

SEMIPRIMITIVITY OF GROUP ALGEBRAS: PAST RESULTS AND RECENT PROGRESS

D. S. PASSMAN

University of Wisconsin-Madison

ABSTRACT. Let K be a field and let G be a multiplicative group. The group ring $K[G]$ is an easily defined, rather attractive algebraic object. As the name implies, its study is a meeting place for two essentially different algebraic disciplines. Indeed, group ring results frequently require a blend of group theoretic and ring theoretic techniques. A natural, but surprisingly elusive, group ring problem concerns the semiprimitivity of $K[G]$. Specifically, we wish to find necessary and sufficient conditions on the group G for its group algebra to have Jacobson radical equal to zero. More generally, we wish to determine the structure of the ideal $\mathcal{J}K[G]$. In the case of infinite groups, this problem has been studied with reasonable success during the past 45 years, and our goal here is to survey what is known. In particular, we describe some of the techniques used, discuss a number of the results which have been obtained, and mention several tantalizing conjectures.

§1. INTRODUCTION

Consider the following construction of the polynomial ring in two variables, say x and y , over a field K . To start with, form the set $S = \{x^a y^b \mid a, b = 0, 1, 2, \dots\}$ of monomials in x and y , and define multiplication in S by $x^a y^b \cdot x^c y^d = x^{a+c} y^{b+d}$. In this way, we see that S becomes an associative semigroup with identity element $1 = x^0 y^0$. Next, let $K[x, y] = K[S]$ be the K -vector space with basis consisting of the elements of S . In other words, every element of $K[x, y]$ is a formal finite sum $\sum k_{a,b} x^a y^b$ with coefficients $k_{a,b} \in K$. Of course, the addition in $K[x, y]$ is the usual vector space addition, and multiplication in $K[x, y]$ is defined distributively using the multiplication in S . Since the associative law for multiplication in S clearly carries over to $K[S]$, it follows that $K[x, y]$ is an associative K -algebra. Similarly, we could construct the Laurent polynomial ring $K[x, y, x^{-1}, y^{-1}]$ by taking S to be the multiplicative group $S = \{x^a y^b \mid a, b = 0, \pm 1, \pm 2, \dots\}$ and again forming $K[S]$. Indeed, this is our first example of a group ring.

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More generally, let K be a field and let G be any multiplicative group. Then the *group algebra* or *group ring* $K[G]$ is a K -vector space with basis consisting of the elements of G . Thus every element of $K[G]$ is a formal finite sum

$$\alpha = \sum_{g \in G} k_g g$$

with coefficients $k_g \in K$. Again, addition in $K[G]$ is the obvious vector space operation, and we define multiplication distributively using the given multiplication of G . In this way, $K[G]$ becomes an associative K -algebra, with structure highly dependent on the nature of G . Basic references for group algebras include the books [MZ], [Pa], [P3], [P10], [Se1] and [Se2].

As is well known, group rings are important tools in both group theory and ring theory. For example, they provide the correct framework to study and understand the ordinary and modular character theory of finite groups. Furthermore, when G is a polycyclic-by-finite group, then $K[G]$ is a right and left Noetherian K -algebra and hence it is a useful testing ground for the rich theory of noncommutative Noetherian rings. In turn, the module theory of the latter group algebra can feed back into group theory to yield information on the structure of abelian-by-polycyclic groups. But, group rings are more than just useful tools. They are easily defined, rather attractive algebraic objects which are worthy of being considered in their own right. Their study is necessarily ring theoretic in nature, but the techniques and proofs exhibit a strong group theoretic flavor. The goal of this paper is to survey the progress made on a rather elusive group ring problem.

If R is an associative ring with 1, then a (right) R -module V is just a right R -vector space. Thus V is an additive abelian group which admits right multiplication by R , and such that this scalar multiplication satisfies the usual axioms. Of course, these rules are precisely equivalent to the existence of a natural ring homomorphism $\theta_V: R \rightarrow \text{End}(V)$, where $\text{End}(V)$ is the ring of endomorphisms of the additive abelian group V . We say that $V \neq 0$ is *irreducible* if V has no proper R -submodule. In other words, the irreducible R -modules are the natural analogs of the 1-dimensional vector spaces over fields. For convenience, we let $\text{Irr}(R)$ denote the set of all such irreducible R -modules.

A ring R is said to be *primitive* if it has a faithful irreducible module. In other words, R is primitive if there exists $V \in \text{Irr}(R)$ with θ_V a one-to-one map. Such rings have a nice, rather natural structure; they are dense sets of linear transformations over division rings. Unfortunately, primitive rings are fairly scarce, so the next best situation is to study the ring R by looking at all its irreducible modules. But there is still a fundamental obstruction here, namely

$$\mathcal{J}R = \bigcap_{V \in \text{Irr}(R)} \ker \theta_V = \{r \in R \mid Vr = 0 \text{ for all } V \in \text{Irr}(R)\}.$$

This characteristic ideal is called the *Jacobson radical* of R , and we say that R is *semiprimitive* precisely when $\mathcal{J}R = 0$. Thus R is semiprimitive if and only if it is a subdirect product of primitive rings. In particular, such rings are reasonably well understood.

It is therefore of some interest and importance to determine those groups G with semiprimitive group algebras $K[G]$. More generally, we would like to describe the structure of the Jacobson radical $\mathcal{J}K[G]$ for any group G . In the case of finite groups, the semiprimitivity problem has the following classical solution, dating from the work of Maschke in 1898.

Theorem 1.1. [M] *Let G be a finite group and let K be a field.*

- i. *If $\text{char } K = 0$, then $K[G]$ is semiprimitive.*
- ii. *If $\text{char } K = p > 0$, then $K[G]$ is semiprimitive if and only if G has no elements of order p .*

The goal now is to extend this result, or some variant of it, to the case of infinite groups, and in this survey, which is a revised and updated version of [P18] and [P19], we will discuss the progress which has been made in this direction.

§2. FIELDS OF CHARACTERISTIC 0

It is not surprising that the early advances on the semiprimitivity problem for infinite groups concerned fields of characteristic 0, and indeed the field C of complex numbers. The first significant result appeared in 1950, with a proof using analytic methods, including the spectral norm and the auxiliary norm of $C[G]$.

Theorem 2.1. [R] *If C is the field of complex numbers, then every group algebra $C[G]$ is semiprimitive.*

This result intrigued a number of ring theorists who rightly felt that it should have an algebraic proof. Thus, for example, the semiprimitivity problem for fields of characteristic 0 appeared in the Ram's Head Inn problem list [K1] (see also [K2]), and an algebraic argument for Theorem 2.1 was quickly discovered. It is instructive to consider some of the ingredients of this new proof. Recall that an ideal I of any ring R is said to be *nil* if all elements of I are nilpotent. Since every nil ideal of R is contained in $\mathcal{J}R$, a first step in proving that $K[G]$ is semiprimitive might be to show that it has no nonzero nil ideal. In this direction we have

Lemma 2.2. *Let K be a subfield of the complex numbers which is closed under complex conjugation. If G is any group, then $K[G]$ has no nonzero nil ideal.*

Proof. Define a map $*$: $K[G] \rightarrow K[G]$ by

$$\left(\sum_g k_g g \right)^* = \sum_g \bar{k}_g g^{-1}$$

where $\bar{}$ indicates complex conjugation. It is easy to see that $(\alpha\beta)^* = \beta^*\alpha^*$, $\alpha^{**} = \alpha$, and $(\alpha + \beta)^* = \alpha^* + \beta^*$. Furthermore, if $\alpha = \sum_g k_g g$, then the identity coefficient of $\alpha\alpha^*$ is equal to $\sum_g k_g \bar{k}_g = \sum_g |k_g|^2$. Hence $\alpha\alpha^* = 0$ if and only if $\alpha = 0$.

Let I be a nonzero ideal of $K[G]$ and choose $0 \neq \alpha \in I$. Then, by the above, $\beta = \alpha\alpha^*$ is a nonzero element of I , and β is easily seen to be $*$ -symmetric. In other words, any nonzero ideal of $K[G]$ contains a nonzero $*$ -symmetric element. Next, we claim that 0 is the unique $*$ -symmetric nilpotent element. Indeed, if γ is $*$ -symmetric and nilpotent, then so is any power of γ . Thus it suffices to assume that $\gamma^2 = 0$. But then $0 = \gamma^2 = \gamma\gamma^*$, so $\gamma = 0$ as required, and the result follows immediately from the latter two observations. \square

The second ingredient holds over any field. Note that if H is a subgroup of G , then $K[H]$ is naturally embedded in $K[G]$. Indeed, this is just the group ring analog of the obvious polynomial ring inclusion $K[x] \subseteq K[x, y]$. Furthermore, since $K[G]$ is a free right and left $K[H]$ -module, using coset representatives for H in G as a free basis, we have

Lemma 2.3. *Let K be any field and let H be a subgroup of G .*

- i. *If W is an irreducible $K[H]$ -module, then there exists an irreducible $K[G]$ -module V with W a submodule of V_H , the restriction of V to $K[H]$.*
- ii. $\mathcal{J}K[G] \cap K[H] \subseteq \mathcal{J}K[H]$.

The remainder of the argument is of less interest. To start with, the Hilbert Nullstellensatz asserts that if A is a finitely generated commutative algebra over a field K , then $\mathcal{J}A$ is a nil ideal. Furthermore, recall that there is a trivial proof of this result in case K is nondenumerable. Indeed, the same proof shows, without the commutativity assumption, that if A is a countable dimensional algebra over a nondenumerable field, then $\mathcal{J}A$ is nil. In particular, it follows from this and Lemma 2.2 that if H is a countable group, then the complex group algebra $C[H]$ is semiprimitive. Finally, if G is any group and if $\alpha \in \mathcal{J}C[G]$, then there exists a finitely generated and hence countable subgroup H of G with $\alpha \in C[H]$. But then Lemma 2.3(ii) yields

$$\alpha \in C[H] \cap \mathcal{J}C[G] \subseteq \mathcal{J}C[H] = 0,$$

and Theorem 2.1 is proved.

Much more important is the later work of Amitsur on the behavior of the radical under field extensions. If A is a K -algebra and if F is a field containing K , then we denote the F -algebra $F \otimes_K A$ by A^F . Thus A^F is the largest ring generated by its commuting subrings F and A , with the two copies of K identified.

Theorem 2.4. [A1] *Let $F \supseteq K$ be fields and let A be a K -algebra.*

- i. $\mathcal{J}(A^F) \cap A \subseteq \mathcal{J}A$ with equality when F/K is algebraic.

- ii. If F/K is a finite separable extension, then $\mathcal{J}(A^F) = F \otimes_K \mathcal{J}A$.
- iii. If F is a nontrivial purely transcendental extension of K , then $\mathcal{J}(A^F) = F \otimes_K I$ for some nil ideal I of A .

Since $K[G]^F = F \otimes_K K[G] = F[G]$, the preceding result and Lemma 2.2 applied to the field Q of rational numbers yield

Theorem 2.5. [A2] *Let K be a field of characteristic 0 so that K contains the rational numbers Q , and let G be an arbitrary group.*

- i. *If K/Q is not algebraic, then $K[G]$ is semiprimitive.*
- ii. *If K/Q is algebraic, then $\mathcal{J}K[G] = K \otimes_Q \mathcal{J}Q[G]$ and $K[G]$ has no nonzero nil ideal.*

In particular, the semiprimitivity problem for algebraic extensions of Q reduces to Q itself. Presumably $Q[G]$ is always semiprimitive, but unfortunately the above result marks the extent of our knowledge. Indeed, there has been no significant progress on the characteristic 0 problem since Theorem 2.5 appeared in 1959. We remark that the semiprimitivity of $Q[G]$ would follow quite easily if one knew that finitely generated algebras necessarily have nil Jacobson radicals. However, as was shown in [B], this is not always the case.

§3. FIELDS OF CHARACTERISTIC $p > 0$

Now let us turn to modular fields and assume for the remainder of this paper that $\text{char } K = p > 0$. In view of Theorem 1.1, it is reasonable to suppose $K[G]$ is semiprimitive if and only if G is a p' -group, that is a group with no elements of order p . One direction of this is most likely true, but as we will see, the other direction is decidedly false. We begin with an interesting trace argument.

For any group G , let $\text{tr}: K[G] \rightarrow K$ be the map which reads off the identity coefficient, so that $\text{tr}(\sum k_g g) = k_1$. Then tr is obviously a K -linear functional, and it is easy to see that $\text{tr } \alpha\beta = \text{tr } \beta\alpha$ for all $\alpha, \beta \in K[G]$. Next, we note that if A is any K -algebra and if $\alpha_1, \alpha_2, \dots, \alpha_s \in A$, then

$$(\alpha_1 + \alpha_2 + \dots + \alpha_s)^{p^n} = \alpha_1^{p^n} + \alpha_2^{p^n} + \dots + \alpha_s^{p^n} + \beta$$

for some $\beta \in [A, A]$, where the latter subspace is the span of all Lie products $[\gamma, \delta] = \gamma\delta - \delta\gamma$ with $\gamma, \delta \in A$.

