GROUPS WITH CERTAIN NORMALITY CONDITIONS

CHIA-HSIN LIU AND D. S. PASSMAN

Abstract. We classify two types of finite groups with certain normality conditions, namely SSN groups and groups with all noncyclic subgroups normal. These conditions are key ingredients in the study of the multiplicative Jordan decomposition problem for group rings.

1. Introduction

In this paper, we classify two types of finite groups with certain normality conditions.

The first type is groups with SSN. Specifically, we say that a finite group G has property SN if for any subgroup Y of G and normal subgroup N of G, we have either $Y \supseteq N$ or $YN \triangleleft G$. Furthermore, we say that a finite group G has SSN if every subgroup of G has SN. This type of group originated in the study of the multiplicative Jordan decomposition property for integral group rings and was used extensively in [LP1], [LP2] and [L]. In particular, [LP1, Proposition 2.5] and [L, Lemma 1.2] prove that, for any finite group G, if the integral group ring $\mathbb{Z}[G]$ has the multiplicative Jordan decomposition property, then G has SSN.

The second type is groups with all noncyclic subgroups normal. For convenience, we say that these groups satisfy NCN. While groups with SSN is a fairly new concept, groups with NCN have been studied for many years. Indeed, [Lim] classified 2-groups with NCN in 1968 and, in a different context, [P1] classified finite p-groups with NCN, except for certain small 2-groups, in 1970. A complete classification of finite p-groups with NCN was given in [BJ]. More recently, [L] found an interesting relation between SSN groups and NCN groups. Namely, finite p-groups with SSN are groups with NCN. For this reason, we are interested in studying both SSN groups and NCN groups.

In the case of finite groups, [SSZ, Theorem 4.2] purported to classify nilpotent non-p-groups with NCN, but that result is clearly wrong. In addition, [ZGQX, Theorem 3.4] claimed to classify the non-nilpotent groups with NCN. That paper comes close, but unfortunately is a bit careless at a certain point and eventually states a wrong result.

In this paper, we classify solvable finite SSN groups in Section 2. For non-solvable finite SSN groups, first note that nonabelian simple groups certainly have SN. Also, it is straight forward to check that the alternating group Alt_5 of degree 5 is a group with SSN. So it is natural to expect that more non-solvable SSN groups can be found among the nonabelian simple groups. Surprisingly, it turns out that

---

2010 *Mathematics Subject Classification.* 20D15, 20F19.

Key words and phrases. Dedekind group, normality condition, SN group, SSN group, noncyclics normal, NCN group.

The first author's research was supported in part by an MOST grant.

Alt_5 is the unique nonabelian finite simple group with SSN, and indeed the only non-solvable finite group with SSN. This is proved in Section 3. In Section 4, we point out the errors in [SSZ] and give a correct classification of the nilpotent finite NCN groups. Finally, in Section 5, we point out the inaccuracies in [ZGQX] and offer a complete, detailed and somewhat different proof for the classification of non-nilpotent finite NCN groups.

2. Solvable Groups with SSN

In this section, we classify solvable finite groups satisfying SSN. Indeed Proposition 2.2, Proposition 2.3 and Theorem 2.7 together give this complete classification.

Throughout this paper, we use $\mathfrak{C}_G(H)$ for the centralizer of H in G, $\mathfrak{N}_G(H)$ for the normalizer of H in G, and $\mathfrak{Z}(G)$ for the center of the group G.

We start with some basic lemmas on groups with SN. Recall that G is said to be Dedekind if all its subgroups are normal. As is well known, such groups are either abelian or the direct product of the quaternion group of order 8, an elementary abelian 2-group, and an abelian group of odd order. In particular, if $G = A \times B$ is the direct product of groups of relatively prime order, then G is Dedekind if and only if both A and B are Dedekind.

