Chapter 8

- Throughout, let Γ be bipartite DRG w/ $D \geq 3$, $k \geq 3$.
- Goal: Express dual eigenvalues + intersection numbers c_i + b_i via formulas w/ a single parameter q in the 2-homogeneous case.
- Recall in ch. 7 we saw that Γ is 2-homogeneous $\Rightarrow$ $\forall$ nontrivial eigenvalue θ, $\exists\ \beta = \beta(\theta)$ s.t. $\theta^*_{i-1} - \beta\theta^*_i + \theta^*_{i+1} = 0$ for $1 \leq i \leq D-1$, where θ^*s are the dual eigenvalues corresponding to θ (Lemma 27). Lemma 28 gave formulas for θ^*_i in terms of single parameter q for $\beta \neq \pm 2$. Lemmas 30 + 31 give simpler formulas for θ^*_i in special cases $\beta = 2$, $\beta = -2$.
- First handle special case $\beta = \pm 2$, which is a k-cube:

Thm 34: TFAE:

(i) $\Gamma = H(k,2) = H(D,2)$ is a k-cube

(ii) $c_i = i$ and $b_i = D-i$ for $0 \leq i \leq D$

(iii) Γ 2-homogeneous and $\gamma_i = 1$ for $0 \leq i \leq D-1$ (γ_i defined in Thm 13)

(iv) '' $\gamma_2 = 1$

(v) '' $\mu = 2$

(vi) '' $\beta(\theta_1) = 2$

(vii) '' $\beta(\theta_{D-1}) = -2$

(viii) $\theta^*_i = D-2i$ for $0 \leq i \leq D$ for some set of θ^*s corresponding to some θ

(ix) $\theta^*_i = (-1)^i(D-2i)$ for $0 \leq i \leq D$ ''

(x) Γ 2-homogeneous and $\theta_1 = k-2$

(xi) '' $\theta_{D-1} = -k+2$

Pf:

- (i)$\Leftrightarrow$(ii): saw (i) $\Rightarrow$ (ii) in class. (ii) $\Rightarrow$ (i) also holds
- (ii) $\Rightarrow$ (iii): Γ (combinatorially) 2-homogeneous $\Leftrightarrow$ $(b_{i-1}-1)(c_{i+1}-1) = (\mu-1)p^i_{2i}$ (Thm 13) $\Leftrightarrow$ $b_i c_i = p^i_{2i}$ by (ii) (recall $\mu = c_2$). $\uparrow(*)$

Recall $p^i_{2i} = \frac{b_i(c_{i+1}-1)+c_i(b_{i-1}-1)}{\mu}$ (**) $\forall$ DRG (26). By (ii), (**) becomes $p^i_{2i} = \frac{b_ic_i + c_ib_i}{2} = b_ic_i$. By definition (thm 13), $\gamma_i = \frac{c_i(b_{i-1}-1)}{p^i_{2i}}$ $= \frac{c_ib_i}{b_ic_i} = 1$.

- (iii) $\Rightarrow$ (iv): (iv) is special case of (iii).
- (iv) $\Rightarrow$ (v): Let $i=2$ in $\gamma_i = \frac{c_i(b_{i-1}-1)}{p^i_{2i}}$ to get $p^2_{22} = \mu(b_1-1) = \mu(k-2)$ (using $c_1=1$ and $b_i+c_i=k$). Γ 2-hom $\Rightarrow$ (*). Set $i=2$ in (*) to get $(k-2)(c_3-1) = (\mu-1)p^i_{2i} = (\mu-1)\mu(k-2) \Rightarrow c_3-1 = (\mu-1)\mu$. $i=2$ in (**): $\mu(k-2) = p^2_{22} = \frac{b_2(c_3-1)+\mu(b_{i-1}-1)}{\mu} = \frac{(k-\mu)\mu(\mu-1)+\mu(k-2)}{\mu}$ $\Leftrightarrow$ $(\mu-1)(k-2) = (\mu-1)(k-\mu)$. Thm 19 (geometrically 2-hom characterization) gives $\mu \geq 2$ so divide by $\mu-1$ to get $\mu=2$.
- (v) $\Rightarrow$ (vi): Lemma 27 (general β recurrence for 2-hom) gives $\beta^2 = \frac{(\mu-2)(k(\mu-2)+2\mu)}{(\mu-1)(k-\mu)} + 4$ so $\mu=2$ gives $\beta^2=4$. Lemma 27 also gives $\beta(\theta_1) \geq 2$, so $\beta(\theta_1)=2$.
- (vi) $\Leftrightarrow$ (vii): Lemma 33 says $\beta(\theta_1) = -\beta(\theta_{D-1})$.
- (vi) $\Rightarrow$ (viii): Lemma 30 states θ_i^* takes this form for special case $\beta=2$
- (vii) $\Rightarrow$ (ix): Lemma 31 '' $\beta=-2$
- (viii) $\Rightarrow$ (ii): Use $\frac{\theta_1^*}{\theta_0^*} = \frac{\theta}{k}$ (20), $c_i = \frac{\theta\theta_i^* - k\theta_{i+1}^*}{\theta_{i-1}^* - \theta_{i+1}^*}$ (24), $k = \frac{\theta_0^*(\theta_0^*-\theta_2^*)}{\theta_1^{*2} - \theta_0^*\theta_2^*}$ (25), plug in $\theta_i^* = D-2i$, use $b_i = k-c_i$
- (ix) $\Rightarrow$ (ii): Similar to (viii) $\Rightarrow$ (ii).
- (vi) $\Leftrightarrow$ (x): Lemma 27 gives $\beta(\theta_1) = \frac{k^2-2k+\theta_1^2}{(k-1)\theta_1}$. Plug in & solve.
- (vii) $\Leftrightarrow$ (ix): similar w/ θ_{D-1}.