Lemma 3.1. *If G is a p' -group, then $K[G]$ has no nonzero nil ideal.*

Proof. Suppose $\alpha = \sum k_g g \in K[G]$ is nilpotent, and choose n sufficiently large so that $\alpha^{p^n} = 0$. Then by the preceding formula,

$$0 = \alpha^{p^n} = \sum_{g \in G} (k_g)^{p^n} g^{p^n} + \beta$$

for some $\beta \in [K[G], K[G]]$. In particular, since tr annihilates all Lie products, we have $\text{tr } \beta = 0$ and hence

$$0 = \sum_{g \in G} (k_g)^{p^n} \text{tr } g^{p^n}.$$

But G is a p' -group, so $g^{p^n} = 1$ if and only if $g = 1$, and therefore $\text{tr } g^{p^n} = 0$ for all $g \neq 1$. It follows that $0 = (k_1)^{p^n}$, and we conclude that if α is nilpotent, then $0 = k_1 = \text{tr } \alpha$.

Finally, let I be a nil ideal of $K[G]$ and let $\gamma = \sum c_g g \in I$. Then $\gamma x^{-1} \in I$ is nilpotent for any $x \in G$, so the above yields $0 = \text{tr } \gamma x^{-1} = c_x$. Thus $\gamma = 0$, and hence $I = 0$, as required. \square

Since any finitely generated field extension of $\text{GF}(p)$ is separably generated, it is a simple matter to translate the argument of Theorem 2.5 to this context. In particular, Theorem 2.4 and Lemma 3.1 yield

Theorem 3.2. *Let K be a field of characteristic $p > 0$, write $K_0 = \text{GF}(p)$, and let G be a p' -group.*

- i. *If K/K_0 is not algebraic, then $K[G]$ is semiprimitive.*
- ii. *If K/K_0 is algebraic, then $\mathcal{J}K[G] = K \otimes_{K_0} \mathcal{J}K_0[G]$.*

If G is a p' -group, then $K[G]$ is presumably always semiprimitive. But the converse is certainly not true; there are numerous groups G having elements of order p , but with $\mathcal{J}K[G] = 0$. For example, we have

- (1) $p = 2$ and $G = \langle x, y \mid y^{-1}xy = x^{-1}, y^2 = 1 \rangle$ is infinite dihedral.
- (2) $G = Z \wr Z_p$ is the wreath product of the infinite cyclic group Z by the cyclic group Z_p of order p .
- (3) $G = Z_p \wr Z$ is again a wreath product and has a normal infinite elementary abelian p -subgroup.
- (4) $G = \text{FSym}_\infty$, the countably infinite finitary symmetric group.

Note that (1), which appeared in [Wa], was the first such example, and (4) is a result of [F]. Furthermore, we know that the groups in (3) and (4) have primitive group algebras. The real answer to the semiprimitivity problem is most likely

Conjecture 3.3. *Let K be a field of characteristic $p > 0$ and let G be a group. Then $\mathcal{J}K[G] \neq 0$ if and only if G has an element of order p “well placed” in G .*

Of course, before this can be proved, we must first determine what “well placed” means. To do this, it is necessary to compute numerous examples. However, we can get some idea of the possible meaning by considering a slightly different problem. For any ring R , let $\mathcal{N}R$ denote the join of all its nilpotent ideals. Thus $\mathcal{N}R$ is a characteristic nil ideal called the *nilpotent radical* of R . For general rings, it is neither nilpotent nor a radical, but we do have $\mathcal{N}R \subseteq \mathcal{J}R$.

Next, if A and B are subgroups of a group G , then the *finitary centralizer* of B in A is defined by

$$\mathbb{D}_A(B) = \{a \in A \mid |B : \mathbb{C}_B(a)| < \infty\}.$$

In other words, $\mathbb{D}_A(B)$ consists of all elements of A which almost centralize B , and consequently it is a subgroup of A normalized by $\mathbb{N}_G(A) \cap \mathbb{N}_G(B)$. Corresponding to this finitary centralizer is a finitary center, the *f.c.* or *finite conjugate center* of G , given by

$$\Delta(G) = \mathbb{D}_G(G) = \{x \in G \mid |G : \mathbb{C}_G(x)| < \infty\}.$$

Thus $\Delta(G)$ consists of all elements of G having only finitely many conjugates, and it is easy to see that $\Delta = \Delta(G)$ is a characteristic subgroup of G . Furthermore, we let $\Delta^+(G)$ be the set of torsion elements of Δ , that is the elements of finite order in the group. Surprisingly, $\Delta^+ = \Delta^+(G)$ is also a characteristic subgroup of G . Indeed, Δ/Δ^+ is a torsion free abelian group and Δ^+ is the join of all finite normal subgroups of G .

The following result is proved using a powerful coset counting argument known as the Δ -method.

Theorem 3.4. [P1], [P2] *Let $\mathcal{D}(G)$ denote the set of finite normal subgroups of G , and let $\Delta^+ = \Delta^+(G) = \langle D \mid D \in \mathcal{D}(G) \rangle$. If $\text{char } K = p > 0$, then*

- i. $\mathcal{N}K[G] = \mathcal{J}K[\Delta^+] \cdot K[G]$.
- ii. $\mathcal{J}K[\Delta^+] = \bigcup_{D \in \mathcal{D}(G)} \mathcal{J}K[D]$.
- iii. $\mathcal{N}K[G] \neq 0$ if and only if Δ^+ contains an element of order p and hence if and only if G has a finite normal subgroup of order divisible by p .

Note that (i) asserts that $\mathcal{J}K[\Delta^+]$ is contained in $\mathcal{N}K[G]$ and generates it as a right ideal. Furthermore, (iii) is an immediate consequence of parts (i) and (ii), along with Theorem 1.1. Thus “well placed” for this radical means that the element of order p is contained in $\Delta^+(G)$ or equivalently in some finite normal subgroup of G . We close this section with a simple, but quite useful, observation.

Lemma 3.5. [V] *If H is a normal subgroup of G of finite index n , then*

$$\mathcal{J}K[G]^n \subseteq \mathcal{J}K[H] \cdot K[G] \subseteq \mathcal{J}K[G].$$

Furthermore, if p does not divide n , then $\mathcal{J}K[H] \cdot K[G] = \mathcal{J}K[G]$.

In particular, if $\mathcal{J}K[H] = 0$ in the above, then $\mathcal{J}K[G]$ is nilpotent and Theorem 3.4 can come into play. With this observation, it is now a simple exercise to prove that $K[G]$ is semiprimitive when $G = Z \wr Z_p$ or when $p = 2$ and G is infinite dihedral.

§4. SOLVABLE GROUPS AND LINEAR GROUPS

This brings us to the early 1970's; it was time to compute some examples. We looked for families of groups which were sufficiently diverse to give us meaningful answers, yet simple enough to be dealt with effectively. Two obvious candidates were the families of solvable groups and linear groups. As it turned out, the solvable case yielded the most information and required the more interesting techniques. Therefore we begin our exposition with these groups. We will ignore some earlier special case considerations and just deal with the general problem.

First, recall that G is said to be an *f.c. group* if $G = \Delta(G)$, or equivalently if all conjugacy classes of G are finite. Next, let G be any group, let H be a subgroup of G , and let I be a nonzero ideal of $K[G]$. Then an *intersection theorem* is a result which guarantees that $I \cap K[H] \neq 0$ under suitable assumptions on H , G/H , or I . There are numerous results of this nature in the literature, and Zalesskiĭ proved a marvellous one for solvable groups. Specifically, he showed

Theorem 4.1. [Z1] *If G is a solvable group, then G has a characteristic f.c. subgroup $\mathfrak{Z}(G)$ with the following property. If K is any field and if I is a nonzero ideal of $K[G]$, then $I \cap K[\mathfrak{Z}(G)] \neq 0$.*

This *Zalesskiĭ subgroup* $\mathfrak{Z}(G)$ is the f.c. center of a finitary analog of the Fitting subgroup of a finite solvable group. Of course, if G is solvable and if $\mathcal{J}K[G] \neq 0$, then the preceding theorem implies that $\mathcal{J}K[G] \cap K[\mathfrak{Z}(G)] \neq 0$. Thus, the next step in the solution of the semiprimivity problem for these groups is to deal with this intersection. For this, we require an interesting general result which is a noncommutative analog of the argument used to prove Theorem 2.4(iii).

Lemma 4.2. [Wa] *Let G be an arbitrary group, let $H \triangleleft G$, and suppose that $\alpha \in \mathcal{J}K[G] \cap K[H]$. If x is any element of G of infinite order modulo H , then there exists a positive integer n such that*

$$\alpha \alpha^x \alpha^{x^2} \cdots \alpha^{x^n} = 0.$$

Here, of course, $\alpha^y = y^{-1}\alpha y$ for any $y \in G$. Now, if x has infinite order modulo H , then so does x^s for any positive integer s . Thus, each such x gives rise to a family of equations, with varying s and varying $n = n(s)$. These *Wallace equations* are rather unwieldy in general. Nevertheless, we were able to use them effectively when H is a solvable f.c. group.

For any element $\beta = \sum b_g g \in K[G]$, let us write $\text{supp } \beta = \{g \in G \mid b_g \neq 0\}$. In particular, the *support* of β is a finite subset of G which is nonempty when $\beta \neq 0$. Furthermore, let $\text{quot } \beta$ denote the set of *quotients* xy^{-1} with $x, y \in \text{supp } \beta$, and for any prime p let $p\text{-quot } \beta$ denote the set of nonidentity elements of $\text{quot } \beta$ having order a power of p . Finally, if L is any subgroup of G , we write

$$\sqrt{L} = \{x \in G \mid x^n \in L \text{ for some } n \neq 0\}.$$

Obviously, $\sqrt{L} \supseteq L$, but this *root set* need not be a subgroup of G .

Proposition 4.3. [HP] *Let G be an arbitrary group, let H be a normal solvable f.c. subgroup of G , and let $\alpha \in \mathcal{J}K[G] \cap K[H]$. Then*

$$G = \bigcup_{x \in p\text{-quot } \alpha} \sqrt{\mathbb{C}_G(x)}.$$

It remained to translate the latter set theoretic union into a more understandable condition. To start with, notice that $G = \sqrt{\langle 1 \rangle}$ is equivalent to G being a periodic group, and therefore the preceding root set equation is related to the Burnside problem. Fortunately, the Burnside problem is quite simple to deal with when G is solvable, and paper [P4] handled this more general situation. Specifically, it showed that if $G = \bigcup_1^n \sqrt{L_i}$ is a finite union of root sets of subgroups and if G is finitely generated and solvable, then some L_i must have finite index in G . By combining all these ingredients, we obtained

Theorem 4.4. [HP], [P4], [Z1] *Let G be a solvable group and let K be a field of characteristic $p > 0$. Then $\mathcal{J}K[G] \neq 0$ if and only if $\mathfrak{Z}(G)$ contains an element x of order p which has only finitely many conjugates under the action of each finitely generated subgroup of G .*

Note that the latter condition on x is equivalent to the assertion that if $x \in H \subseteq G$ with H finitely generated, then $x \in \Delta^+(H)$. In particular, if G is a finitely generated group, then this condition reduces to the assumption that $x \in \Delta^+(G)$, and of course this is precisely equivalent to the nonvanishing of $\mathcal{N}K[G]$. In fact, fairly soon afterwards, Zalesskiĭ built upon the preceding, added an additional intersection theorem of sorts, and proved

Theorem 4.5. [Z2] *If G is a finitely generated solvable group and K is a field of characteristic $p > 0$, then $\mathcal{J}K[G] = \mathcal{N}K[G]$.*

In particular, in the above situation, we not only know when $K[G]$ is semiprimitive, we actually know the complete structure of $\mathcal{J}K[G]$ by applying Theorem 3.4. Most of these results have now been extended to groups which have a finite normal series with f.c. factor groups. But these generalizations offer nothing new in the way of ideas or techniques. Now let us move on to consider linear groups over a field F . Here there are actually three different problems according to whether $\text{char } F = 0$, $\text{char } F = p = \text{char } K$, or $\text{char } F = q > 0$ with $q \neq \text{char } K$. The first two cases were completely settled in the 1970's, but the third was only finished quite recently.

The linear groups in characteristic p actually turned out to be the most interesting of the three possibilities. Here, the proof consisted of a complicated trace argument, along with the solution of another variant of the Burnside problem for linear groups. The answer is quite similar to that for solvable groups and requires that we first define a particular characteristic f.c. subgroup $\mathcal{L}(G)$. This is done in a fairly simple manner, so $\mathcal{L}(G)$ is by no means as important as $\mathfrak{Z}(G)$.