Lemma 2.1. *Let G be a group with SN and let $1 \neq N \triangleleft G$.*

- *i. If N is not cyclic, then G/N is a Dedekind group.*
- *ii. If H is a subgroup of G disjoint from N, then $NH \triangleleft G$ and H is a Dedekind group. In particular, this holds if $|H|$ and $|N|$ are relatively prime.*
- *iii. If H is a subgroup of G generated by elements of prime order, then $NH \triangleleft G$.*

Proof. (i) Let J be a cyclic subgroup of G. The $J \not\supseteq N$ since N is not cyclic. Thus SN implies that $NJ \triangleleft G$. In particular, all cyclic subgroups of G/N are normal and G/N is a Dedekind group.

(ii) Let K be a subgroup of H. Then $H \cap N = 1$ implies that $K \not\supseteq N$. Thus $NK \triangleleft G$ and $K = NK \cap H \triangleleft H$.

(iii) Let J be a cyclic generator of H of prime order. If $J \not\supseteq N$, then $NJ \triangleleft G$ by SN. On the other hand, if $J \supseteq N$, then $|J|$ prime implies that $J = N$ and again $NJ = N \triangleleft G$. It follows that all generators of NH/N are normal in G/N and hence $NH/N \triangleleft G/N$. □

Now we consider finite p-groups with SSN.

Proposition 2.2. *Let G be a finite p-group for some prime p. Then G satisfies SSN if and only if G satisfies NCN.*

Proof. [L, Propositions 1.3 and 3.3] show that if G satisfies SSN, then all noncyclic subgroups of G are normal.

Conversely, suppose that all noncyclic subgroups of G are normal and let H be any subgroup of G. We want to show that if Y is a subgroup of H and $N \triangleleft H$, then either $Y \supseteq N$ or $YN \triangleleft H$. Indeed, if YN is noncyclic, then $YN \triangleleft G$ and hence $YN \triangleleft H$. On the other hand, suppose YN is cyclic. Since Y and N are subgroups of the finite cyclic p-group YN, we have either $Y \supseteq N$ or $Y \subseteq N$. The latter yields $YN = N \triangleleft H$ and the proof is complete. □

Note that finite p-groups with NCN have been studied in [Lim, P1] and [BJ]. A complete classification can be found in the latter paper.

Next, we consider finite nilpotent SSN groups that are not p-groups.

Proposition 2.3. *Let G be a finite nilpotent group that is not a p-group for any prime p. Then G satisfies SSN if and only if G is Dedekind.*

Proof. By assumption, we have $G = P \times H$ where P is a nontrivial Sylow p-subgroup of G, H is a nontrivial nilpotent subgroup of G and the orders of P and H are relatively prime.

Suppose that G has SSN. Since both P and H are normal in G, Lemma 2.1(ii) implies that both H and P are Dedekind. Since they have relatively prime orders, it follows that $G = P \times H$ is Dedekind.

Conversely, if G is Dedekind, then any subgroup H of G is Dedekind and hence satisfies SN. Therefore, G satisfies SSN. □

Next, we consider a special type of non-nilpotent group with SN.

Lemma 2.4. *Let $G = PQ$ be a finite group with SN. Here P is a normal p-subgroup of G for some prime p, and Q is a p'-subgroup that acts nontrivially on P.*

i. P is an elementary abelian p-group and Q acts irreducibly on P.
ii. If, in addition, Q does not act faithfully on P, then P is cyclic of order p, and Q is a cyclic q-group for some prime $q \neq p$. Furthermore, $|Q| \geq q^2$.

Proof. (i) Let $F = \Phi(P)$ be the Frattini subgroup of P. Then $F \operatorname{char} P$, so $F \triangleleft G$. If $F \neq 1$, then part (ii) of Lemma 2.1 implies that $FQ/F \triangleleft G/F$. Thus since $P/F \triangleleft G/F$, we see that the two disjoint normal subgroups FQ/F and P/F commute. In other words, Q centralizes the Frattini quotient P/F of P and therefore [P, Theorem 11.7] implies that Q centralizes P, a contradiction. Thus $F = 1$ and P is elementary abelian.

