4/9/15

Fall 2015

MATH 846

Quantum Groups + Hopf Algebras

MWF 11:00 AM

Birge 346

Paul Terwilliger

101A V.V

office hrs : after class or

contact me in 101A, 322

Text: Kassel. Quantum Groups

we will follow text, more or less

Eval: Each nondissertator will give

a 50 min lecture near end of semester

Topic of your choice, from text or elsewhere.

I can suggest topics

Lec 1 wed Sept 2

9/2/15

3

I. Preliminaries

I assume you know

groups, rings, fields, vector spaces ..

Recall a ring R with data $0, 1, +, \mu_R$

$R, 0, +$ is abelian group

Multiplication map

$$\mu_R : \begin{array}{ccc} R \times R & \to & R \\ a \quad b & \to & ab \end{array}$$

satisfies

R1 $(ab)c = a(bc)$

R2 $a(b+c) = ab + ac$

R3 $(a+b)c = ac + bc$

R4 $1a = a1 = a$

For rings R, R'

a map $f : R \to R'$ is a

<u>ring map</u> (= ring morphism = ring homomorphism)

whenever

M1 $f(0) = 0$

M2 $f(a+b) = f(a) + f(b)$

M3 $f(1) = 1$

M4 $f(ab) = f(a)f(b)$

Algebras (2 views)

Given field k.

I: An <u>algebra over k</u> is a ring A

together with a ring map

$$\eta_A : k \to A$$

whose image is in the center of A

II An algebra over k is a ring A together with a scalar product

$$k \times A \to A$$
$$\alpha \quad a \to \alpha a$$

that makes A a vector space over k,

and

$$* \quad \alpha(ab) = (\alpha a)b = a(\alpha b) \qquad \forall \alpha \in k \quad \forall a, b \in A$$

$*$ means the mult map

$$\mu_A : A \times A \to A$$
$$a \; b \to ab$$

is k-bilinear

Views I, II are equiv (ex)

I $\to$ II :

Define scalar mult :

$$k \times A \longrightarrow A$$
$$\alpha \quad a \longrightarrow \eta_A(\alpha)\, a$$

II $\to$ I :

Define ring map

$$\eta_A : \begin{array}{ccc} k & \longrightarrow & A \\ \alpha & \longrightarrow & \alpha 1 \end{array}$$

From now on, fix a field k.

Unless otherwise stated, all vector spaces, algebras, etc are meant to be over k.

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<u>Note</u> Given any ring map

$$\begin{aligned} k &\to k \\ \alpha &\to \bar{\alpha} \end{aligned}$$

$A = k$ becomes an algebra over k with

$$\eta_A : \begin{aligned} k &\to k \\ \alpha &\to \bar{\alpha} \end{aligned}$$

When we discuss <u>the algebra k</u>, it is understood the ring map $\eta_A = \text{ident}$ unless otherwise stated.

LEM 1 Given algebras A, B and a ring map $f : A \to B$

TFAE:

(i) f is k-linear

(ii) $\eta_B = f \circ \eta_A$ ← composition

(iii) this diagram commutes:

$$\begin{array}{ccc} k & \xrightarrow{\eta_A} & A \\ {\scriptstyle id}\downarrow & & \downarrow{\scriptstyle f} \\ k & \xrightarrow{\eta_B} & B \end{array}$$

pf $\forall \alpha \in k$ $\forall a \in A$ compare

$$\begin{array}{ccc} f(\alpha a) & & \alpha f(a) \\ \| & & \| \\ f(\eta_A(\alpha)\, a) & & \eta_B(\alpha)\, f(a) \\ \| & & \\ f(\eta_A(\alpha))\, f(a) & & \end{array}$$

$\star$

(i) $\to$ (ii) By $\star$ with $a=1$

$$f(\eta_A(\alpha)) = \eta_B(\alpha)$$

(ii) $\to$ (i) by $\star$

(ii) $\Leftrightarrow$ (iii) clear $\square$

DEF 2 Given algebras A, B

A map $f : A \to B$ is an

<u>algebra morphism</u> (= algebra homomorphism)

whenever f is a ring map and (i) – (iii)

hold in LEM 1.

DEF 3 Given algebras A, B

$\mathrm{Hom}_{alg}(A, B)$ = set of all algebra morphisms from A to B

DEF 4 For algebras A, B

A is a <u>subalgebra</u> of B whenever

$\exists$ injective algebra morphism $A \to B$

— o —

Given ring R with mult map μ_R

Replace μ_R by

$$\mu_{R^{op}} : \begin{array}{ccc} R \times R & \to & R \\ a \quad b & \to & ba \end{array}$$

Result is ring (ex) called R^{op}

— o —

Given algebra A. keep η_A and replace μ_A by $\mu_{A^{op}}$. Result is algebra (ex) called A^{op}

By cnstr
$$\eta_{A^{op}} = \eta_A$$

Define

$$\tau_{A,A} : \begin{array}{ccc} A \times A & \longrightarrow & A \times A \\ (a,b) & \longrightarrow & (b,a) \end{array}$$

$\mu_{A^{op}}$ is composition

$$A \times A \xrightarrow{\tau_{A,A}} A \times A \xrightarrow{\mu_A} A$$

Obs A is commutative iff $\mu_{A^{op}} = \mu_A$

Given algebra A.

