Lec 10 Friday Sept 25 9/25/15

More on tensor products..

Given an algebra A

the vector space $U = A$ becomes an A-module with action

$$\begin{array}{ccc} A \times U & \longrightarrow & U \\ a \quad u & \rightarrow & au \end{array}$$

Now let V denote any vector space.

Next goal: turn $A \otimes V$ into an A-module.

LEM 15 For a vector space V, there exists an A-module structure on $A \otimes V$ with action

$$\begin{array}{ccc} A \times A \otimes V & \longrightarrow & A \otimes V \\ a \quad b \otimes v & \longrightarrow & (ab) \otimes v \end{array}$$

pf The multiplication map

$$\mu : A \times A \longrightarrow A$$

is bilinear.

So μ induces a linear map

$$\bar{\mu} : A \otimes A \to A$$

Via LEM 14 this induces a linear map

$$\begin{tikzcd}
(A \otimes A) \otimes V \arrow[r, "\bar{\mu} \otimes id"'] \arrow[d, phantom, "\text{IS}"] & A \otimes V \\
A \otimes (A \otimes V) &
\end{tikzcd}$$

Gives a linear map

$$\varphi : A \otimes (A \otimes V) \to A \otimes V$$

this gives bilinear map

$$\begin{array}{ccccc} A & \times & A \otimes V & \to & A \otimes V \\ a & & w & \to & \varphi(a \otimes w) \end{array} \qquad ☆$$

By constr ☆ sends

$$a, \quad b \otimes v \longrightarrow (ab) \otimes v$$

$\forall a, b \in A \quad \forall v \in V.$

☆ gives desired A-module structure. $\square$

DEF 16 An A-module is called <u>free</u> whenever it is isomorphic to an A-module of the form $A \otimes V$ as in LEM 15.

— ∘ —

Given an algebra A and a set I

The vector space $\bigoplus_{i \in I} A$ becomes an A-module with action

$$\begin{aligned} A \quad \times \quad \bigoplus_{i\in I} A &\longrightarrow \bigoplus_{i\in I} A \\ a \qquad \bigoplus_i a_i &\longrightarrow \bigoplus_i (a a_i) \end{aligned}$$

LEM 17 The above A-module $\bigoplus_{i \in I} A$ is free.

pf Define a vector space

$$V = \bigoplus_{i \in I} k$$

We have vector space isomorphisms

$$\begin{aligned} A \otimes V &= A \otimes \Big(\bigoplus_{i \in I} k \Big) \\ &\underset{\text{LEM 12}}{\simeq} \bigoplus_{i \in I} (A \otimes k) \\ &\underset{\text{LEM 9}}{\simeq} \bigoplus_{i \in I} A \end{aligned}$$

One checks each iso above is an A-module iso.

Result follows by Def 16. $\square$

Let W denote any A-module.

Pick any elements $\{w_i\}_{i\in I}$ in W

The map

$$\bigoplus_{i\in I} A \longrightarrow W \qquad *$$

$$\bigoplus_i a_i \longrightarrow \sum_i a_i w_i$$

is an A-module hom.

Call $\{w_i\}_{i\in I}$ an <u>A-module basis</u> for W whenever

$*$ is a bijection.

Ex 18 The A-module $\bigoplus_{i\in I} A$ has an A-module basis $\{e_i\}_{i\in I}$ where

$$e_i = (0,\dots,0,\underset{i}{1},0,\dots,0)$$

Ex 19 Let V denote a vs with basis $\{v_i\}_{i\in I}$ Then the free A-module $A\otimes V$ has an A-module basis

$$\{1\otimes v_i\}_{i\in I}$$

— o —

Given vector spaces U, U', V, V'

Given $f \in Hom(U, U')$, $g \in Hom(V, V')$

By LEM 14 $\exists$ lin map

$$\begin{aligned} U \otimes V &\longrightarrow U' \otimes V' \qquad (*) \\ u \otimes v &\longrightarrow f(u) \otimes g(v) \end{aligned}$$

the composition

$$\begin{aligned} V \otimes U &\xrightarrow{T_{V,U}} U \otimes V \xrightarrow{*} U' \otimes V' \qquad (**) \\ v \otimes u &\to u \otimes v \longrightarrow f(u) \otimes g(v) \end{aligned}$$

is linear. So

$$** \in Hom(V \otimes U, U' \otimes V')$$

this gives bilinear map

$$\begin{aligned} Hom(U, U') \times Hom(V, V') &\to Hom(V \otimes U, U' \otimes V') \qquad \bigstar \\ f \qquad\qquad g \qquad &\longrightarrow \quad ** \end{aligned}$$