Now suppose that Q does not act irreducibly on P. Then, by Maschke's theorem (see [P, Theorem 15.1]), $P = X \times Y$ where X and Y are nonidentity Q-stable subgroups of P. Now $1 \neq X \triangleleft G$, so Lemma 2.1(ii) implies that $XQ \triangleleft G$. Hence the commutator $[Y, Q]$ satisfies $[Y, Q] \subseteq [Y, XQ] \subseteq Y \cap XQ = 1$, and Q centralizes Y. Similarly, Q centralizes X, and Q acts trivially on P, a contradiction.

(ii) Let $C = \mathfrak{C}_Q(P)$. Then $C \triangleleft Q$ and C centralizes P, so $C \triangleleft G$. Furthermore, $1 \neq C$ by assumption. Now let P_0 be a subgroup of P of order p. Then $P_0 \cap C = 1$, so $CP_0 \triangleleft G$, by Lemma 2.1(ii). Hence $P_0 = CP_0 \cap P \triangleleft G$ and P_0 is a Q-stable subgroup of P. Part (i) above now implies that $P = P_0$ is cyclic. In particular, since $\mathrm{Aut}(P)$ is cyclic, we see that Q/C is cyclic.

Now let J be any subgroup of Q. If J acts trivially on P, then $J \subseteq C$. On the other hand, if J acts nontrivially, then CJ cannot be normal in G and hence, by the SN condition, we have $J \supseteq C$. Of course, the latter inclusion must be proper, so $J > C$.

Since Q acts nontrivially on P, we can choose J above to be a cyclic q-group acting nontrivially on P for some prime q. Then $J > C$ and hence $1 \neq C$ is a cyclic q-group. Next, if J is a Sylow r-subgroup of Q with $r \neq q$, then we cannot have $J > C$. Thus $J \subseteq C$ and $J = 1$. It therefore follows that Q is a q-group.

Now let J be a subgroup of Q of order q. Then we cannot have $J > C$, so $J \subseteq C$ and J is the unique subgroup of order q of the cyclic group C. It therefore follows that the q-group Q has a unique subgroup of order q and hence, by [P, Corollary 9.8], either Q is cyclic, or $q = 2$ and Q is generalized quaternion. In the latter case, Q is generated by its subgroups of order 4, so there exists such a subgroup J that acts nontrivially on P. Then $J > C$ implies that $|C| = 2$ and $C = \mathfrak{Z}(Q)$. But Q/C

is cyclic, so Q is abelian, and this is a contradiction. Thus Q can only be cyclic. Since Q acts nontrivially on P, we have $Q > C > 1$ and $|Q| \geq q^2$. □

Now we use the stronger SSN property.

Lemma 2.5. *Let G be a finite group with SSN and suppose G has a nonidentity normal p-subgroup P_0. Let P be a Sylow p-subgroup of G so that P contains P_0.*

i. G has a nilpotent p-complement Q with $W = P_0Q \lhd G$. In particular, G is solvable and $G = PW = PQ$.

ii. At least one of P or Q is normal in G. Consequently, G has a normal Sylow r-subgroup $R \neq 1$ for some prime r.

Proof. (i) If $r \neq p$ is a prime and R is a Sylow r-subgroup of G, then since G has SN, Lemma 2.1(ii) implies that $P_0R \lhd G$. Thus $R \cong P_0R/P_0 \lhd G/P_0$, so G/P_0 has a normal Sylow subgroup for all primes different from p. It follows that G/P_0 has a normal nilpotent p-complement W/P_0. Since W is clearly solvable, we can choose Q to be a Hall p'-subgroup of W by [P, Theorem 11.1]. Thus $W = P_0Q$ and $Q \cong W/P_0$ is nilpotent. Furthermore, G/W is a p-group, so G is solvable and $G = PW = PQ$.