Theorem 4.6. [P5], [P6] *Let G be an F -linear group and assume that $\text{char } F = p = \text{char } K$. Then $\mathcal{J}K[G] \neq 0$ if and only if $\mathcal{L}(G)$ has an element x of order p which has only finitely many conjugates under the action of each finitely generated subgroup of G .*

Now let us assume that G is a finitely generated F -linear group. If $\text{char } F \neq p$, then it follows quite easily that G has a normal subgroup H of finite index which is residually a finite q -group for some prime $q \neq p$. Consequently, $\mathcal{J}K[H] = 0$ and Lemma 3.5 implies that $\mathcal{J}K[G]$ is nilpotent. On the other hand, if $\text{char } F = p$, then it follows from the preceding theorem and a certain amount of work that $\mathcal{J}K[G]$ is at least a locally nilpotent ideal. In other words, we have

Corollary 4.7. [P7] *If G is a finitely generated linear group and $\text{char } K = p > 0$, then $\mathcal{J}K[G] = \mathcal{N}K[G]$.*

Thus a pattern began to emerge and we were led to

Conjecture 4.8. *If G is any finitely generated group and if $\text{char } K = p > 0$, then $\mathcal{J}K[G] = \mathcal{N}K[G]$.*

There was even some corroborating evidence which held for arbitrary groups. Recall that the nilpotent radical is not a radical in general. Indeed, there exists a finitely generated K -algebra A with $\mathcal{N}(A/\mathcal{N}A) \neq 0$. But this cannot happen for group rings of finitely generated groups if the preceding conjecture is to hold. Fortunately, we were able to show

Theorem 4.9. [P7] *If G is any finitely generated group, then $K[G]$ is a finitely generated K -algebra and $\mathcal{N}(K[G]/\mathcal{N}K[G]) = 0$. Furthermore, if H is a subgroup of finite index in G , then $\mathcal{J}K[H] = \mathcal{N}K[H]$ if and only if $\mathcal{J}K[G] = \mathcal{N}K[G]$.*

We remark that this result, Theorem 4.5, and Corollary 4.7 were all proved using the following quite surprising radical-like property of the Δ^+ operator.

Lemma 4.10. *Let G be a finitely generated group and let H be a normal subgroup of G . If $H \subseteq \Delta^+(G)$, then $\Delta^+(G/H) = \Delta^+(G)/H$.*

It is easy to see that this lemma requires G to be finitely generated, and it does not hold for the Δ operator or indeed for the operator \mathbb{Z} , where $\mathbb{Z}(G)$ is the center of G . Unfortunately, this marks the extent of our knowledge of the semiprimitivity question for finitely generated groups. There has been no significant progress made on this problem since the above theorems appeared in 1973 and 1974.

§5. LOCALLY FINITE GROUPS

The obvious next step is to deal with arbitrary groups G under the assumption that we know the answer in the finitely generated case. For convenience, let

$\mathcal{F}(G)$ denote the set of finitely generated subgroups of G . Then, motivated by Theorems 4.4 and 4.6, we define a local version of the f.c. center by

$$\Lambda(G) = \{ x \in G \mid |H : \mathbb{C}_H(x)| < \infty \text{ for all } H \in \mathcal{F}(G) \}.$$

In other words,

$$\Lambda(G) = \bigcap_{H \in \mathcal{F}(G)} \mathbb{D}_G(H)$$

consists of all elements of G which have only finitely many conjugates under the action of each finitely generated subgroup of G . If we also let $\Lambda^+ = \Lambda^+(G)$ be the set of torsion elements of $\Lambda = \Lambda(G)$, then the known structure of Δ and Δ^+ translate to

Lemma 5.1. *Let G be an arbitrary group.*

- i. Λ and Λ^+ are characteristic subgroups of G .
- ii. Λ/Λ^+ is torsion free abelian, and Λ^+ is a locally finite group.
- iii. If $H \triangleleft G$ with $H \subseteq \Lambda^+$, then $\Lambda^+(G/H) = \Lambda^+(G)/H$.

Of course, a group G is *locally finite* if every finitely generated subgroup is finite. For such groups, it follows easily that $\Lambda^+(G) = G$. Thus, the assertion of part (ii) that $\Lambda^+(G)$ is locally finite cannot be further sharpened. Notice also that part (iii) above asserts that the operator Λ^+ exhibits radical-like properties. This is clearly a local version of Lemma 4.10.

Now suppose $\alpha \in \mathcal{JK}[G]$ and let H be any finitely generated subgroup of G with $\text{supp } \alpha \subseteq H$. Then $\alpha \in \mathcal{JK}[G] \cap K[H] \subseteq \mathcal{JK}[H]$ and hence, if we happen to know that $\mathcal{JK}[H] = \mathcal{NK}[H]$, then we can use the structure of $\mathcal{NK}[H]$, as described in Theorem 3.4, to better understand α . Specifically, we obtain

Theorem 5.2. [P7] *Let G be an arbitrary group and let K be a field of characteristic $p > 0$. If $\mathcal{JK}[H] = \mathcal{NK}[H]$ for all $H \in \mathcal{F}(G)$, then*

$$\mathcal{JK}[G] = \mathcal{JK}[\Lambda^+(G)] \cdot K[G].$$

In particular, it follows from Theorem 4.5 and Corollary 4.7 that if G is either locally solvable or locally linear, then $\mathcal{JK}[G] = \mathcal{JK}[\Lambda^+(G)] \cdot K[G]$. This is, in fact, how the semiprimitivity problem for characteristic 0 linear groups was settled. Namely, if G is such a group, then $\mathcal{JK}[G]$ is generated by $\mathcal{JK}[\Lambda^+(G)]$, and $\Lambda^+(G)$ is a locally finite characteristic 0 linear group. Thus $\Lambda^+(G)$ is abelian-by-finite and, with this, we can easily obtain a result quite similar to Theorem 4.6.

Notice also that if Conjecture 4.8 holds, then Theorem 5.2 reduces the semiprimitivity problem to the case of locally finite groups. In other words, this result splits the general problem into two parts. Specifically, we must first study the finitely generated case and show that $\mathcal{JK}[G] = \mathcal{NK}[G]$ for such groups. Then we must

settle the problem for locally finite groups. In particular, this means that the locally finite case is also of crucial importance, and the remainder of this survey will be devoted to a discussion of this situation.

To start with, let us take another look at Theorems 4.4 and 4.6 in the context of locally finite groups. In each case, we have a normal f.c. subgroup H of G and an element $x \in H$ of order p . Since H is generated by its finite normal subgroups, it follows that x is contained in such a subgroup M . In particular, M is a finite subnormal subgroup of G of order divisible by p , and it began to appear that these finite subnormal subgroups might be the key to the solution. But inclusion in the Jacobson radical is a local property, so a local version of subnormality was really more appropriate.

Let G be a locally finite group and let A be a finite subgroup of G . We say that A is *locally subnormal* in G , and write $A \text{ lsn } G$, if A is subnormal in B for all finite subgroups B of G with $A \subseteq B$. For example, if G is locally nilpotent, then every finite subgroup is locally subnormal. Basic properties are as follows.

Lemma 5.3. *Let G be a locally finite group and let K be a field.*

- i. $\mathcal{JK}[G]$ is a nil ideal.
- ii. If $A \triangleleft \triangleleft G$, then $\mathcal{JK}[A] \subseteq \mathcal{JK}[G]$.
- iii. If $A \text{ lsn } G$, then $\mathcal{JK}[A] \subseteq \mathcal{JK}[G]$.

Proof. We sketch the argument. For part (i), let $\alpha \in \mathcal{JK}[G]$ and choose a finite subgroup H of G which contains the support of α . Then $\alpha \in \mathcal{JK}[G] \cap K[H] \subseteq \mathcal{JK}[H]$ by Lemma 2.3(ii), and $\mathcal{JK}[H]$ is nilpotent since H is finite. Thus α is nilpotent, and $\mathcal{JK}[G]$ is indeed a nil ideal. For part (ii), it suffices to assume that $A \triangleleft G$, and to show that $\mathcal{JK}[A] \cdot K[G]$ is a nil right ideal of $K[G]$. To this end, let $\gamma \in \mathcal{JK}[A] \cdot K[G]$ and write $\gamma = \sum_1^n \alpha_i \beta_i$ with $\alpha_i \in \mathcal{JK}[A]$ and $\beta_i \in K[G]$. Since G/A is locally finite, there exists a finite subgroup B/A of G/A with $\text{supp } \beta_i \subseteq B$ for all i . Then, by Lemma 3.5, $\gamma = \sum_1^n \alpha_i \beta_i \in \mathcal{JK}[A] \cdot K[B] \subseteq \mathcal{JK}[B]$, and hence we conclude from (i) that γ is nilpotent. Part (iii) follows in a similar manner. \square

If K is a field of characteristic $p > 0$, and if P is a locally finite p -group, then it follows from part (iii) above that $\mathcal{JK}[P]$ is the *augmentation ideal* of $K[P]$, namely the kernel of the natural homomorphism $K[P] \rightarrow K$ given by $P \mapsto 1$. In particular, if $P = \mathbb{O}_p(G)$ is the largest normal p -subgroup of G , then $\mathcal{JK}[P] \cdot K[G]$ is the kernel of the natural homomorphism $K[G] \rightarrow K[G/P]$, and this kernel is contained in $\mathcal{JK}[G]$ by (ii) above. In other words, we have

$$\mathcal{JK}[G]/(\mathcal{JK}[P] \cdot K[G]) \cong \mathcal{JK}[G/P],$$

and obviously $\mathbb{O}_p(G/P) = \langle 1 \rangle$. Because of this, it usually suffices to assume that $\mathbb{O}_p(G) = \langle 1 \rangle$.

As we will see, if $\mathbb{O}_p(G) = \langle 1 \rangle$, then the differences between locally subnormal subgroups, finite subnormal subgroups, and finite subgroups of normal f.c. subgroups essentially disappear. Note that we are interested in the p -elements of such a finite subgroup A , and hence our real concern is with $\mathbb{O}^{p'}(A)$, the characteristic subgroup of A generated by its Sylow p -subgroups. In other words, we can usually assume that $A = \mathbb{O}^{p'}(A)$. In the following definition, $\text{len } A$ denotes the *composition length* of A , namely the common length of all composition series for A . Since A is finite, $\text{len } A$ is certainly finite.

Now for any locally finite group G and fixed prime p , let $\mathbb{S}^p(G)$ be the characteristic subgroup of G generated by all $A \text{ lsn } G$ with $A = \mathbb{O}^{p'}(A)$. Furthermore, for each integer $n \geq 1$, let $\mathbb{S}_n^p(G)$ be the subgroup of G generated by all $A \text{ lsn } G$ with $A = \mathbb{O}^{p'}(A)$ and $\text{len } A \leq n$. Then we have

Theorem 5.4. [P8] *Let G be a locally finite group with $\mathbb{O}_p(G) = \langle 1 \rangle$. Then $\mathbb{S}^p(G)$ is the ascending union of its characteristic f.c. subgroups $\mathbb{S}_n^p(G)$.*

Suppose, in the above situation, that $A \text{ lsn } G$, $A = \mathbb{O}^{p'}(A)$, and say $\text{len } A = n$. Then $A \subseteq \mathbb{S}_n^p(G)$, and the latter is a normal f.c. subgroup of G . In particular, since $\mathbb{S}_n^p(G)$ is generated by its finite normal subgroups, there exists such a subgroup B with $A \subseteq B \triangleleft \mathbb{S}_n^p(G)$. But $|B| < \infty$, so $A \triangleleft \triangleleft B$ and therefore $A \triangleleft \triangleleft G$. Furthermore, if we take B to be the normal closure of A in $\mathbb{S}_n^p(G)$, then $B = \mathbb{O}^{p'}(B)$ and $B \triangleleft \triangleleft G$ with subnormal depth at most 2. Thus these several concepts all merge into one.

To handle groups having normal p -subgroups, it is natural to define $\mathbb{T}^p(G) \supseteq \mathbb{O}_p(G)$ so that

$$\mathbb{T}^p(G)/\mathbb{O}_p(G) = \mathbb{S}^p(G/\mathbb{O}_p(G)).$$

Then $\mathbb{T}^p(G)$ is a characteristic subgroup of G with a fairly nice structure which can be read off from the preceding theorem. Furthermore, we have

Lemma 5.5. *Let $\text{char } K = p$, and write $T = \mathbb{T}^p(G)$ and $P = \mathbb{O}_p(G)$.*

- i. $\mathcal{J}K[T] \cdot K[G] \subseteq \mathcal{J}K[G]$.
- ii. $\mathcal{J}K[T]/(\mathcal{J}K[P] \cdot K[T]) = \mathcal{J}K[\mathbb{S}^p(G/P)] = \bigcup \mathcal{J}K[A]$, where the union is over all $A \text{ lsn } G/P$ with $A = \mathbb{O}^{p'}(A)$.
- iii. $\mathcal{J}K[T] \neq 0$ if and only if $T \neq \langle 1 \rangle$, or equivalently if and only if G has a locally subnormal subgroup of order divisible by p .

For a number of reasons, we suspected that the set theoretic inclusion in (i) above might always be an equality. For example, we knew that it held for G a locally finite solvable group or an F -linear group with $\text{char } F = 0$ or p . Furthermore, there was some additional corroborating evidence which will be discussed in the next section. With all of this, we were led to

Conjecture 5.6. *If G is a locally finite group and K is a field of characteristic $p > 0$, then*

$$\mathcal{J}K[G] = \mathcal{J}K[\mathbb{T}^p(G)] \cdot K[G].$$

§6. LOCALLY SOLVABLE GROUPS

Before we proceed further, it is worthwhile to see what the latter two conjectures say about the semiprimitivity problem for group rings of arbitrary groups. To this end, let G be any group and let K be a field of characteristic $p > 0$. If H is a finitely generated subgroup of G , then according to Conjecture 4.8, $\mathcal{J}K[H] = \mathcal{N}K[H]$, and therefore Theorem 5.2 yields $\mathcal{J}K[G] = \mathcal{J}K[\Lambda^+(G)] \cdot K[G]$. But $\Lambda^+(G)$ is locally finite, so Conjecture 5.6 implies that $\mathcal{J}K[\Lambda^+(G)] = \mathcal{J}K[\mathbb{T}^p(\Lambda^+(G))] \cdot K[\Lambda^+(G)]$, and hence we have

$$\mathcal{J}K[G] = \mathcal{J}K[\mathbb{T}^p(\Lambda^+(G))] \cdot K[G].$$

Furthermore, Lemma 5.5 contains an appropriate description of $\mathcal{J}K[\mathbb{T}^p(\Lambda^+(G))]$. In particular, it follows from the above and Lemma 5.5(iii) that $\mathcal{J}K[G] \neq 0$ if and only if $\mathbb{T}^p(\Lambda^+(G)) \neq \langle 1 \rangle$, and hence if and only if G has an element x of order p contained in a locally subnormal subgroup of $\Lambda^+(G)$. With this, we now know what “well placed” should mean in Conjecture 3.3.