☆ induces linear map

$$\lambda : \operatorname{Hom}(U, U') \otimes \operatorname{Hom}(V, V') \to \operatorname{Hom}(V \otimes U, U' \otimes V')$$

$$f \otimes g \to \lambda(f \otimes g)$$

s.t.

$$\lambda(f \otimes g)(v \otimes u) = f(u) \otimes g(v) \qquad \forall u \in U \ \forall v \in V$$

Describe λ

Ex 20 U, U', V, V' all have dim 2

vectorspace	basis		desc
U	$u_1 \quad u_2$		
U'	$u_1' \quad u_2'$		
V	$v_1 \quad v_2$		
V'	$v_1' \quad v_2'$		
$Hom(U,U')$	e_{ij}	$1 \le i,j \le 2$	$e_{ij} \quad u_r \to \delta_{jr} u_i'$
$Hom(V,V')$	E_{ij}	$1 \le i,j \le 2$	$E_{ij} \quad v_r \to \delta_{jr} v_i'$
$Hom(U,U') \otimes Hom(V,V')$	$e_{ij} \otimes E_{rs}$	$1 \le i,j,r,s \le 2$	
$V \otimes U$	$v_i \otimes u_j$	$1 \le i,j \le 2$	
$U' \otimes V'$	$u_i' \otimes v_j'$	$1 \le i,j \le 2$	
$Hom(V \otimes U, U' \otimes V')$	ε_{ijrs}	$1 \le i,j,r,s \le 2$	

$$\varepsilon_{ijrs}(v_\ell \otimes u_m) = \delta_{\ell r}\delta_{ms}\, u_i' \otimes v_j'$$

For $1 \le i, j, r, s \le 2$,

$$\lambda(e_{ij} \otimes E_{rs})(v_\ell \otimes u_m) = e_{ij}(u_m) \otimes E_{rs}(v_\ell)$$

$$= \delta_{jm}\, u_i' \otimes \delta_{\ell s}\, v_r'$$

$$= \delta_{jm}\,\delta_{\ell s}\; u_i' \otimes v_r'$$

So

$$\lambda(e_{ij} \otimes E_{rs}) = \mathcal{E}_{irsj}$$

λ sends basis

$$e_{ij} \otimes E_{rs} \qquad 1 \le i, j, r, s \le 2$$

to (a perm of) basis

$$\mathcal{E}_{irsj} \qquad 1 \le i, j, r, s \le 2$$

λ is an isomorphism of vector spaces. $\square$

In the next example λ is not an iso of vector spaces

Ex 21 Take $U = k[x]$, $V = k[x]$ $x = \text{indet}$

$U' = k$, $V' = k$.

Describe λ.

We have vector space isomorphisms

$$U \simeq \bigoplus_{i \in \mathbb{N}} k$$

$$\begin{aligned} \mathrm{Hom}(U, U') &\simeq \mathrm{Hom}\left(\bigoplus_{i \in \mathbb{N}} k, k\right) \\ &\simeq \prod_{i \in \mathbb{N}} \mathrm{Hom}(k, k) \\ &\simeq \prod_{i \in \mathbb{N}} k \end{aligned}$$

View $\mathrm{Hom}(U, U')$ as set of ∞ sequences

$a_0 a_1 a_2 \cdots$ $a_i \in k$

Similarly view $\mathrm{Hom}(V, V')$ as set of ∞ sequences

$b_0 b_1 b_2 \cdots$ $b_i \in k$

Describe $\mathrm{Hom}(U, U') \otimes \mathrm{Hom}(V, V')$

let

$M =$ vector space of all matrices over k with rows/cols indexed by $\mathbb{N}$

let $\tilde{M} = \{ m \in M \,/\, \operatorname{rank}(m) < \infty \}$

Have bilinear map

$$\theta : \begin{array}{ccc} \operatorname{Hom}(U,U') \times \operatorname{Hom}(V,V') & \longrightarrow & \tilde{M} \\ \{a_i\}_{i\in\mathbb{N}} \quad \{b_j\}_{j\in\mathbb{N}} & \longrightarrow & \underset{\substack{\uparrow \\ \operatorname{rank} \leq 1}}{(a_i b_j)_{i,j \in \mathbb{N}^2}} \end{array}$$

θ induces linear map

$$\bar{\theta} : \quad \operatorname{Hom}(U,U') \otimes \operatorname{Hom}(V,V') \longrightarrow \tilde{M}$$

One checks $\bar{\theta}$ is bijection.