(ii) If Q acts trivially on P_0, then $W = P_0 \times Q$. Thus $Q \operatorname{char} W$ and $Q \lhd G$, as required. We can therefore assume that Q acts nontrivially on P_0, and since W has SN, we conclude that P_0 is elementary abelian by Lemma 2.4(i). If P_0 is not cyclic, then G/P_0 is a Dedekind group by Lemma 2.1(i), so $P/P_0 \lhd G/P_0$ and $P \lhd G$. Thus it suffices to assume that P_0 is cyclic and hence that $|P_0| = p$. This of course implies that $P_0 \subseteq \mathfrak{Z}(P)$.

If Q acts faithfully on P_0, then $\mathfrak{C}_G(P_0)$ contains no p'-elements and hence is a p-group. It follows in this case that $P = \mathfrak{C}_G(P_0) \lhd G$. On the other hand, if Q does not act faithfully, then since $W = P_0Q$ has SN, Lemma 2.4(ii) implies that Q is a cyclic q-group for some prime $q \neq p$. Furthermore, since $|\mathrm{Aut}(P_0)| = p - 1$, we see that $q \mid (p-1)$ and hence $q < p$. In particular, G is a $\{p, q\}$-group and Q is a cyclic Sylow q-subgroup with q the smallest prime dividing $|G|$. Burnside's theorem [P, Theorem 12.7] now implies that G has a normal q-complement, which must clearly be P.

It remains to show that G has a nonidentity Sylow subgroup that is normal. To this end, it suffices to assume that P is not normal in G. Then we must have $Q \neq 1$ and $Q \lhd G$. But Q is nilpotent, so any nonidentity Sylow subgroup R of Q is characteristic in Q and hence normal in G. Since Q is a Hall subgroup of G, we see that R is a Sylow subgroup of G. □

Lemma 2.6. *Let $G = PQ$ be a finite group with SSN. Here P is a normal p-subgroup of G for some prime p and Q is a nilpotent p'-subgroup that acts nontrivially on P. If Q acts faithfully on P, then P is elementary abelian, Q is cyclic and every nontrivial subgroup of Q acts irreducibly on P.*

Proof. By Lemma 2.4(i), P is elementary abelian. Since Q acts faithfully on P, we have $\mathfrak{C}_Q(P) = 1$. For any nontrivial subgroup K of Q, PK is an SN subgroup of G since G has SSN. Furthermore, K acts nontrivially on P since Q acts faithfully. By Lemma 2.4(i), K acts irreducibly on P.

Now we want to show that Q is cyclic. To this end, let $\mathrm{End}(P)$ be the endomorphism ring of the abelian group P. Then, for each $u \in Q$, we have an endomorphism θ_u of P given by $x \mapsto x^u$. Since Q is nilpotent, we can let $Z = \langle z \rangle$ be a nontrivial cyclic central subgroup of Q. If $E = \{f \in \mathrm{End}(P) \mid f\theta_z = \theta_z f\}$ is the centralizer of θ_z in $\mathrm{End}(P)$ and if $U = \{\theta_u \mid u \in Q\}$, then clearly $U \subseteq E$ since z is central in Q. Now Z acts irreducibly on P, so Schur's lemma implies that E is a division ring. But P is finite, so E is a finite division ring and hence a field. In particular, U is a finite subgroup of the multiplicative group of E, and therefore U is cyclic. Furthermore, $\mathfrak{C}_Q(P) = 1$ implies that the map $\theta : Q \to U$ given by $u \mapsto \theta_u$ is an isomorphism of groups. Thus Q is indeed cyclic. □

We can now classify solvable SSN groups that are not nilpotent.

Theorem 2.7. *Let G be a finite solvable group that is not nilpotent. Then G has SSN if and only if G is one of the following two types of groups.*

i. $G = P \rtimes Q$ where P is a normal elementary abelian p-subgroup of G for some prime p, Q is a cyclic p'-subgroup of G which acts faithfully on P and every nontrivial subgroup of Q acts irreducibly on P.

ii. $G = P \rtimes Q$ where P is a normal subgroup of G of order p for some prime p, Q is a cyclic q-group for some prime $q \neq p$ with $|Q| \geq q^2$ and Q does not act faithfully on P.