Recall the center

$$Z(A) = \{ a \in A \mid ab = ba \ \forall b \in A \}$$

the inclusion map $Z(A) \to A$

is an algebra morphism (ex)

So $Z(A)$ is a subalgebra of A

obs $Z(A^{op}) = Z(A)$

Given algebra A

Recall a (2-sided) ideal of A is a nonempty subset $J \subseteq A$ such that

$$J + J \subseteq J$$

$$AJ \subseteq J \qquad \text{ie } \mu_A(A \times J) \subseteq J$$

$$JA \subseteq J \qquad \mu_A(J \times A) \subseteq J$$

Assume J is an ideal of A

$$J = A \Leftrightarrow 1 \in J$$

Assume $J \neq A$

Consider set of cosets A/J

Canonical map

$$\text{can}: \begin{array}{l} A \to A/J \\ a \to a + J \end{array} \qquad *$$

A/J has unique algebra structure s.t. $*$ is algebra morphism (ex). Here

$$\eta_{A/J} = \text{can} \circ \eta_A$$

Direct products and direct sums

Given vector spaces over k

$$\{V_i\}_{i \in I}$$

I = index set, pos ∞

the <u>direct product</u>

$$\prod_{i \in I} V_i$$

is the vector space over k consisting of the sequences

$$\{v_i\}_{i \in I} \qquad v_i \in V_i \qquad i \in I$$

For $i \in I$ the map

$$\pi_i \quad \begin{aligned} \prod_{j \in I} V_j &\to V_i \\ \{v_j\}_{j \in I} &\to v_i \end{aligned}$$

"ith projection map"

is k-linear

the <u>direct sum</u>

$$\bigoplus_{i\in I} V_i$$

is the subspace of $\prod_{i\in I} V_i$ consisting of the elements with finitely many non 0 coordinates.

For $|I|<\infty$

$$\prod_{i\in I} V_i = \bigoplus_{i\in I} V_i$$

Given algebras $\{A_i\}_{i\in I}$

the direct product

$$A = \prod_{i\in I} A_i$$

is a ring with mult

$$A \times A \to A$$

$$\{a_i\}_{i\in I} \quad \{b_i\}_{i\in I} \to \{a_i b_i\}_{i\in I}$$

so

$$1_A = \{1\}_{i\in I}$$

$$0_A = \{0\}_{i\in I}$$

there exists a unique algebra str on A

s.t. fr $i \in I$

$\pi_i : A \to A_i$ is algebra morphism (ex)

Here

$$\eta_A : \begin{array}{ccc} k & \to & A \\ \alpha & \to & \{\eta_{A_i}(\alpha)\}_{i\in I} \end{array}$$

Given algebra A and indeterminate x,

the algebra $A[x]$ consists of the polynomials

in x that have all coefs in A.

Here

$$\eta_{A[x]} : \begin{array}{ccc} k & \to & A[x] \\ \alpha & \mapsto & \eta_A(\alpha) \end{array}$$

The algebra $A[x, x^{-1}]$ of Laurent polynomials in x

is similarly defined.

Given algebra A and integer $n \geq 1$

$M_n(A)$ is the algebra of $n \times n$ matrices with entries in A

Here

$$\eta_{M_n(A)} \qquad \begin{array}{ccc} k & \to & M_n(A) \\ \alpha & \to & \begin{pmatrix} \eta_A(\alpha) & & 0 \\ & \eta_A(\alpha) & \\ 0 & & \ddots \\ & & & \eta_A(\alpha) \end{pmatrix} \end{array}$$

Given non 0 vector space V over k

the algebra $End(V)$ consists of the k-linear maps $V \to V$.

Multiplication is composition.

Here

$$\eta_{End(V)} : \begin{array}{ccc} k & \to & End(V) \\ \alpha & \to & \alpha I \end{array}$$

Given algebra A

An <u>A-module</u> is a vector space V over k

and a k-bilinear map

$$A \times V \to V$$
$$a \quad v \to av$$

s.t

$$(ab)v = a(bv) \qquad \forall a, b \in A \quad \forall v \in V$$
$$1v = v$$

— o —

Given non 0 vector space V over k

A <u>representation of A on V</u> is an algebra morphism

$$A \to \mathrm{End}(V)$$

Natural bijection between A-module structures on V
and reps of A on V. (ex)

Given A-modules V, W

a map $f : V \to W$ is called

A-linear ($=$ A-module homomorphism $=$ A-module morphism)

whenever f is k-linear and

$$f(av) = af(v) \qquad \forall a \in A \quad \forall v \in V$$

Above f is A-module isomorphism whenever f is bijection.

Given A-module V

Given subspace $W \subseteq V$ s.t. $AW \subseteq W$

Action of A on W turns W into an A-module, called

an <u>A-submodule of V</u>

Another view of submodules:

Given A-modules V, W

Assume $W \subseteq V$ as sets, but no assumption about algebra str.

then the incl map $W \to V$ is A-linear iff

W is an A-submodule of V (ex)

Given algebra A and A-modules

$V_1, V_2, \ldots, V_n$

the dir sum $V = \bigoplus_{i=1}^{n} V_i$

becomes an A-module with action

$$a\left(\bigoplus_i v_i\right) = \bigoplus_i (a v_i) \qquad \forall a \in A \quad v_i \in V_i$$

This is the unique A-module str on V s.t. for $1 \le i \le n$

$\pi_i : V \to V_i$ is A-linear (ex)

DEF 5 Given algebra A and A-module V

(i) V is <u>simple</u> (or <u>irreducible</u>) whenever $V \neq 0$ and V has no A-submodule besides $0, V$

(ii) V is <u>semisimple</u> whenever V is iso to a direct sum of simple A-modules

(iii) V is <u>decomposable</u> whenever V is iso to a direct sum of non 0 A-modules.

(iv) V is <u>indecomposable</u> whenever V is not decomposable.