Of course, neither Conjecture 3.3 nor 4.8 has been proved, and we seem to be quite far from the general solution. Nevertheless, significant progress has been made in the case of locally finite groups, so we return to this situation now. Indeed, until further notice, G will always denote a locally finite group and K will be a field of characteristic $p > 0$. As we remarked, Conjecture 5.6 was shown in [P7] to hold for solvable groups and F -linear groups with $\text{char } F = 0$ or p . Furthermore, we have

Theorem 6.1. [P9] *Let G be a locally finite group.*

- i. $\mathcal{J}K[\mathbb{T}^p(G)] \cdot K[G]$ is a semiprime ideal of $K[G]$, and it is a prime ideal when $\Delta^+(G/\mathbb{O}_p(G)) = \langle 1 \rangle$.
- ii. If H is a subgroup of finite index in G , then $\mathcal{J}K[G] = \mathcal{J}K[\mathbb{T}^p(G)] \cdot K[G]$ if and only if $\mathcal{J}K[H] = \mathcal{J}K[\mathbb{T}^p(H)] \cdot K[H]$.

Of course, an ideal I of a ring R is said to be *semiprime* if $\mathcal{N}(R/I) = 0$, and $\mathcal{J}R$ must necessarily have this property. Thus the above result at least partially corroborates Conjecture 5.6. We remark that Theorem 6.1 was surprisingly difficult to prove. It required intersection theorems from [DZ], and a significant amount of group theory. Specifically, a *generalized Fitting subgroup* $\mathbb{F}^*(G)$ was defined in [P9] and shown to have the following minimax property.

Theorem 6.2. [P9], [P20] *Let $G = \mathbb{S}^p(G)$ with $\mathbb{O}_p(G)$ finite, and set $F = \mathbb{F}^*(G)$.*

- i. $G = \mathbb{D}_G(F) = \{g \in G \mid |F : \mathbb{C}_F(g)| < \infty\}$, and hence F is a characteristic f.c. subgroup of G .
- ii. Suppose $G \triangleleft GB$ where B is a finite group with $|F : \mathbb{C}_F(B)| < \infty$. Then GB is generated by its locally subnormal subgroups.

In other words, part (i) shows that F is small enough to be almost central in G , while part (ii) implies that it is big enough to control certain types of automorphisms

of G . We remark that the definition of \mathbb{F}^* was changed to a more natural one in [P20], and the reference to that paper in the preceding theorem refers to this new formulation. Next, we state and prove the following elementary, but extremely powerful consequence of Theorems 3.4 and 4.9.

Lemma 6.3. [P11] *Let $H \triangleleft G$ with $\mathcal{J}K[H] = \mathcal{N}K[H]$. If $D = \mathbb{D}_G(H)$, then*

$$\mathcal{J}K[G] = \mathcal{J}K[D] \cdot K[G].$$

Proof. Since $D \triangleleft G$, Lemma 5.3(ii) implies that $\mathcal{J}K[D] \cdot K[G] \subseteq \mathcal{J}K[G]$. For the reverse inclusion, let $\alpha \in \mathcal{J}K[G]$ and choose any subgroup $B \supseteq H$ with $|B/H| < \infty$ and $\alpha \in K[B]$. Then $\alpha \in \mathcal{J}K[G] \cap K[B] \subseteq \mathcal{J}K[B]$, and $\mathcal{J}K[B] = \mathcal{N}K[B] = \mathcal{J}K[\Delta^+(B)] \cdot K[B]$ by Theorems 4.9 and 3.4(i). But $|B : H| < \infty$ and B is periodic, so $\Delta^+(B) = \mathbb{D}_B(H) = D \cap B$, and hence $\alpha \in \mathcal{J}K[D \cap B] \cdot K[B] \subseteq \mathcal{J}K[D \cap B] \cdot K[G]$. Since this holds for all such B , it follows easily that $\alpha \in \mathcal{J}K[D] \cdot K[G]$. \square

The final result of this section deals with locally p -solvable groups. Its proof uses Δ -methods applied to finite subgroups of G , a rather surprising idea, and makes crucial use of Theorem 6.1(ii) and the preceding result applied to $H = \mathbb{O}_{p'}(G)$. In addition, it requires Hall-Higman methods (see [HH]) and a number of observations on p -solvable finitary linear groups.

Theorem 6.4. [P11] *If G is a locally finite, locally p -solvable group and if K is a field of characteristic $p > 0$, then*

$$\mathcal{J}K[G] = \mathcal{J}K[\mathbb{T}^p(G)] \cdot K[G].$$

With this result, proved in 1979, we completed an intensive ten year attack on the semiprimitivity problem in characteristic $p > 0$. At this point, it seemed appropriate to move on to other tasks. The general locally finite case would surely require a better understanding of the finite simple groups, and the classification was not to be completed for several more years. But before we leave the 1970's, we should mention two special cases of Conjecture 5.6 which would serve as later test problems. To start with, if G is infinite simple, then it follows easily that $\mathbb{T}^p(G) = \langle 1 \rangle$. Furthermore, if $|G|_p < \infty$, that is if there is a bound on the orders of the finite p -subgroups of G , then $\mathbb{T}^p(G)$ is a finite normal subgroup of G . Thus we were led to

Conjecture 6.5. *Let G be a locally finite group.*

- i. *If G is an infinite simple group, then $\mathcal{J}K[G] = 0$.*
- ii. *If $|G|_p < \infty$, then $\mathcal{J}K[G]$ is nilpotent.*

These were not considered at all during the decade of the 1980's, but they were solved in the affirmative in the early 1990's using the known structure of infinite simple groups. It turned out that the wait was necessary.

§7. INFINITE SIMPLE GROUPS

Finally, we can begin our discussion of recent progress on semiprimitivity. Again we assume that G is a locally finite group and that K is a field of characteristic $p > 0$. If π is any set of primes, we say that g is a π -element if $|g|$, the order of g , has all its prime factors in π . For convenience, we let G_π denote the set of π -elements of G , so that $1 \in G_\pi$ for all π . If X is a finite subset of $G^\# = G \setminus \{1\}$, we say that $z \in G$ is a π -insulator of X if $z \in G_\pi$ and $zX \cap G_\pi = \emptyset$. Furthermore, we say that G is π -insulated if every finite subset of $G^\#$ has a π -insulator. Note that, if $\pi = \{p\}$ consists of the single prime p , then we use p -element and p -insulated instead of the more cumbersome $\{p\}$ -element and $\{p\}$ -insulated. The following is proved by a simple trace argument.

Lemma 7.1. *Let π be a set of primes containing $p = \text{char } K$. If G is π -insulated, then $K[G]$ is semiprimitive.*

Surprisingly, this is all the group ring theory we need to settle Conjecture 6.5(i). The remainder of the long argument is entirely group theoretic in nature and requires a close look at the structure of locally finite simple groups. For our purposes, it suffices to assume that all such groups are countably infinite.

Suppose, for example, that $G = \text{FAlt}_\infty$ is the finitary alternating group on the set of positive integers. If Alt_n denotes the subgroup of G moving points in $\{1, 2, \dots, n\}$ and fixing the rest, then G is the ascending union of the groups Alt_n with $n \geq 5$, and hence G is an ascending union of finite simple groups. Unfortunately, this property is not always true. More typical is the case where G is the finitary special linear group $\text{FSL}_\infty(F)$ with F a finite field. Here G consists of all countably infinite square F -matrices

$$g = \begin{bmatrix} \bar{g} & 0 \\ 0 & I \end{bmatrix}$$

where $\bar{g} \in \text{SL}_n(F)$ for some n and I is the identity matrix on the remaining rows and columns. Notice that $\text{FSL}_\infty(F)$ contains no nonidentity scalar matrix, so there is no need to form the projective group. Now it is clear that G is the ascending union of the finite subgroups $G_n \cong \text{SL}_n(F)$ with $n \geq 4$, but this time the groups G_n are not simple. Instead, G_n has a normal subgroup M_n , corresponding to the scalar matrices, and $G_n/M_n \cong \text{PSL}_n(F)$ is simple. Furthermore, the combined map

$$G_{n-1} \rightarrow G_n \rightarrow G_n/M_n \cong \text{PSL}_n(F)$$

is easily seen to be an embedding. This is indicative of the following fundamental result.

Lemma 7.2. [Ke] *Let G be a locally finite, countably infinite simple group. Then G has finite subgroups G_i for $i = 0, 1, 2, \dots$ satisfying*

- i. $G_i \subseteq G_{i+1}$ and $G = \bigcup_{i=0}^{\infty} G_i$,

- ii. $M_i \triangleleft G_i$ with $G_i/M_i = S_i$ a nontrivial simple group, and
- iii. for all $i < j$, the composite map

$$G_i \rightarrow G_j \rightarrow G_j/M_j = S_j$$

is an embedding.

In the above situation, we say that G is a *limit of the approximating sequence* S_0, S_1, \dots and we write $G = \lim_{i \rightarrow \infty} S_i$. Of course, G is not uniquely determined by the simple groups S_i , but the approximating sequence does encode a surprising amount of information on the structure of G . To start with, the Classification Theorem (see [G]) asserts that the collection of finite simple groups is divided into finitely many infinite families and finitely many exceptions, the *sporadic groups*. Thus, since any subsequence of the triples (G_i, M_i, S_i) also determines an approximating sequence for G , we can assume that all S_i belong to the same infinite family. Now most of these families have a prime power parameter and all of them have an integer parameter n . Furthermore, it turns out that G is a linear group if and only if the parameter n is uniformly bounded. The nonlinear case was settled first.

Theorem 7.3. [PZ] *Let G be a locally finite simple group which is not a linear group. Then G is p -insulated for any prime p , and consequently every group algebra $K[G]$ is semiprimitive.*

One aspect of the proof of this result deals with the maps $G_i \rightarrow G_j \rightarrow S_j$ which are by no means the obvious inclusions. Fortunately, this difficulty can be overcome with a simple idea implemented in a fairly tedious manner. The more interesting aspect of the argument really concerns the infinite groups FAlt_∞ , $\text{FSL}_\infty(F)$, $\text{FSU}_\infty(F)$, $\text{FSp}_\infty(F)$, and $\text{F}\Omega_\infty(F)$ where F is a finite field. Note that the latter four groups correspond to those families of finite simple groups of Lie type for which the integer parameter n can become unbounded. The finitary alternating group had actually been considered by Formanek in 1972, and we sketch his clever argument.

Lemma 7.4. [F] *If $G = \text{FAlt}_\infty$ or FSym_∞ , then G is p -insulated for any prime p .*

Proof. If X is a finite subset of $G^\#$, then we can choose an even integer k so that the elements of $X \subseteq \text{FSym}_\infty$ move only points in the set $\{1, 2, \dots, k\}$. Now define

$$z = (1 * \dots *) (2 * \dots *) \dots (k * \dots *)$$

where the $*$'s denote distinct points in $\{k+1, k+2, \dots\}$ and where $(j * \dots *)$ is a cycle of length p^j . Clearly $z \in \text{FSym}_\infty$ is a p -element, and hence $g \in \text{FAlt}_\infty$ if p is odd. On the other hand, if $p = 2$, then z is the product of an even number of odd cycles, so again $z \in \text{FAlt}_\infty \subseteq G$.

Finally, let $x \in X$, and write x as a product of disjoint cycles which, by assumption, involve only the first k points. If $(j_1 j_2 \dots j_r)$ is such a nontrivial cycle in x , then zx (acting on the right) contains the cycle

$$(j_1 * \dots * j_2 * \dots * \dots j_r * \dots *)$$

which is the juxtaposition of the corresponding cycles in z . Since the j_i are distinct, the latter displayed cycle has length $p^{j_1} + p^{j_2} + \dots + p^{j_r}$ and this is not a power of p . Thus zx is not a p -element, so $zX \cap G_p = \emptyset$ and G is p -insulated. \square

The corresponding proof for the infinite size matrix groups is much more complicated. In some sense, these groups divide naturally into the four cases

Case 1:	$G = \text{FSL}_\infty(F)$	$\text{char } F \neq p$
Case 2:	$G = \text{FSU}_\infty(F), \text{FSp}_\infty(F), \text{F}\Omega_\infty(F)$	$\text{char } F \neq p$
Case 3:	$G = \text{FSL}_\infty(F)$	$\text{char } F = p$
Case 4:	$G = \text{FSU}_\infty(F), \text{FSp}_\infty(F), \text{F}\Omega_\infty(F)$	$\text{char } F = p$

and these are dealt with in turn. The difficulty increases as we go down the list and reaches a crescendo when we hit the bottom.