Also, we have vector space isomorphisms

$$V \otimes U \simeq \left(\bigoplus_{i \in \mathbb{N}} k \right) \otimes \left(\bigoplus_{j \in \mathbb{N}} k \right)$$

$$\simeq \bigoplus_{i,j \in \mathbb{N}^2} k \otimes k$$

$$\simeq \bigoplus_{i,j \in \mathbb{N}^2} k$$

and

$$U' \otimes V' \simeq k \otimes k$$

$$\simeq k$$

So

$$\mathrm{Hom}(V \otimes U, U' \otimes V') \simeq \mathrm{Hom}\left(\bigoplus_{i,j \in \mathbb{N}^2} k, k \right)$$

$$\simeq \prod_{i,j \in \mathbb{N}^2} \mathrm{Hom}(k, k)$$

$$\simeq \prod_{i,j \in \mathbb{N}^2} k$$

$$\simeq M$$

Obs

$$\lambda : \underset{\tilde{M}}{\underset{\wr\mathrm{s}}{\mathrm{Hom}(U, U') \otimes \mathrm{Hom}(V, V')}} \to \underset{M}{\underset{\wr\mathrm{s}}{\mathrm{Hom}(V \otimes U, U' \otimes V')}}$$

is not an iso

$\square$

Given vector spaces U, U', V, V'

Recall Linear map

$$\lambda: \quad \begin{array}{ccc} Hom(U,U') \otimes Hom(V,V') & \longrightarrow & Hom(V\otimes U, U'\otimes V') \\ f \otimes g & \longrightarrow & \lambda(f\otimes g) \end{array} \qquad *$$

$$\lambda(f\otimes g)(v\otimes u) = f(u)\otimes g(v)$$

Thm 22 With the above notation,

λ is an isomorphism of v.s. provided that at least one of the following hold:

(i) U and V are fin dim'l

(ii) U and U' are fin dim'l

(iii) V and V' are fin dim'l

pf (i) Write

$$U \simeq \bigoplus_{i \in I} U_i \qquad U_i \simeq k \qquad |I| < \infty$$

$$V \simeq \bigoplus_{j \in J} V_j \qquad V_j \simeq k \qquad |J| < \infty$$

We have vector space isomorphisms

$$\begin{aligned} \mathrm{Hom}(U, U') &\simeq \mathrm{Hom}\Big(\bigoplus_{i \in I} U_i, U'\Big) \\ &\simeq \bigoplus_{i \in I} \mathrm{Hom}(\underset{\wr\, k}{U_i}, U') \\ &\simeq \bigoplus_{i \in I} U' \end{aligned}$$

Similarly

$$\mathrm{Hom}(V, V') \simeq \bigoplus_{j \in J} V'$$

So

$$\mathrm{Hom}(U, U') \otimes \mathrm{Hom}(V, V') \simeq \bigoplus_{i,j \in I \times J} U' \otimes V'$$

Also

$$V \otimes U \simeq \bigoplus_{i,j \in I \times J} V_j \otimes U_i$$

$$\simeq \bigoplus_{i,j \in I \times J} k$$

So

$$\mathrm{Hom}(V \otimes U,\ U' \otimes V') \simeq \mathrm{Hom}\Big(\bigoplus_{i,j \in I \times J} k\ ,\ U' \otimes V'\Big)$$

$$\simeq \bigoplus_{i,j \in I \times J} \mathrm{Hom}(k,\ U' \otimes V')$$

$$\simeq \bigoplus_{i,j \in I \times J} U' \otimes V'$$

$*$ becomes

$$\lambda : \bigoplus_{i,j \in I \times J} U' \otimes V' \longrightarrow \bigoplus_{i,j \in I \times J} U' \otimes V' \qquad **$$

<u>Claim</u> $**$ is the identity map.

<u>pf cd</u>