- Thm 35: (General case $\beta \neq \pm 2$) Suppose equivalent conditions of thm 34 do not hold. Then TFAE:

(i): Γ is 2-homogeneous

(ii): $\exists q \in \mathbb{C} \setminus \{0, 1, -1\}$ s.t. $\theta_i^* = \dfrac{(q^{D-2}+1)(q^{D-2i}-1)}{q^{D-i-2}(q^2-1)}$ $(0 \le i \le D)$

is a dual eigenvalue sequence corresponding to some θ.

(iii): $\exists q \in \mathbb{C}$ s.t. $c_i = \dfrac{(q^D+q^2)(q^{2i}-1)}{(q^D+q^{2i})(q^2-1)}$ & $b_i = c_{D-i}$ $(0 \le i \le D)$

Furthermore, suppose (i)–(iii) hold. Then (A)–(C) hold:

(A): The set Φ of q satisfying (ii) coincides w/ the set of q satisfying (iii)

(B): Fix $q \in \Phi$ & let θ be the eigenvalue corresponding to θ^*s in (ii). Then $\theta = \dfrac{(q^D+q^2)(q^{D-2}-1)}{q^{D-1}(q^2-1)}$ and $\beta(\theta) = q + q^{-1}$.

(C): $\gamma_i = \dfrac{(q^D+q^2)(q^D+q^{2i+2})}{(q^D+q^4)(q^D+q^{2i})}$ $(1 \le i \le D-1)$, for any $q \in \Phi$.

Pf: ∘ (i) ⇒ (ii): Let $\theta_0^*, \dots, \theta_D^*$ correspond to θ_1. By lemma 28,

$\theta_i^* = \pm \dfrac{(q^{D-2}-1)(q^{D-2i}+1)}{q^{D-i-2}(q^2-1)}$. But if $\theta_i^* < 0$ (minus case),

lemma 28 says $\theta_i^* = \theta_{D-i}^*$ $\forall i$, which contradicts that θ^*s are distinct (which follows from lemma 10 b/c these θ^*s are w.r.t. θ_1).

∘ (ii) ⇒ (iii): (ii) satisfies lemma 28 so for $\beta = q + q^{-1} \neq \pm 2$: $\theta_{i-1}^* - \beta\theta_i^* + \theta_{i+1}^* = 0$ and $|\beta| \ge 2$ (lemma 27), so $|\beta| > 2$, so $q \in \mathbb{R}$ by lemma 29.

$q + q^{-1} \neq \pm 2 \Rightarrow q \neq \pm 1$ so denominator in (iii) nonzero.

Lemma 29 also gives θ^*s distinct. (24): $c_i = \dfrac{\theta\theta_i^* - k\theta_{i+1}^*}{\theta_{i-1}^* - \theta_i^*}$

Substitute $\frac{\theta_1^*}{\theta_0^*} = \frac{a}{k}$ (20), $k = \frac{\theta_0^*(\theta_0^* - \theta_2^*)}{\theta_1^{*2} - \theta_0^*\theta_2^*}$ and (ii) to get expression for c_i in (iii).

$b_i = k - c_i = c_D - c_i = c_{D-i}$, where last equality follows from expressions in (iii) we just found for c_D, c_i, c_{D-i}.