Now on to the simple linear groups. Here, we have the following wonderful characterization of such groups based on Lemma 7.2 and the Classification Theorem.

Theorem 7.5. [Be], [Bo], [HS], [T] *Let G be a locally finite simple group. If G is an infinite linear group, then G is a group of Lie type over a locally finite field F .*

Of course, the field F is *locally finite* if $\text{char } F = q > 0$ and F is contained in the algebraic closure of $\text{GF}(q)$. It follows from the above characterization that G contains a 1-parameter family of q -elements and, using this and the Zariski topology on G , we obtain

Theorem 7.6. [P14] *Let G be a locally finite simple group. If G is an infinite linear group over a locally finite field F of characteristic $q > 0$, then G is $\{p, q\}$ -insulated for any prime p . In particular, every group algebra $K[G]$ is semiprimitive.*

Thus Theorems 7.3 and 7.6 settle Conjecture 6.5(i) in the affirmative. Furthermore, with a little more work and a knowledge of the Schur multipliers of the groups of Lie type, we can prove that if G is infinite simple, then any twisted group algebra $K^t[G]$ is semiprimitive. This is not merely of academic interest; the twisted result is actually needed to proceed further.

§8. EXTENSION PROBLEMS

The second test problem, namely Conjecture 6.5(ii), turned out to be less important. However it did motivate us to study certain extension problems which must necessarily be part of the general solution. To start with, let G be a locally finite

group, and let $|G|_p$ denote the supremum of the orders of its finite p -subgroups. In view of the Sylow theorems, it is clear that $|G|_p < \infty$ if and only if G satisfies the ascending chain condition on finite p -subgroups and hence if and only if G has no infinite p -subgroup. Of course, $|G|_p = 1$ is equivalent to G being a p' -group. Now if $|G|_p < \infty$, then we have a finite parameter to induct on, and by so doing, Lemma 7.2 and Theorem 7.5 yield

Lemma 8.1. *Let G be a locally finite group with $|G|_p < \infty$. Then G has a finite subnormal series*

$$\langle 1 \rangle = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_n = G$$

with each quotient $\bar{G}_i = G_i/G_{i-1}$ either

- i. a p' -group,
- ii. a finite simple group, or
- iii. an infinite simple group of Lie type.

In particular, for each subscript i , we know the solution to the semiprimitivity problem for $K[\bar{G}_i]$, and thus we should be able to find the solution for $K[G]$ from the preceding lemma provided we can handle the *extension problem*. To this end, let N be a normal subgroup of the arbitrary group G . Then we know that G is an extension of N by G/N , and therefore $K[G]$ is an extension of $K[N]$ by G/N . As we will see below, this structure is best understood in the context of crossed products.

Let R be any ring and let G be any group. Then a *crossed product* $R * G$ of G over R is an associative ring having a copy \bar{G} of G as a left R -basis. In other words, every element α of $R * G$ is uniquely a finite sum $\alpha = \sum_{x \in G} r_x \bar{x}$ with coefficients $r_x \in R$ and with *support* defined by $\text{supp } \alpha = \{x \in G \mid r_x \neq 0\}$. Addition in $R * G$ is as expected, and multiplication is determined by the rules

$$(\text{twisting}) \quad \bar{x}\bar{y} = \tau(x, y) \overline{xy} \quad \text{for all } x, y \in G,$$

where τ is a map from $G \times G$ to the group of units $U(R)$ of R , and

$$(\text{action}) \quad r\bar{x} = \bar{x}r^{\sigma(g)} \quad \text{for all } r \in R, x \in G,$$

where σ is a map from G to $\text{Aut}(R)$. Note that τ and σ are not group homomorphisms in general. The relations they are assumed to satisfy are precisely equivalent to the associativity of the ring, and we may also suppose that $\bar{1} = 1$ is the identity element of $R * G$. Obviously, any group algebra is a crossed product with trivial twisting and action.

Now suppose $K[G]$ is given with $N \triangleleft G$ and let S be a transversal for N in G . Observe that the elements of S act on $R = K[N]$ via conjugation by $r^s = s^{-1}rs$, and that if $s_1, s_2 \in S$, then there exists $s_3 \in S$ and $u \in N$ with $s_1s_2 = us_3$. Since

$K[G] = \bigoplus \sum_{s \in S} K[N]s$ and since there is a natural one-to-one correspondence between the elements of S and those of G/N , it is now clear that $K[G] = R*(G/N)$ is a crossed product of G/N over $R = K[N]$. In fact, there is a more general result here. Namely, if $R*G$ is any crossed product and if $N \triangleleft G$, then $R*G = (R*N)*(G/N)$ where $R*N$ is the uniquely determined sub-crossed product of $R*G$ consisting of those elements having support in N (see [P12]).

It would be nice if the extension aspects of the semiprimitivity problem followed directly from crossed product considerations. However, this is not the case, as can be seen from the following example.

Lemma 8.2. *Let G be an arbitrary group containing an element of prime order p . Then there exists a semiprimitive commutative algebra R over a field of characteristic p and a crossed product $R*G$, such that $R*G$ is not semiprime. In particular, $R*G$ is not semiprimitive.*

Proof. Let H be the given subgroup of G of order p and let K be a field of characteristic p . If Ω denotes the set of right cosets of H in G , then G permutes Ω by right multiplication, and we let $\omega_0 \in \Omega$ correspond to the coset H . Consequently, $H = G_{\omega_0} = \{g \in G \mid \omega_0 g = \omega_0\}$.

Now let R be the (complete) direct product $\prod_{\omega \in \Omega} K_{\omega}$, where each K_{ω} is a copy of K . Then R is a semiprimitive K -algebra, it is in fact von Neumann regular, and the permutation action of G on Ω extends to an action of G on R . In this way we obtain a homomorphism $\sigma: G \rightarrow \text{Aut}(R)$ and we use σ to form the *skew group ring* $R*G$. In other words, $R*G$ is a crossed product with parameter σ , as above, and with trivial twisting. One knows (see [P12]) that such a construction always leads to an associative ring.

For each $\omega \in \Omega$, let e_{ω} denote the idempotent of R which has a 1 for its ω -coordinate and zeros elsewhere. Then $\bar{g}^{-1}e_{\omega}\bar{g} = e_{\omega g}$ and $e_{\omega}e_{\omega'} = 0$ if $\omega \neq \omega'$. In particular, e_{ω_0} commutes with \bar{H} , and if $g \in G \setminus H$ and $r \in R$, then

$$e_{\omega_0}(r\bar{g})e_{\omega_0} = r\bar{g}(\bar{g}^{-1}e_{\omega_0}\bar{g})e_{\omega_0} = r\bar{g}e_{\omega_0 g}e_{\omega_0} = 0.$$

It now follows easily that $e_{\omega_0}(R*G)e_{\omega_0} = e_{\omega_0}R*\bar{H} \cong K[\bar{H}]$ since $e_{\omega_0}R \cong K$ and the twisting is trivial. In particular, since K has characteristic p and $|\bar{H}| = p$, we conclude that $e_{\omega_0}(R*G)e_{\omega_0}$ is not semiprime, and therefore neither is $R*G$. \square

While crossed product methods are sometimes useful in studying semiprimitivity, it turns out that twisted group algebras are absolutely crucial. Recall that a *twisted group algebra* $K^t[G]$ is a crossed product $K*G$ of G over K with trivial action. In particular, $K^t[G]$ is an associative K -algebra with K -basis \bar{G} and with $\bar{x}\bar{y} = \tau(x, y)\bar{xy}$ for all $x, y \in G$. Here $\tau: G \times G \rightarrow K \setminus \{0\}$ is the twisting function, and associativity is equivalent to τ being a 2-cocycle.

As we will see at the end of this section, twisted group algebras come into play because they are homomorphic images of ordinary group algebras. For example, let

Z be a central subgroup of G and let I be an ideal of $K[Z]$ with $K[Z]/I \cong K$. Then $I \cdot K[G] \triangleleft K[G]$ and it is easy to see that $K[G]/(I \cdot K[G])$ is naturally isomorphic to some twisted group algebra $K^t[G/Z]$ of G/Z .

Now, let us return to Conjecture 6.5(ii). Since the work on this problem actually occurred before Theorem 7.6 was proved, it was necessary to use an earlier special case of the latter result contained in [Z3]. By dealing with the extension problem, we were then able to obtain the affirmative solution

Theorem 8.3. [P13] *Let G be a locally finite group with $|G|_p < \infty$. If $\text{char } K = p > 0$, then $\mathcal{JK}[G]$ is nilpotent.*

The proof of this result starts with a simple reduction which allows us to assume that G has no finite normal subgroup of order divisible by p , and we are left with the task of showing that $\mathcal{JK}[G] = 0$. Some aspects of the latter semiprimitivity argument will be discussed in the more general context of

Theorem 8.4. [P14] *Let $K[G]$ be the group algebra of a locally finite group G over a field K of characteristic $p > 0$. Suppose that G has a finite subnormal series*

$$G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_n = G$$

with each quotient G_i/G_{i-1} either

- i. *a locally p -solvable group,*
- ii. *an infinite nonabelian simple group, or*
- iii. *generated by its locally subnormal subgroups.*

If $\mathcal{JK}[G_0] = 0$, then $K[G]$ is semiprimitive if and only if G has no locally subnormal subgroup of order divisible by p .

A brief outline of the proof of semiprimitivity here is as follows. First, we can assume that K is algebraically closed and that $n = 1$. Indeed, by Lemma 6.3, we can suppose that G has a normal subgroup N with $|N : \mathbb{C}_N(g)| < \infty$ for all $g \in G$, and such that $G/N = H$ is a group satisfying condition (i), (ii) or (iii). In particular, N is an f.c. group and therefore, by hypothesis, it must be a p' -group. With this, case (i) now follows from Theorem 6.4, while case (iii) is an immediate consequence of Theorem 3.4 and Lemma 3.5. Finally, let H be an infinite simple group. Then, we may suppose that H is countably infinite and not a p' -group, and that G has no nontrivial f.c. homomorphic images. In other words, the pair (G, N) is what we call a p' -f.c. cover of H . Now if N is central in G , then G is a central cover of H and it follows from our previous comments that $K[G]$ is a subdirect product of various twisted group algebras $K^t[H]$. Thus, the twisted analogs of Theorems 7.3 and 7.6 apply here and yield the result.

On the other hand, if N is not central in G , then we show that $H = G/N$ is contained in $\text{FGL}(\mathcal{V})$, the *finitary general linear group* on \mathcal{V} , where \mathcal{V} is a vector space over the Galois field $\text{GF}(q)$ for some prime q involved in the subgroup N . Furthermore, with a good deal of effort, this implies that H cannot be a linear group, and therefore the following key result of J. Hall applies.

Theorem 8.5. [H1], [H2], [H3] *Let G be a countably infinite, locally finite simple group which is not a linear group, and suppose that $G \subseteq \text{FGL}(\mathcal{V})$, where \mathcal{V} is a vector space over some field F_0 .*

- i. *If F_0 has characteristic 0, then $G \cong \text{FAlt}_\infty$.*
- ii. *If $\text{char } F_0 = q > 0$, then G is isomorphic to one of the stable finitary groups FAlt_∞ , $\text{FSL}_\infty(F)$, $\text{FSU}_\infty(F)$, $\text{FSp}_\infty(F)$, or $\text{F}\Omega_\infty(F)$, where F is some locally finite field of characteristic q .*

We remark that the uncountable groups have also been classified, but the appropriate analogs of $\text{FSL}_\infty(F)$ are somewhat more complicated to describe. Finally, we define a stronger version of p -insulation and we show that if H is *strongly p -insulated*, then any p' -f.c. cover G of H is p -insulated and hence satisfies $\mathcal{J}K[G] = 0$. Thus all that remains is to prove that the stable groups H , as listed in Theorem 8.5(ii), are strongly p -insulated, and this is achieved in [P14].

§9. THE LOCAL SUBNORMAL CLOSURE

In some sense, the results associated with Conjecture 6.5 are all global in nature. Namely, they involve global assumptions on the locally finite group G like being simple or having a particular type of finite subnormal series. Obviously, the next step is to move on to more local assumptions. However, by some strange quirk of fate, this earlier work is not wasted. It turns out that the infinite simple groups and the locally p -solvable groups (of Theorem 6.4) are the critical factors in the general solution. We will consider this phenomenon in more detail in the next section.

For now, let $H \subseteq X$ be finite groups. Since the set of subnormal subgroups of X is closed under intersection, it follows that there is a unique smallest subnormal subgroup S of X which contains H . This is called the *subnormal closure* of H in X , and we denote it by $S = H^{[X]}$. If H^S is the normal closure of H in S , then $H \subseteq H^S \triangleleft S \triangleleft\triangleleft X$, so the minimal nature of S implies that $S = H^S$. In fact, S is characterized by the two properties

- i. $H \subseteq S \triangleleft\triangleleft X$, and
- ii. $S = H^S$

since (ii) implies that H cannot be contained in a proper normal subgroup of S , and hence it is not in a proper subnormal subgroup of S . In general, subnormal closures do not exist for arbitrary subgroups of infinite groups.