Proof. Suppose first that G has SSN. Since G is solvable, it has a nontrivial normal s-subgroup for some prime s. Thus, by Lemma 2.5(ii), G has a nontrivial normal Sylow p-subgroup P for some prime p. Next, by Lemma 2.5(i), G has a nilpotent p-complement Q and $G = PQ$. But G is not nilpotent, so Q acts nontrivially on P, and Lemma 2.4(i) implies that P is an elementary abelian p-group and Q acts irreducibly on P.

If Q acts faithfully on P, then by Lemma 2.6, Q is cyclic and every nontrivial subgroup of Q acts irreducibly on P. Thus G is a group of type (i).

On the other hand, if Q does not act faithfully on P, then by Lemma 2.4(ii), P is cyclic of order p and Q is a cyclic q-group for some prime $q \neq p$ with $|Q| \geq q^2$. Thus G is a group of type (ii).

Conversely, assume now that G is a group of type (i) or (ii). We want to show that G has SSN. First assume that G is a group of type (i). For any subgroup H of $G = P \rtimes Q$, we need to show that H has SN. Since G is solvable, it follows that H is solvable, so the basic results of P. Hall can apply. Now let $P_0 = P \cap H$. If $P_0 = 1$, then H is conjugate to a subgroup of Q by Hall's theorem. So H is abelian and hence has SN. Now assume $P_0 \neq 1$. Of course, $P_0 \triangleleft H$ since $P \triangleleft G$. By Hall's theorem, H has a Hall p'-subgroup K with $H = P_0 \rtimes K$. If $K = 1$, then $H = P_0 \subseteq P$, so H is abelian and hence has SN. If $K \neq 1$, then $1 \neq K^t \subseteq Q$ for some $t \in G$ by Hall's theorem, and hence K^t acts irreducibly on P. But K^t stabilizes P_0^t and $1 \neq P_0^t \subseteq P$, so we have $P_0^t = P$ and hence $P_0 = P^{t^{-1}} = P$. In other words, $H = PK$.

If $1 \neq N \triangleleft H$ and Y is a subgroup of H, we want to show that $Y \supseteq N$ or $YN \triangleleft H$. Since $N \triangleleft H$ and $P \triangleleft H$, we see that $N \cap P \triangleleft H$ and hence K acts on $N \cap P$ by conjugation. Thus K^t stabilizes $(N \cap P)^t \subseteq P^t = P$ and this implies that $(N \cap P)^t = 1$ or P, since K^t acts irreducibly on P. It follows that $N \cap P = 1$ or P.

If $N \cap P = 1$, then N is a p'-group and N centralizes P since both N and P are normal in H. By Hall's theorem, $N^u \subseteq Q$ for some $u \in G$. Moreover, N^u centralizes $P^u = P$. But Q acts faithfully on P, so $N^u = 1$ and hence $N = 1$, a contradiction. Thus $N \cap P = P$, so $NY \supseteq P$ and $NY \triangleleft H$ since H/P is abelian. We conclude that H has SN and hence G satisfies SSN.

Finally assume that G is a group of type (ii) and let H be any subgroup of G. If $1 \neq N \triangleleft H$ and Y is a subgroup of H, we want to show that either $Y \supseteq N$ or $YN \triangleleft H$. Since P is a group of order p, we see that $YN \cap P = 1$ or P. If $YN \cap P = P$, then $YN \supseteq P$ and $YN \triangleleft H$ since H/P is abelian. On the other hand, if $YN \cap P = 1$, then YN is a p'-subgroup of G, and Hall's theorem implies that $YN \subseteq Q^v$ for some $v \in G$. Since G is a group of type (ii), Q is a cyclic q-group and hence so is YN. It follows that for the two subgroups Y and N of YN, we have either $Y \supseteq N$ or $N \supseteq Y$. If the latter holds, then $YN = N \triangleleft H$. So H has SN and hence G satisfies SSN. □

