \otimes v \otimes g & \longrightarrow & w \otimes g \; f(v) \\ & & \downarrow \\ \downarrow & & \phi \\ & & \overset{?}{\parallel} \\ F \otimes G & \longrightarrow & F \circ G \end{array}$$

$F(v) = f(v)w$

$G(u) = g(u)v$

show $\phi = F \circ G$

Fa $u \in U$ $\quad \phi(u) = f(v)\, g(u)\, w$

$$\begin{aligned} (F \circ G)(u) &= F(G(u)) \\ &= F(g(u)v) \\ &= g(u)\, F(v) \\ &= f(v)\, g(u)\, w \\ &= \phi(u) \; \checkmark \end{aligned}$$

$\square$

Given a finite dim'l vector space V

Given basis $\{v_i\}_{i \in I}$ for V

the dual basis $\{v^i\}_{i \in I}$ for V^* satisfies

$$\langle v^i, v_j \rangle = \delta_{ij} \qquad \forall i, j \in I$$

$$\text{''} \; v^i(v_j)$$

One checks, for $v \in V$

$$v = \sum_{i \in I} \langle v^i, v \rangle v_i$$

Also for $f \in V^*$,

$$f = \sum_{i \in I} \langle f, v_i \rangle v^i$$

Given f.d. vector spaces U, V

Given

basis $\{u_i\}_i$ for U

basis $\{v_j\}_j$ for V

Get

basis $\{u_i \otimes v_j\}_{i,j}$ for $U \otimes V$ ✩

Recall vs iso

$$(U \otimes V)^* \simeq V^* \otimes U^*$$

$$H \leftrightarrow g \otimes f$$

$$H(u \otimes v) = f(u)\, g(v)$$

Identify $(U \otimes V)^*$ with $V^* \otimes U^*$ via above iso.

From this pt of view the evaluation map becomes

$$\langle\, ,\, \rangle : \begin{array}{ccccc} V^* \otimes U^*, & U \otimes V & \to & k \\ g \otimes f & u \otimes v & \to & f(u)\, g(v) \\ & & & \shortparallel \\ & & & \langle f, u\rangle \langle g, v \rangle \end{array}$$

Also from this pt of view, the dual basis for ✩ is

basis $\{v^j \otimes u^i\}_{i,j}$ for $V^* \otimes U^*$.