Observe that if $H \subseteq X \subseteq Y$ are all finite, then $H \subseteq H^{[Y]} \cap X \triangleleft\triangleleft X$. Thus the minimal nature of $H^{[X]}$ implies that $H^{[X]} \subseteq H^{[Y]} \cap X \subseteq H^{[Y]}$, and this inclusion allows us to define a *local subnormal closure* for finite subgroups of locally finite groups. Specifically, if H is a finite subgroup of the locally finite group G , then we write

$$H^{[G]} = \bigcup_X H^{[X]}$$

where the union is over all finite subgroups X of G containing H . Note that, if G is finite, then the inclusion $H^{[X]} \subseteq H^{[Y]}$ immediately implies that the two possible meanings for $H^{[G]}$ are, in fact, the same. Some basic properties are as follows.

Lemma 9.1. *Let H be a finite subgroup of G , and set $S = H^{[G]}$.*

- i. S is a subgroup of G containing H .
- ii. If $A \text{ lsn } S$, then $A \text{ lsn } G$.
- iii. $S = H^S$ is the normal closure of H in S .

Obviously, part (ii) above allows us to reduce semiprimivity questions from $K[G]$ to $K[S]$, and when we do this, the conclusion $S = H^S$ of (iii) comes into play. Surprisingly, this latter fact turns out to be a rather crucial property. For example, consider the following lovely observation of Wielandt.

Theorem 9.2. [W2], [W3] *The only primitive, finitary permutation groups on an infinite set Ω are FSym_Ω and FAlt_Ω .*

Then, by adding the hypothesis $G = H^G$, we can quickly extend this result to finitary permutation groups which are not even transitive. Indeed, we have

Lemma 9.3. *Let $G \subseteq \text{Sym}_\Omega$ and suppose that $G = H^G$ for some finite subgroup H . If $H \subseteq \text{FSym}_\Omega$, then G has a finite subnormal series*

$$\langle 1 \rangle = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_n = G$$

with each factor $\bar{G}_i = G_i/G_{i-1}$ either an f.c. group or isomorphic to FAlt_{Λ_i} for some infinite set Λ_i .

Proof. Since $H \subseteq \text{FSym}_\Omega \triangleleft \text{Sym}_\Omega$, it follows that $G = H^G \subseteq \text{FSym}_\Omega$. Now suppose that H moves k points of Ω . Then H can act nontrivially on at most k orbits of G and thus $G = H^G$ implies that G has at most k nontrivial orbits.

For simplicity, let us just consider the case where G is transitive on the infinite set Ω , and let Γ be a block of imprimitivity for G . If $|\Gamma| > k$, then Γ contains a point fixed by H and hence $\Gamma = \Gamma H$. Furthermore, each conjugate H^g of H also moves k points, so $\Gamma = \Gamma H^g$. Thus Γ is stabilized by $\langle H^g \mid g \in G \rangle = H^G = G$, so Γ is an orbit of G and hence $\Gamma = \Omega$. In other words, all nontrivial blocks have size $\leq k$ and therefore we can choose one, say Γ , of maximal size.

Now if Λ denotes the set $\{\Gamma g \mid g \in G\}$ of distinct translates of Γ , then it follows that $|\Lambda| = \infty$ and that G acts in a primitive manner on Λ . In particular, if N is the kernel of this action, then Theorem 9.2 implies that $G/N \cong \text{FSym}_\Lambda$ or FAlt_Λ . Furthermore, N stabilizes all Γg and acts faithfully on the disjoint union $\Omega = \bigcup \Gamma g$ with $\Gamma g \in \Lambda$. Thus, since $N \subseteq G \subseteq \text{FSym}_\Omega$, it follows that N embeds in the direct sum of the finite symmetric groups $\text{Sym}_{\Gamma g}$ and therefore N is an f.c. group. \square

Notice how nicely this fits in with the hypothesis of Theorem 8.4. Similar results hold for finitary automorphism groups. To start with, we say that G acts in a

finitary manner on the group V if $|V : \mathbb{C}_V(x)| < \infty$ for all $x \in G$. Furthermore, G acts in a *strongly finitary manner* if the action is finitary and if all G -stable subgroups of V are normal in V . In particular, both of these concepts include the usual notion of a finitary action of a group G on a vector space V over a finite field. Notice that we do not assume, at this point, that G acts faithfully on V . Note further that if G is strongly finitary on V and if W is a G -stable subgroup of V , then G acts in a strongly finitary manner on both W and V/W . Of course, G acts *irreducibly* on V if and only if V has no proper G -stable normal subgroup.

Lemma 9.4. *Let G act in a strongly finitary manner on the group V , and assume that $G = H^G$ is the normal closure of some finite subgroup H . Then V has a finite chain*

$$\langle 1 \rangle = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n = V$$

of G -stable normal subgroups such that, for each i , either G acts irreducibly on $\bar{V}_i = V_i/V_{i-1}$ or it acts trivially on this quotient.

Now suppose, in addition, that V is a locally finite f.c. group, and assume that G acts irreducibly on the infinite quotient $\bar{V}_i = V_i/V_{i-1}$. If \bar{V}_i is nonabelian, then it follows easily from the f.c. property that it is a *semisimple group*, namely isomorphic to a (weak) direct product of finite nonabelian simple groups. Furthermore, G permutes these direct factors transitively, and therefore Theorem 9.2 enables us to describe $\bar{G}_i = G/\mathbb{C}_G(\bar{V}_i)$. On the other hand, if \bar{V}_i is abelian, then it is an elementary abelian q -group for some prime q , and again we can describe \bar{G}_i if the representation is imprimitive. Fortunately, when the representation is primitive, we can apply the following key result of Phillips.

Theorem 9.5. [Ph1], [Ph2] *Let G be a primitive, locally finite subgroup of $\text{FGL}(\mathcal{V})$, where \mathcal{V} is an infinite dimensional vector space. Then G contains a normal infinite simple subgroup D , such that G/D is solvable of derived length ≤ 6 .*

As a consequence, we obtain

Lemma 9.6. *Let G act faithfully and in a strongly finitary manner on the locally finite f.c. group V . If G is the normal closure of a finite subgroup H , then G has a finite subnormal series*

$$\langle 1 \rangle = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_m = G$$

with each quotient $\bar{G}_i = G_i/G_{i-1}$ either infinite simple or an f.c. group. Furthermore, each such infinite simple quotient is a finitary linear group over a finite field $GF(q)$ for some prime q involved in V .

In particular, by using Theorem 8.5(ii), we can obtain a precise automorphism group analog of Lemma 9.3.

§10. LOCAL RESULTS

The recent series of local results began in [P16] where it was shown that if G has no nonidentity locally subnormal subgroup, then $\mathcal{JK}[G] = 0$. Obviously, this was close to the precise necessary and sufficient conditions for semiprimitivity. Indeed, all that was missing was the relationship between the orders of the locally subnormal subgroups and the characteristic of the field. To proceed further, it was again necessary to work in the more general context of twisted group algebras. In particular, Theorem 6.4 had to be extended to this context, and then the methods used to prove the preceding local result were generalized to yield

Theorem 10.1. [P16], [P17] *Let G be a locally finite group and let K be a field of characteristic $p > 0$. Then $K[G]$ is semiprimitive if and only if G has no locally subnormal subgroup of order divisible by p .*

This is, of course, the semiprimitivity consequence of Conjecture 5.6. As we mentioned in the last section, the earlier results on locally p -solvable groups and on infinite simple groups, as encapsulated in Theorem 8.4, are crucial to the proof of Theorem 10.1, and we give some indication below of this phenomenon. We begin by discussing an ultraproduct argument suggested by the work of [H2] and [H3].

Let $G_1 \subseteq G_2 \subseteq \dots$ be finite subgroups of G with $G = \bigcup_{i=1}^{\infty} G_i$ and let $\mathcal{N} = \{1, 2, \dots\}$ be the set of natural numbers. Then we can choose an ultrafilter \mathcal{F} on \mathcal{N} containing the cofinite subsets, and we note that all members of \mathcal{F} are infinite. Now suppose that each G_i acts as permutations (on the right) on a set Ω_i with kernel N_i . Then the ultraproduct $\prod_{\mathcal{F}} G_i$ acts on $\Omega = \prod_{\mathcal{F}} \Omega_i$ via $\otimes_i w_i \cdot \otimes_i g_i = \otimes_i w_i g_i$. Furthermore, we can define a homomorphism $\theta : G \rightarrow \prod_{\mathcal{F}} G_i$ by $\theta(g) = \otimes_i \theta_i(g)$ where $\theta_i(g) = g$ if $g \in G_i$ and $\theta_i(g) = 1$ otherwise. In this way, we obtain a permutation action of G on Ω which satisfies

Lemma 10.2. *Let G and Ω be as above.*

- i. *If N is contained in the kernel of the action of G on Ω , then there exists a subsequence $\mathcal{M} \subseteq \mathcal{N}$ such that N is the ascending union of the subgroups $N \cap N_i$ with $i \in \mathcal{M}$.*
- ii. *If $g \in G$ and if $\theta_i(g)$ moves at most k points of Ω_i for each i , then g moves at most k points of Ω and hence is finitary on Ω .*

Proof. (i) Suppose $x \in N$ and let $\mathcal{S}(x) = \{i \mid \theta_i(x) \text{ acts nontrivially on } \Omega_i\}$. For each $i \in \mathcal{S}(x)$ choose $w_i \in \Omega_i$ moved by $\theta_i(x)$, and if $i \notin \mathcal{S}(x)$ let $w_i \in \Omega_i$ be arbitrary. Then $w = \otimes_i w_i \in \Omega$ and, since $x \in N$, we have

$$\otimes_i w_i \equiv_{\mathcal{F}} (\otimes_i w_i)x = \otimes_i w_i \theta_i(x),$$

where $\equiv_{\mathcal{F}}$ indicates that the two elements of the (complete) direct product agree on a member of \mathcal{F} . In other words, $w_i = w_i \theta_i(x)$ almost everywhere and consequently

$\mathcal{S}(x)$ has measure 0, that is $\mathcal{S}(x) \notin \mathcal{F}$. Furthermore, if X is any finite subset of N , then $\mathcal{S}(X) = \bigcup_{x \in X} \mathcal{S}(x)$ also has measure 0 and therefore the complement of $\mathcal{S}(X)$ is contained in \mathcal{F} and is infinite. Thus we can choose $i \in \mathcal{N} \setminus \mathcal{S}(X)$ sufficiently large so that $X \subseteq G_i$. But then $\theta_i(X) = X$ acts trivially on Ω_i and hence $X \subseteq N_i$. It follows that every finite subset of N is contained in some $N \cap N_i$ and, since each such N_i is finite, the subsequence \mathcal{M} is easily seen to exist.

(ii) Suppose for example that $k = 3$ so that $\theta_i(g)$ moves at most 3 points of Ω_i . Then, for each i , we can choose $a_i, b_i, c_i \in \Omega_i$, not necessarily distinct, with $\theta_i(g)$ fixing the remaining points. Now let $a = \otimes_i a_i$, $b = \otimes_i b_i$ and $c = \otimes_i c_i$ be the elements of Ω determined by these choices. We claim that these are the only possible points moved by g . To this end, let $w = \otimes_i w_i \in \Omega$ and define $A = \{i \mid w_i = a_i\}$, $B = \{i \mid w_i = b_i\}$, $C = \{i \mid w_i = c_i\}$, and $D = \{i \mid w_i \neq a_i, b_i, c_i\}$. Then $A \cup B \cup C \cup D = \mathcal{N}$ and hence at least one of these four sets must have measure 1. Now, if $A \in \mathcal{F}$, then $w = \otimes_i w_i \equiv_{\mathcal{F}} \otimes_i a_i = a$ and similarly $B \in \mathcal{F}$ yields $w = b$ and $C \in \mathcal{F}$ yields $w = c$. Finally, if $D \in \mathcal{F}$, then since $\theta_i(g)$ acts trivially on $\Omega_i \setminus \{a_i, b_i, c_i\}$, we have $wg = \otimes_i w_i \theta_i(g) \equiv_{\mathcal{F}} \otimes_i w_i = w$ and g fixes w . \square

Now, what might the groups G_i act on? To understand our choice, let us first assume that $G_i = W$ is a finite group with no nonidentity solvable normal subgroup. Let $S = \text{soc } W$ be the *socle* of W , so that S is generated by the minimal normal subgroups of W . Since any two distinct minimal normal subgroups commute, it follows that $\text{soc } W$ is the direct product of certain of these subgroups. Furthermore, any minimal normal subgroup is either an elementary abelian q -group for some prime q , or it is *semisimple*, namely a direct product of nonabelian simple groups. This proves (i) below and, of course, parts (ii) and (iii) are routine consequences.

Lemma 10.3. *Let W be a finite group with no nonidentity solvable normal subgroup and set $S = \text{soc } W$.*

- i. $S = M_1 \times M_2 \times \cdots \times M_k$ is a finite direct product of the nonabelian simple groups M_i . Thus S is semisimple.
- ii. $\mathbb{C}_W(S) = \langle 1 \rangle$, so W acts faithfully as automorphisms on S .
- iii. The groups M_i are precisely the minimal normal subgroups of S . Thus W permutes the set $\Omega = \{M_1, M_2, \dots, M_k\}$ by conjugation.
- iv. If N is the kernel of the action of W on Ω , then $S = N^{(4)}$ where the latter is the fourth derived subgroup of N .