3. Non-solvable Groups with SSN

In this section we classify the non-solvable finite groups satisfying SSN. We will first consider the case where G is nonabelian and simple. To this end, we need

Proposition 3.1. *Let G be a group with SSN and let P be a Sylow p-subgroup of G. If G does not have a normal p-complement, then P is elementary abelian and not central in its normalizer.*

Proof. Obviously, we can assume that $P \neq 1$. When p is odd, we use Thompson's theorem on normal p-complements. When $p = 2$, we are surprisingly able to use Frobenius' theorem. We first show that $\mathfrak{N}_G(P)$ contains a q-subgroup Q, with $q \neq p$, that acts nontrivially on P.

Case 1. $p > 2$.

Proof. Let $Z = \mathfrak{Z}(P)$ so that $Z \neq 1$. We first note that $C = \mathfrak{C}_G(Z)$ has a normal p-complement. To this end, let Q be a Sylow q-subgroup of C with $q \neq p$. Then, since C satisfies SN and $1 \neq Z \triangleleft C$, Lemma 2.1(ii) implies that $ZQ \triangleleft C$. But Q centralizes Z, so $ZQ = Z \times Q$. Thus $Q \operatorname{char} ZQ$ and hence $Q \triangleleft C$. Since this is true for all $q \neq p$, we conclude that C has a normal nilpotent p-complement.

Next, let $T = \mathfrak{T}(P)$ be the Thompson subgroup of P, so that $T \neq 1$ is the characteristic subgroup of P generated by all abelian subgroups of P of largest rank. Since G does not have a normal p-complement, $p > 2$ and $C = \mathfrak{C}_G(Z)$ has a normal p-complement, Thompson's theorem (see [P, Theorem 14.6]) implies that $N = \mathfrak{N}_G(T)$ does not have a normal p-complement. In particular, N must have a Sylow q-subgroup Q, with $q \neq p$, that is not normal. But N has SN, so we know from Lemma 2.1(ii) that $TQ \triangleleft N$. Thus Q cannot be characteristic in TQ, and hence Q acts nontrivially on T.

Suppose first that T is cyclic. Then the largest rank of an abelian subgroup of P is 1, and T is generated by all cyclic subgroups of P. Thus $T = P$ and Q acts nontrivially on P, as required. On the other hand, if T is not cyclic, then Lemma 2.1(i) implies that N/T is Dedekind. In particular, $P/T \triangleleft N/T$ and $P \triangleleft N$. Thus Q acts on P and this action is certainly nontrivial since Q is nontrivial on $T \subseteq P$. □

Case 2. $p = 2$.

Proof. Since G does not have a normal p-complement, Frobenius' theorem ([P, Theorem 13.3]) implies that there exists a subgroup P_0 of P and a q-subgroup $Q \subseteq \mathfrak{N}_G(P_0) = N$, with $q \neq p$, such that Q acts nontrivially on P_0. Among all such pairs (P_0, Q), choose P_0 to have maximal order. Observe that P_0 is not cyclic, since the automorphism group of a cyclic 2-group is a 2-group and Q could not exist. Thus P_0 is a noncyclic normal subgroup of N, a group with SN, and Lemma 2.1(i) now implies that N/P_0 is a Dedekind group. In particular, if $P_1 = \mathfrak{N}_P(P_0)$, then $P_1/P_0 \triangleleft N/P_0$ and $P_1 \triangleleft N$. But then Q acts on P_1, and Q acts nontrivially since it is nontrivial on $P_0 \subseteq P_1$. The maximality of $|P_0|$ now implies that $P_1 = P_0$. But normalizers grow in p-groups, so we must have $P = P_0$. □

We have shown in all cases that there exists a q-subgroup $Q \subseteq \mathfrak{N}_G(P)$, with $q \neq p$, such that Q acts nontrivially on P. Since the group PQ has SN, Lemma 2.4(i) implies that P is an elementary abelian p-group. □

With this in hand, we now show that a nonabelian simple group G with SSN can have only trivial cyclic subgroups.

