The Diameter of Bipartite Distance-Regular Graphs

Lecture Notes

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Def: The girth of a graph Γ is the length of a shortest cycle.

We will show that the number of biparitite DRG with fixed valency and girth is finite.

Let $\Gamma(X,R)$ be any bipartite DRG

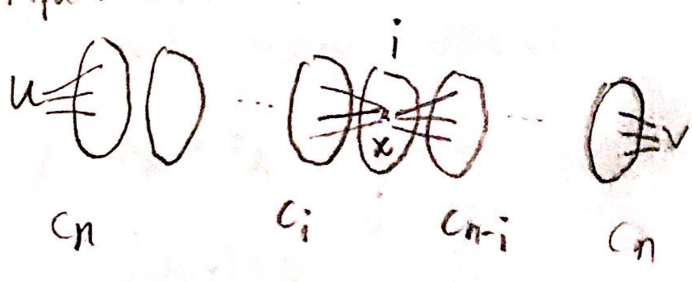
Def: Let $u, v \in X$ with $\partial(u,v) = n \geq 1$

Let B be vertex subgraph of Γ with vertex set

$$X_B = \{u\} \cup (\Gamma_1(u) \cap \Gamma_{n-1}(v)) \cup (\Gamma_2(u) \cap \Gamma_{n-2}(v)) \cdots \cup (\Gamma_{n-1}(u) \cap \Gamma_1(v)) \cup \{v\}$$

Lemma 1. If $x \in X_B$, and if $\partial(x,u) = i$ ($1 \leq i \leq n-1$), then the valency of x in B is $c_i + c_{n-i}$. The valency of u and v in B is c_n.

Pf: Bipartite:



$$k_B(x) = c_i + c_{n-i}$$

$$k_B(u) = k_B(v) = c_n \quad \blacksquare$$

Scanned with CamScanner

Lemma 2. For any pair of vertices a, b in B

$$\partial(a,b) \le \partial(u,v) = n$$

where the distances are measured in Γ.

Pf: By def of B, $\begin{cases} \partial(a,v) + \partial(a,u) = n \\ \partial(b,v) + \partial(b,u) = n \end{cases}$

Sum them. $\partial(a,u) + \partial(b,u) \le n \ (*)$ or $\partial(a,v) + \partial(b,v) \le n$.

WLOG $(*)$, $\partial(a,b) \le \partial(a,u) + \partial(u,b) \le n$. ∎

Thm 1. Let $\Gamma(X,R)$ be any bipartite DRG.

Then for any positive integer n. $n \le d$ (diameter of Γ),

$c_n > 1$ implies that there exists an i $(1 \le i \le n-1)$

such that $c_n \ge c_i + c_{n-i}$

Pf: Assume that for some n and for all i $(1 \le i \le n-1)$.

we have $c_i + c_{n-i} > c_n$. NTS: $c_n = 1$

Let $u, v \in X$ having $\partial(u,v) = n$. Let B be subgraph of Γ as above.

Define $f: X_B \times X_B \to \mathbb{Z}$ by $f(a,b) = \partial(a,b) - |\partial(a,u) - \partial(b,u)|$

Obs: $f(u,v) = 0$ $\quad f(a,a) = 0$.

By Lem 2 and triangle inequality: $0 \le f(a,b) \le n$ $\quad a, b \in X_B$.

We can let $f_0 = \max\{f(a,b) \mid a, b \in X_B\}$.

Claim: $f_0 = 0$. and hence $f(a,b) = 0$ for all $a, b \in X_B$.

Pf of claim: Let $S \subset X_B \times X_B$ be $S = \{(a,b) \in X_B \times X_B \mid f(a,b) = f_0\}$.

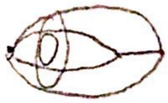
Choose $(a,b) \in S$ s.t. $|\partial(a,u) - \partial(b,u)|$ attains maximum.

Suppose $(a,b) \neq (u,v)$ or (v,u):

WLOG $a \neq u$ or v. So $\partial(a,u) = j$ $\quad 1 \leq j \leq n-1$

$$K_B(a) = C_j + C_{n-j}$$

Let $\partial(a,b) = l \leq n$ by lemma 2.

We have $C_n \geq C_l$. 

So $C_j + C_{n-j} - C_l \geq C_j + C_{n-j} - C_n > 0$ by assumption.

By $\partial(a,b) = l$ and def C_l, $\exists a' \in X_B$. s.t a' is adjcent to a

and $\partial(a',b) = \partial(a+b) + 1$

So $f(a',b) = \partial(a,b) + 1 - |\partial(a',u) - \partial(b,u)| \leq f(a,b)$ by construction

$\Rightarrow |\partial(a',u) - \partial(b,u)| \geq |\partial(a,u) - \partial(b,u)| + 1$ (*)

Since a' is adjacent to a.

we have $|\partial(a',u) - \partial(b,u)| \leq |\partial(a,u) - \partial(b,u)| + 1$.

Therefore, '=' holds. $\quad f(a',b) = f(a,b) \Rightarrow (a',b) \in S$.

But (*) is a contradiction, because (a,b) is s.t $|\partial(a,u) - \partial(b,u)|$ is maximum

So $a = u$ or v.

Similarly, $b = u$ or v $\Rightarrow (a,b) = (u,v)$ or $(v,u) \Rightarrow f_0 = 0$ /

Then the claim is proved

If $c_n \neq 1$. u is adjacent to $x \neq y$ in B.

$f(x,y) = \partial(x,y) > 0$. Contradiction $\Rightarrow c_n = 1$ ∎

Corollary 1. Any bipartite DRG $\Gamma(X,R)$ with at least one cycle is finite, with diameter d, valency k, and girth g satisfying $d \le \frac{(k-1)(g-2)}{2} + 1$

Pf: Γ is bipartite $\Rightarrow g$ is even, and $c_{\frac{g}{2}} > 1$

Claim: $c_i \geq \frac{2i}{g-2}$, $i \geq 1$

Induction on i: If $1 \le i < \frac{g}{2}$, $c_i = 1$ by def of girth

So $c_i \geq \frac{2i}{g-2}$.

If $i \geq \frac{g}{2}$, assume that $c_l \geq \frac{2l}{g-2}$ for all $1 \le l < i$

$i \geq \frac{g}{2} \Rightarrow c_i \geq c_{\frac{g}{2}} > 1$

Apply Thm 1, $\exists j$, $1 \le j \le i-1$ with $c_j + c_{i-j} \le c_i$

$$c_i \geq c_j + c_{i-j} \geq \frac{2j}{g-2} + \frac{2(i-j)}{g-2} = \frac{2i}{g-2}$$ by inductive assumption.

If Γ is infinite, then $c_i \to \infty$ as $i \to \infty$, but $k < \infty$.

So d must $< \infty$

We also have $k-1 \geq c_{d-1} \geq \frac{2(d-1)}{g-2}$

So $d \le \frac{(k-1)(g-2)}{2} + 1$ ∎

Corollary 2. For any integers k and g, $k, g > 2$, there are finitely many bipartite DRG with valency k and girth g. ∎