Proof. (iv) Note that $N = \bigcap_i \mathbb{N}_W(M_i)$, so $N \supseteq S$ and $N^{(4)} \supseteq S^{(4)} = S$. Furthermore, since $\mathbb{C}_W(S) = \langle 1 \rangle$, it follows that N embeds in $\prod_i \text{Aut}(M_i)$. But under this embedding, S corresponds to $\prod_i \text{Inn}(M_i)$, so N/S embeds in $\prod_i \text{Out}(M_i)$. Finally, the precise version of the Schreier conjecture (see [G]), using the Classification of Finite Simple Groups, implies that each outer automorphism group $\text{Out}(M_i)$ is solvable of derived length ≤ 4 , and hence $N^{(4)} \subseteq S$, as required. \square

If W is an arbitrary finite group, we let $\text{sol } W$ denote the unique largest normal solvable subgroup of W . Then $\bar{W} = W/\text{sol } W$ has no nonidentity solvable normal subgroup, so the above lemma applies to this group. In particular, if we define $\text{rad } W \supseteq \text{sol } W$ by $\text{rad } W/\text{sol } W = \text{soc } \bar{W}$, then $\text{rad } W$ is *solvable-by-semisimple* and W permutes the set $\Omega(W)$ of simple factors of $\text{rad } W/\text{sol } W$ by conjugation. For convenience, we call $|\Omega(W)|$ the *width* of W .

Now let us turn to the proof of Theorem 10.1. In view of Lemma 9.1, it suffices to assume that $G = H^G$ for some finite subgroup H of G . Furthermore, we may suppose that G is countably infinite. In particular, we can write $G = \bigcup_{i=1}^{\infty} G_i$ where the G_i are finite subgroups of G satisfying $H \subseteq G_1 \subseteq G_2 \subseteq \dots$. Now, as we indicated above, each G_i acts as permutations on the set $\Omega_i = \Omega(G_i)$ of simple factors of $\text{rad } G_i/\text{sol } G_i$. Indeed, if N_i is the kernel of this action, then Lemma 10.3(iv) implies that $N_i^{(4)}$ is a normal subgroup of $\text{rad } G_i$ and hence it is solvable-by-semisimple. Furthermore, if we choose the ultrafilter \mathcal{F} on \mathcal{N} to contain the cofinite subsets, then G acts as permutations on the ultraproduct $\Omega = \prod_{\mathcal{F}} \Omega_i$ and Lemma 10.2 comes into play. If N denotes the kernel of G on Ω , then we study the structure of G by considering N and $\bar{G} = G/N \subseteq \text{Sym}_{\Omega}$ in turn.

To start with, Lemma 10.2(i) implies that there exists a subsequence \mathcal{M} of the natural numbers $\mathcal{N} = \{1, 2, \dots\}$ such that $L = N^{(4)}$ is the ascending union of its finite subgroups $L \cap N_i^{(4)}$ with $i \in \mathcal{M}$. Furthermore, note that $(L \cap N_i^{(4)}) \triangleleft N_i^{(4)}$ and that $N_i^{(4)}$ is solvable-by-semisimple. Thus $L \cap N_i^{(4)}$ is also solvable-by-semisimple, and $N^{(4)} = L = \bigcup_{i \in \mathcal{M}} (L \cap N_i^{(4)})$ is *locally solvable-by-semisimple*. There are now two cases to consider according to whether the widths which occur here are bounded or not. For the bounded case, we have

Lemma 10.4. *Let L be the ascending union of the finite subgroups $L_1 \subseteq L_2 \subseteq \dots$ and suppose that each L_i is solvable-by-semisimple. If the widths of the various subgroups L_i are uniformly bounded, then L has a finite subnormal series*

$$\langle 1 \rangle = M_0 \triangleleft M_1 \triangleleft \dots \triangleleft M_n = L$$

with each factor M_{i+1}/M_i either simple or locally solvable.

This follows easily by induction on the given upper bound for the widths. For example, if all L_i are solvable, which occurs when all widths are equal 0, then L is certainly locally solvable. On the other hand, if each L_i is a simple group, then clearly the same is true of L .

Using this lemma and Theorem 8.4, we can easily settle the semiprimivity problem for $N^{(4)} = L$ in the case of bounded widths. The unbounded case builds upon this, but also requires some techniques from the proof of Theorem 7.3 to construct a particular p -insulator.

Finally, consider $\bar{G} = G/N \subseteq \text{Sym}_{\Omega}$, and notice that $\bar{G} = \bar{H}^{\bar{G}}$. Again, there are two cases to deal with according to the nature of the action of H on the various

Ω_i . Suppose first that H moves a bounded number of points in each Ω_i . Then Lemma 10.2(ii) implies that $\bar{H} \subseteq \text{FSym}_\Omega$ and we conclude from Lemma 9.3 that \bar{G} has a finite subnormal series with factors which are either f.c. groups or isomorphic to FAlt_∞ . In particular, the result again follows from Theorem 8.4.

The last case, where H moves arbitrarily large numbers of points in its various actions, requires an entirely new approach based on the representation theory of finite wreath products. Nevertheless, it should be clear from the above remarks that Theorem 8.4 does indeed play a crucial role in this proof.

§11. THE CONJECTURE

Approximately 20 years after it was posed, Conjecture 5.6 was finally solved in the affirmative. Specifically, we have

Theorem 11.1. [P20] *If G is a locally finite group and K is a field of characteristic $p > 0$, then*

$$\mathcal{J}K[G] = \mathcal{J}K[\mathbb{T}^p(G)] \cdot K[G]$$

where $\mathbb{T}^p(G)/\mathbb{O}_p(G) = \mathbb{S}^p(G/\mathbb{O}_p(G))$ is the subgroup of $\bar{G} = G/\mathbb{O}_p(G)$ generated by those locally subnormal subgroups \bar{A} with $\bar{A} = \mathbb{O}^{p'}(\bar{A})$.

In particular, in view of Lemma 5.5(ii), this yields a precise description of $\mathcal{J}K[G]$. As usual, the proof of the above result requires that we work in the more general context of twisted group algebras. Obviously, Theorem 10.1 is needed here, and several new ideas also come into play. To start with, we mention another application of the subnormal closure in finite groups.

Let $K[G]$ be given, and recall that if $\alpha = \sum a_x x \in K[G]$, then the support of α is the finite subset of G given by $\text{supp } \alpha = \{x \in G \mid a_x \neq 0\}$. In addition, we call $H = \langle \text{supp } \alpha \rangle$ the *supporting subgroup* of α . Clearly H is the smallest subgroup of G with $\alpha \in K[H]$ and, since G is locally finite, H is finite. We say that $\beta \in K[G]$ is a *truncation* of α if $\beta = \sum' a_x x$, where \sum' indicates a partial sum of the terms of α . Thus $\text{supp } \beta \subseteq \text{supp } \alpha$, and the coefficients of α and of β agree on the smaller set. Of course, β is a *proper truncation* if $\beta \neq 0$ or α .

Note that, if D is any subgroup of G , then there is a natural $K[D]$ -bimodule projection map $\pi_D: K[G] \rightarrow K[D]$ given by

$$\pi_D: \sum_{x \in G} a_x x \mapsto \sum_{x \in D} a_x x.$$

Thus π_D is the linear extension of the map $G \rightarrow D \cup \{0\}$ which is the identity on D and zero on $G \setminus D$. Clearly, $\pi_D(\alpha)$ is a truncation of α .

Now let $I \triangleleft K[G]$. We say that $0 \neq \alpha$ is a *minimal element* of I if $\alpha \in I$ but no proper truncation of α is contained in I . It is easy to see that I is the linear span of its minimal elements, and that I is the right (or left) ideal generated by those minimal elements having 1 in their support.

Lemma 11.2. *Suppose $\mathcal{JK}[G] \neq 0$ and let α be a minimal element of this ideal having 1 in its support. Then there exists a finite subgroup H of G containing the supporting subgroup $\langle \text{supp } \alpha \rangle$ such that*

- i. $H = \langle \text{supp } \alpha \rangle^H$ and $H = \mathbb{O}^{p'}(H)$.
- ii. α is a minimal element of $\mathcal{JK}[H]$.
- iii. If N is any subgroup of G normalized by H and if $\mathcal{JK}[N] = \mathcal{NK}[N]$, then $H \subseteq \mathbb{D}_G(N)$.
- iv. If N is any subgroup of G normalized by H which satisfies both $\mathbb{O}_p(N) = \langle 1 \rangle$ and $\mathcal{JK}[N] = \mathcal{JK}[\mathbb{S}^p(N)] \cdot K[N]$, then $H \subseteq \mathbb{S}^p(NH)$ and $H \subseteq \mathbb{D}_G(F^*)$, where $F^* = \mathbb{F}^*(N)$.

Proof. Let $\beta_1, \beta_2, \dots, \beta_k$ be the finitely many proper truncations of α . By definition, no β_i is contained in $\mathcal{JK}[G]$, and hence the right ideals $\beta_i K[G]$ are not nil. In other words, we can choose elements $\gamma_i \in K[G]$ with $\beta_i \gamma_i$ not nilpotent. Now G is locally finite, so there exists a finite subgroup $L \subseteq G$ which contains $\langle \text{supp } \alpha \rangle$ and the supports of all γ_i . In particular, $\beta_i \notin \mathcal{JK}[L]$ since $\mathcal{JK}[L]$ is nilpotent.

Now let $H = \langle \text{supp } \alpha \rangle^{[L]}$ be the subnormal closure of $\langle \text{supp } \alpha \rangle$ in L . Then $H = \langle \text{supp } \alpha \rangle^H$ and $\alpha \in \mathcal{JK}[G] \cap K[H] \subseteq \mathcal{JK}[H]$. Furthermore, since $H \triangleleft\triangleleft L$, Lemma 5.3(ii) implies that $\mathcal{JK}[H] \subseteq \mathcal{JK}[L]$. Thus, since $\beta_i \notin \mathcal{JK}[L]$, we have $\beta_i \notin \mathcal{JK}[H]$ and therefore (ii) is proved. Note that $\mathcal{JK}[H] = \mathcal{JK}[\mathbb{O}^{p'}(H)] \cdot K[H]$ by Lemma 3.5, so $\pi_{\mathbb{O}^{p'}(H)}(\alpha) \in \mathcal{JK}[\mathbb{O}^{p'}(H)] \subseteq \mathcal{JK}[H]$. Moreover, $1 \in \text{supp } \alpha$, so $\pi_{\mathbb{O}^{p'}(H)}(\alpha)$ is a nonzero truncation of α contained in $\mathcal{JK}[H]$, and consequently $\pi_{\mathbb{O}^{p'}(H)}(\alpha)$ must equal α . In other words, $\langle \text{supp } \alpha \rangle \subseteq \mathbb{O}^{p'}(H) \triangleleft H$ and, since $H = \langle \text{supp } \alpha \rangle^H$, it follows that $H = \mathbb{O}^{p'}(H)$.

For part (iii), suppose that N is any subgroup of G normalized by H with $\mathcal{JK}[N] = \mathcal{NK}[N]$. If $X = NH$, then N is a normal subgroup of X of finite index, so Theorem 4.9 implies that $\mathcal{JK}[X] = \mathcal{NK}[X]$. In particular, by Theorem 3.4(i), we have $\mathcal{JK}[X] = \mathcal{JK}[D] \cdot K[X]$ where $D = \Delta(X)$. Now $\alpha \in \mathcal{JK}[G] \cap K[X] \subseteq \mathcal{JK}[X]$ and therefore $\pi_D(\alpha)$ is a nonzero truncation of α contained in $\mathcal{JK}[D] \subseteq \mathcal{JK}[X]$. Thus $\pi_D(\alpha) \in \mathcal{JK}[X] \cap K[H] \subseteq \mathcal{JK}[H]$, so the minimal nature of α implies that $\pi_D(\alpha) = \alpha$. In other words, $\langle \text{supp } \alpha \rangle \subseteq D \triangleleft X$ and therefore $H = \langle \text{supp } \alpha \rangle^H \subseteq D$. But $N \subseteq X$, so the definition of D implies that $|N : \mathbb{C}_N(h)| < \infty$ for all $h \in H$, and consequently $H \subseteq \mathbb{D}_G(N)$, as required.