Lemma 3.2. *Let G be a nonabelian simple group with SSN. Then all nonidentity cyclic subgroups have prime order. Furthermore, the centralizers of all nonidentity elements of G are abelian.*

Proof. Since G is nonabelian simple, if p divides $|G|$, then G does not have a normal p-complement. Thus, by Proposition 3.1, the Sylow p-subgroups of G are elementary abelian, and each is acted upon nontrivially by a p'-subgroup in its normalizer. In particular, G has no cyclic subgroups of order p^2. Thus it suffices to show that G has no cyclic subgroups of order pq with distinct primes p and q. In other words, we have to show that no element of prime order p can commute with an element of prime order $q \neq p$.

Let $P \neq 1$ be a Sylow p-subgroup of G. We first show that $\mathfrak{C}_G(P) = P$. To this end, let $N = \mathfrak{N}_G(P)$. Since P is a Sylow p-subgroup of N and since N satisfies condition SSN, Lemma 2.5(i) implies that $N = PQ$ where Q is a nilpotent p-complement. Of course the action of N on P is the same as the action of $Q \cong N/P$ on P. In particular, if $\mathfrak{C}_G(P) > P$, then Q does not act faithfully on P. But then, by Lemma 2.4(ii), Q is a cyclic q-group of order $\geq q^2$, a contradiction.

Now we complete the proof by showing that if $1 \neq P_0$ is a subgroup of P, then $\mathfrak{C}_G(P_0) = P$. Of course, $C = \mathfrak{C}_G(P_0) \supseteq P$ and if it is properly larger than P, then C contains a nonidentity Sylow r-subgroup R for some prime $r \neq p$. Again $P_0 \triangleleft C$, so Lemma 2.1(ii) implies that $P_0R = P_0 \times R \triangleleft C$ and hence $R \triangleleft C$. Also, $P_0 \triangleleft C$ and P is generated by elements of prime order, so Lemma 2.1(iii) implies that $P = P_0P \triangleleft C$. Thus R and P are disjoint normal subgroups of C, so they centralize each other, contradicting the result of the previous paragraph. We conclude that $\mathfrak{C}_G(P_0) = P$, as required. □

It is now a simple matter, using relatively classical results, to obtain

Theorem 3.3. Alt_5*, the alternating group of degree* 5 *and order* 60*, is the unique nonabelian simple group with SSN.*

Proof. Suppose first that G is a nonabelian simple group with SSN. Then by the preceding lemma, all nonidentity cyclic subgroups of G have prime order and G is a CA-group, namely the centralizers of all nonidentity elements of G are abelian. It now follows from a result of Suzuki [Suz] that G has even order. With this, a result of Burnside [B] (or see [G]) implies that $G \cong \mathrm{SL}(2, 2^n)$ with $n > 1$. As is well known, $\mathrm{SL}(2, 2^n)$ has cyclic subgroups of order $2^n - 1$ and $2^n + 1$. Of course, one of these two orders must be divisible by 3 and hence equal to 3. If $2^n+1=3$, then $n=1$ which is not the case. Thus $2^n-1=3$, $n=2$ and $G \cong \mathrm{SL}(2,4) \cong \mathrm{Alt}_5$.

Conversely, let $G \cong \mathrm{Alt}_5$. Since G is simple, it trivially satisfies SN. Furthermore, all abelian subgroups of G satisfy SN. Finally, the proper nonabelian subgroups H of G are isomorphic to Sym_3, Alt_4 or D_{10}, the dihedral group of order 10. Each of the latter groups has a unique proper normal subgroup N, and H/N is abelian. Thus each such H satisfies SN, and therefore $G \cong \mathrm{Alt}_5$ satisfies SSN. □

As a consequence, we settle the case of non-solvable groups with SSN. Indeed, we prove