- (iii) $\Rightarrow$ (i): Plug in (iii)'s expressions for c_i & b_i into (*), using (**) to substitute p_{i2}^2, find (*) holds.
- (i) - (iii) $\Rightarrow$ (A): q satisfies (ii) $\Rightarrow$ q satisfies (iii) by pf of (ii) $\Rightarrow$ (iii). Suppose q satisfies (iii), so $q \not\in \{-1,0,1\}$. Define θ as in (B). Using (iii), substitute $\mu = c_2$, $k = c_D$, θ to get $(\mu-1)\theta^2 = (k-\mu)(k-2)$; Γ 2-hom by (i), so by thm 19,24 (characterization of geometrically 2-hom), $\theta = \theta_1$ or θ_{D-1} (Γ is only geometrically 2-hom / satisfies this w.r.t. θ_1 or θ_{D-1}). $\text{mult}(\theta) = k$ by thm 22 (another 2-hom characterization). Define θ^*s as in (ii). These correspond to θ iff $\theta\theta_i^* = c_i\theta_{i-1}^* + b_i\theta_{i+1}^*$ $(0 \le i \le D)$ & $\theta_0^* = \text{rank}(E)$ by lemma 3. Former satisfied by substituting according to (ii), (iii), latter due to $\text{rank}(E) = \text{mult}(\theta) = k = c_D = \theta_0^*$, where last equality follows from $i = D$ in (iii) and $i=0$ in (ii). So q satisfies (ii).
- (i) - (iii) $\Rightarrow$ (B): Saw in proof of (A) that (B)'s def of θ gives the θ^*s in (ii). $\beta(\theta) = q + q^{-1}$ is from lemma 28.
- (i) - (iii) $\Rightarrow$ (C): Substitute using (iii) & (*) in $\gamma_i = \frac{c_i(b_{i-1}-1)}{p^1_{2i}}$

Corollary 36: Suppose Thm 35 (i)–(iii) hold. Then

(i) if D is even then $\Phi = \{ q \in \mathbb{C} : (q+q^{-1})^2 = \frac{\mu^2}{\mu-1} \cdot \frac{k-2}{k-\mu} \}$
$= \{ a, a^{-1}, -a, -a^{-1} \}$ for some $a \in \mathbb{R}$

(ii) if D is odd then $\Phi = \{ q \in \mathbb{C} : q + q^{-1} = \mu \gamma_r^{-1} \} = \{ a, a^{-1} \}$
for some $a \in \mathbb{R}$, $a > 1$ where $r = \frac{D-1}{2}$.

(iii): For any $q \in \Phi$, let θ be associated w/ the θ^*s in Thm 35 (ii). $q > 0 \Rightarrow \theta = \theta_1$ and $q < 0 \Rightarrow \theta = \theta_{D-1}$.

Pf: ∘ (i): Let Φ' be RHS. If $q \in \Phi$ then substitute $\mu = c_2$, $k = c_D$ in Φ' using Thm 35 (iii) to get $(q-q^{-1})^2$. So $\Phi \subseteq \Phi'$.
Replacing q w/ q^{-1} leaves Thm 35 (iii) invariant (multiply numerator & denominator by q^{D+2+2i}) & so does replacing q w/ $-q$ b/c all powers even since D even. Saw $\Phi \subseteq \mathbb{R}$ in pf of Thm 35 (ii) $\Rightarrow$ (iii) and $\pm 1 \notin \Phi$ b/c Φ satisfies Thm 35 (ii) (denominator non zero) so $q, q^{-1}, -q, -q^{-1}$ are 4 distinct el'ts of Φ. $|\Phi'| \leq 4$ b/c equation defining Φ has ≤ 4 solutions, so $\Phi = \Phi'$.

∘ (ii). Proof similar. Plug in $\mu = c_2$ using Thm 35 (iii) and γ_r using Thm 35 (C) to get $\mu \gamma_r^{-1} = q + q^{-1}$ if $q \in \Phi$, so $\Phi \subseteq \Phi'$.
As in pf of (i), q & q^{-1} are distinct el'ts of Φ and equation defining Φ' has ≤ 2 sol'ns so $\Phi = \Phi'$. $\Phi' \subseteq \mathbb{R}^+$ b/c $\mu, \gamma_r \geq 0$

∘ (iii). Γ can only be geometrically 2-hom w.r.t. θ_1, θ_{D-1}, so $\theta \in \{\theta_1, \theta_{D-1}\}$
$\beta(\theta) = q + q^{-1}$ by Thm 35 (B) so β, q have same sign and by Lemma 27 $\beta \geq 2$ if $\theta = \theta_1$ & $\beta \leq -2$ if $\theta = \theta_{D-1}$.