Finally, suppose H normalizes a group N which satisfies both $\mathbb{O}_p(N) = \langle 1 \rangle$ and $\mathcal{JK}[N] = \mathcal{JK}[\mathbb{S}^p(N)] \cdot K[N]$. If $X = NH$ then Theorem 6.1(ii) easily implies that $\mathcal{JK}[X] = \mathcal{JK}[S] \cdot K[X]$ where $S = \mathbb{S}^p(X)$. Again, $\alpha \in \mathcal{JK}[G] \cap K[X] \subseteq \mathcal{JK}[X]$ and therefore $\pi_S(\alpha)$ is a nonzero truncation of α contained in $\mathcal{JK}[S] \subseteq \mathcal{JK}[X]$. Thus $\pi_S(\alpha) \in \mathcal{JK}[X] \cap K[H] \subseteq \mathcal{JK}[H]$, and the minimal nature of α implies that $\pi_S(\alpha) = \alpha$. In other words, $\langle \text{supp } \alpha \rangle \subseteq S \triangleleft X$, and consequently $H = \langle \text{supp } \alpha \rangle^H \subseteq S = \mathbb{S}^p(NH)$. Furthermore, since $N \triangleleft NH$, we have $F^* = \mathbb{F}^*(N) \subseteq \mathbb{F}^*(NH)$. Thus, since $\mathbb{O}_p(NH)$ is finite, Theorem 6.2(i) applied to the group $\mathbb{S}^p(NH)$ shows that

H acts in a finitary manner on $\mathbb{F}^*(NH)$ and hence also on F^* . \square

Now let us consider some aspects of the proof of Theorem 11.1. Of course, we can assume that $\mathbb{O}_p(G) = \langle 1 \rangle$, and the goal is to show that $\mathcal{JK}[G] \subseteq \mathcal{JK}[\mathbb{S}^p(G)] \cdot K[G]$. In view of our previous comments, it therefore suffices to show that all minimal elements of $\mathcal{JK}[G]$ having 1 in their support are contained in $\mathcal{JK}[\mathbb{S}^p(G)] \cdot K[G]$. To this end, let α be such an element and let $H \supseteq \langle \text{supp } \alpha \rangle$ be given by Lemma 11.2. Furthermore, let $L = H^{[G]}$ be the local subnormal closure of H in G . Note that $\alpha \in \mathcal{JK}[L]$ and, by Lemma 9.1(ii), it suffices to prove that $\mathcal{JK}[L] = \mathcal{JK}[\mathbb{S}^p(L)] \cdot K[L]$.

Let $V = \mathbb{F}^*(\mathbb{S}^p(L))$ and note that V is an f.c. group by Theorem 6.2(i) since $\mathbb{O}_p(L) \subseteq \mathbb{O}_p(G) = \langle 1 \rangle$. Indeed, since H normalizes V and $\mathcal{JK}[V] = \mathcal{NK}[V]$, Lemma 11.2(iii) implies that $H \subseteq \mathbb{D}_L(V) \triangleleft L$. But $L = H^L$, so $L = \mathbb{D}_L(V)$ and therefore $L \supseteq V$ acts in a strongly finitary manner on V . In particular, if we let $C = \mathbb{C}_L(V)$, then it follows from Lemma 9.6 that L has a finite subnormal series

$$C = L_0 \triangleleft L_1 \triangleleft \cdots \triangleleft L_m = L$$

with each quotient $\bar{L}_i = L_i/L_{i-1}$ either an infinite simple finitary linear group or an f.c. group.

Furthermore, since $\mathbb{S}^p(C) \subseteq \mathbb{S}^p(L)$ centralizes V , it follows easily that $\mathbb{S}^p(C)$ is contained in $\mathbb{F}(\mathbb{S}^p(L))$, the *Fitting subgroup* of $\mathbb{S}^p(L)$. But the latter group is a p' -group since $\mathbb{O}_p(L) = \langle 1 \rangle$, and certainly $\mathbb{S}^p(C)$ is generated by p -elements. Thus $\mathbb{S}^p(C) = \langle 1 \rangle$ and Theorem 10.1 implies that $0 = \mathcal{JK}[C] = \mathcal{JK}[\mathbb{S}^p(C)] \cdot K[C]$. In other words, we need only climb the chain $C = L_0 \triangleleft L_1 \triangleleft \cdots \triangleleft L_m = L$ and show that the condition $\mathcal{JK}[L_i] = \mathcal{JK}[\mathbb{S}^p(L_i)] \cdot K[L_i]$ lifts from L_{i-1} to L_i . Of course, $\mathbb{O}_p(L_i) = \langle 1 \rangle$ and therefore Theorem 6.1(ii) easily handles the case where \bar{L}_i is an f.c. group. Thus all that remains is to settle this particular extension problem when \bar{L}_i is infinite simple.

Let us completely change notation and just consider the latter extension problem. By applying Lemma 11.2(iv) and the local subnormal closure, we are quickly faced with the following group theoretic structure. For a fixed prime p , we say that (G, C, H) is a *critical triple* if

1. $C \triangleleft G$ and G/C is an infinite simple group.
2. $H = \mathbb{O}^{p'}(H)$ is a finite subgroup of G with $G = H^G$.
3. $H \subseteq \mathbb{S}^p(CH)$, $\mathbb{O}_p(G) = \langle 1 \rangle$ and $G = \mathbb{D}_G(\mathbb{F}(G))$.

Then, with a good deal of work, and by using Theorems 5.4 and 9.5, we obtain

Lemma 11.3. *If (G, C, H) is a critical triple for the prime p , then there exists a subgroup $\tilde{G} \triangleleft G$ having the following numerous properties. To start with, \tilde{G} has finite index in $G^{(6)}$, the 6th derived subgroup of G . Furthermore, if $\tilde{C} = C \cap \tilde{G}$, then $\tilde{G}/\tilde{C} \cong G/C$ is infinite simple and either (1) \tilde{C} is a nilpotent p' -group, or (2) $G/C = \text{FAlt}_{\mathcal{I}}$ for some infinite set \mathcal{I} , and G has normal subgroups $\tilde{D} \subseteq \tilde{X} \subseteq \tilde{L} \subseteq \tilde{C} \subseteq \tilde{G}$ satisfying*

- i. $\tilde{L} = \mathbb{O}^{p'}(\tilde{C})$, so that \tilde{C}/\tilde{L} is a p' -group.

- ii. \tilde{D} is a finite abelian p' -group which is central in $\mathbb{O}^{p'}(\tilde{G})$.
- iii. \tilde{L} is an f.c. group, and \tilde{L}/\tilde{X} is an abelian p -group.
- iv. There exist finite normal subgroups \tilde{X}_i of \tilde{C} , for all $i \in \mathcal{I}$, with $(\tilde{X}_i)^g = \tilde{X}_{ig}$ where ig is the image of $i \in \mathcal{I}$ under the permutation $Cg \in \text{FAlt}_{\mathcal{I}}$.
- v. $\tilde{D} \subseteq \tilde{X}_i \subseteq \tilde{X}$ and \tilde{X}/\tilde{D} is the (weak) direct product $\prod_{i \in \mathcal{I}} (\tilde{X}_i/\tilde{D})$.
- vi. $\tilde{X}_j/\tilde{D} \subseteq \mathbb{O}^{p'}(\mathbb{C}_{\tilde{L}/\tilde{D}}(\tilde{X}_i/\tilde{D}))$ for all distinct $i, j \in \mathcal{I}$.

This is unfortunately as far as the group theory goes. We must now deal directly with the groups G as described above, compute $\mathcal{JK}[G]$ and verify that the conclusion of Theorem 11.1 is satisfied here. To do this, it suffices to determine $\mathcal{JK}[\tilde{G}]$, since the extension from \tilde{G} to G is easy to handle. Now if case (1) holds and \tilde{C} is a nilpotent p' -group, then $\mathcal{JK}[\tilde{G}] = 0$ by Theorem 8.4 and the result follows quite simply. On the other hand, if case (2) holds, then the only option is to compute $\mathcal{JK}[\tilde{G}]$ by brute force. The ad hoc argument here is fairly long and painful. It does, however, use some interesting crossed product techniques along with the following lemma which allows Theorem 10.1 to again come into play.

Lemma 11.4. *Let Ω be an infinite set and let G be a subgroup of the finitary symmetric group FSym_{Ω} . If the stabilizer $G_{\Delta} = \{g \in G \mid \Delta g = \Delta\}$ of every finite subset $\Delta \subseteq \Omega$ has only infinite orbits on the complementary set $\Omega \setminus \Delta$, then G has no nonidentity locally subnormal subgroups. In particular, this applies when $G \supseteq \text{FAlt}_{\Omega_1} \times \text{FAlt}_{\Omega_2} \times \cdots \times \text{FAlt}_{\Omega_k}$, where $\Omega = \Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_k$ is a disjoint union of infinite sets.*

§12. BURNSIDE GROUPS

To proceed further, we must obviously return to the case of finitely generated groups. Indeed, the next candidates for study should most likely be the finitely generated p -groups, that is the groups associated with the Burnside problem. A natural question here is whether $\mathcal{JK}[G]$ can equal the augmentation ideal $\mathcal{AK}[G]$ of $K[G]$, namely the kernel of the natural epimorphism $K[G] \rightarrow K[G/G] = K$. If Conjecture 4.8 is to hold, then we must have

Conjecture 12.1. *Let G be a finitely generated p -group and let $\text{char } K = p > 0$. Then $\mathcal{JK}[G] = \mathcal{AK}[G]$ if and only if G is finite.*

This is easily seen to be equivalent to the assertion that $\mathcal{JK}[G] = \mathcal{AK}[G]$ if and only if G is a locally finite p -group with $p = \text{char } K$. So far, the only real evidence here of any generality is the following lovely argument of Lichtman.

Lemma 12.2. [L] *Let G be an infinite finitely generated p -group and let K be a field of characteristic $p > 0$. If $\mathcal{JK}[G] = \mathcal{AK}[G]$, then G has an infinite residually finite homomorphic image and, in particular, $G \neq G'$.*

Proof. Let H be the intersection of all normal subgroups of G of finite index. Then G/H is a residually finite homomorphic image of G , and the goal is to show that

this factor group is infinite. Suppose, by way of contradiction, that $|G : H| < \infty$. Since G is finitely generated, it follows that $H = \langle h_1, h_2, \dots, h_n \rangle$ is also finitely generated, and consequently $I = \mathcal{A}K[H] = \sum_{i=1}^n (1-h_i)K[H]$ is a finitely generated right ideal of $K[H]$ and hence a finitely generated right $K[H]$ -module. Furthermore, $I \neq 0$ since otherwise we would have $H = \langle 1 \rangle$ and $|G| < \infty$. Nakayama's lemma now implies that $I \cdot \mathcal{J}K[H]$ is properly contained in I .

By assumption, $\mathcal{J}K[G] = \mathcal{A}K[G]$, and consequently

$$\mathcal{J}K[H] \supseteq \mathcal{J}K[G] \cap K[H] = \mathcal{A}K[G] \cap K[H] = \mathcal{A}K[H].$$

It follows that $\mathcal{J}K[H] = \mathcal{A}K[H] = I$ and, by our previous remarks, I properly contains $I \cdot \mathcal{J}K[H] = I^2$. Now consider the homomorphism $\bar{\cdot} : K[H] \rightarrow K[H]/I^2$. Since $\overline{K[H]} = K + \bar{I}$ and $\bar{I}^2 = 0$, it is clear that this image is a commutative K -algebra properly larger than K . Thus since $\overline{K[H]}$ is spanned over K by \bar{H} , we see that \bar{H} is a nontrivial abelian homomorphic image of H and consequently $H \neq H'$. In other words, H/H' is a nonidentity finitely generated abelian p -group, so $1 < |H/H'| < \infty$ and H' is a normal subgroup of G of finite index properly contained in H . This, of course, contradicts the definition of H . \square

A slight generalization of the above argument shows that every maximal subgroup of G is normal of index p . Note that $\mathcal{J}K[G] = \mathcal{A}K[G]$ if and only if $K[G]$ has precisely one irreducible module, namely the *principal module*. Thus Conjecture 12.1 can be paraphrased as asserting that if G is an infinite finitely generated p -group, then $K[G]$ has a nonprincipal module. For example, if G is a Tarski monster of period p , as constructed in [O], then certainly $G = G'$ and the preceding lemma implies that a nonprincipal irreducible module exists in this case.

On the other hand, many of the remaining Burnside counterexamples are residually finite. One such is the Gupta-Sidki group which is described in [GuS] and [S1] as a certain subgroup of the automorphism group of a 1-rooted regular tree of degree p . For this group, we nevertheless have

Theorem 12.3. [S2] *Let G be a Gupta-Sidki p -group and let $\text{char } K = p$. Then $K[G]$ has a nonprincipal irreducible module.*

Actually, this result is stated in [S2] only for $p = 3$ and for $K = \text{GF}(3)$, but it does hold in the above generality with the same proof. Using this as a starting point, it was then shown in [PT] that $K[G]$ has infinitely many nonisomorphic irreducible modules when the field K is sufficiently large.

Finally, the Golod groups G are described in [Go] and [GoS] as finitely generated subgroups of the group of units of a Golod-Shafarevitch algebra $A = K \oplus N$, where N is an infinite dimensional nil ideal. Furthermore, $\bigcap_{k=0}^{\infty} N^k = 0$, so these groups are residually finite. Now there is a natural epimorphism $\tilde{\cdot} : K[G] \rightarrow A$ which maps $\mathcal{A}K[G]$ onto N . In particular, if $\tilde{\cdot}$ is an isomorphism, then $\mathcal{A}K[G]$ is nil and we have a counterexample to Conjecture 12.1. Fortunately, it was shown in [Si] that

\sim is not an isomorphism, at least when the construction parameters satisfy certain fairly natural conditions.

This is essentially all that is known about the semiprimitivity problem for Burnside groups. Obviously, much remains to be done.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN, MADISON, WISCONSIN 53706
E-mail address: passman@math.wisc.edu