Corollary 3.4. Alt_5 *is the unique non-solvable finite group with SSN.*

Proof. Let G be a non-solvable group with SSN. If $N \neq 1$ is a minimal normal subgroup of G, then N cannot be a p-group by Lemma 2.5(i). Thus [P, Proposition 6.4] implies that N is semisimple, namely a direct product of nonabelian simple groups. Furthermore, if S is the socle of G, namely the subgroup of G generated by all minimal normal subgroups, then S is a direct product of some of these N, and hence S is also semisimple. Suppose S has at least two direct factors and write $S = A \times B$, where A and B are nonidentity semisimple groups. Since S has property SN and $A \triangleleft S$, Lemma 2.1(ii) implies that B is a Dedekind group, certainly a contradiction. Thus S has just one direct factor, so S is nonabelian simple.

Now S inherits property SSN, so the previous theorem implies that $S \cong \mathrm{Alt}_5$. Furthermore, since $S \cap \mathfrak{C}_G(S) = \mathfrak{Z}(S) = 1$ and S is the socle of G, we see that $\mathfrak{C}_G(S) = 1$. Thus G acts faithfully on S and hence G is isomorphic to a subgroup of $\mathrm{Aut}(S)$ containing the group of inner automorphisms, which of course is normal and isomorphic to S. In our case, we have $\mathrm{Aut}(\mathrm{Alt}_5) \cong \mathrm{Sym}_5$ and Alt_5 is maximal in this group. Thus $G \cong \mathrm{Alt}_5$ or $G \cong \mathrm{Sym}_5$. But $\mathrm{Sym}_5 \supseteq \mathrm{Sym}_4$ and Sym_4 does not have property SN. Indeed, if K is the Klein fours subgroup of Sym_4, then $\mathrm{Sym}_4/K \cong \mathrm{Sym}_3$ is not a Dedekind group and this violates Lemma 2.1(i). This leaves only $G \cong \mathrm{Alt}_5$ which we know has SSN. □

4. Nilpotent groups with NCN

Now we discuss nilpotent finite groups with NCN. Note that these groups were studied in [SSZ]. Indeed, [SSZ, Theorem 4.2] claims that if G is a finite nilpotent group, then G has NCN if and only if every Sylow subgroup of G has NCN. This is obviously false. To construct a counterexample, let $G = A \times B$ where $A \cong D_8$ and $B \cong C_3 \times C_3$. Thus every Sylow subgroup of G has NCN. On the other hand, if H is any nonnormal subgroup of A, then the subgroup HB is a noncyclic nonnormal subgroup of G and hence G does not satisfy NCN.

In this section, we quickly offer a correct classification of the nilpotent finite NCN groups. Let us start with some basic properties of NCN groups.

Lemma 4.1. *Let G be a group with NCN.*

i. If H is a subgroup of G, then H has NCN.
ii. If N is a normal subgroup of G, then G/N has NCN.

Proof. (i) Any noncyclic subgroup Y of H is a noncyclic subgroup of G. So $Y \triangleleft G$ and hence $Y \triangleleft H$.

(ii) If Y/N is any noncyclic subgroup of G/N, where Y is a subgroup of G containing N, then Y is noncyclic, so $Y \triangleleft G$ and hence $Y/N \triangleleft G/N$. □

Lemma 4.2. *Suppose that G is a group with NCN, and let H and A be subgroups of this group.*

i. If H is noncyclic, then $H \lhd G$ and the quotient group G/H is Dedekind.
ii. If H is noncyclic and $A \cap H = 1$, then A is Dedekind.

Proof. (i) Since H is noncyclic and G has NCN, we see that H is normal in G. Furthermore, any subgroup of G/H has the form Y/H for some subgroup Y of G containing H. Since H is noncyclic, Y is noncyclic and hence normal in G. Thus Y/H is normal in G/H.

(ii) Since G/H is Dedekind, it follows that HA/H is Dedekind. Thus A is Dedekind since $A \cong A/(A \cap H) \cong HA/H$. □

Obviously, finite Dedekind groups are nilpotent NCN groups. So we need only consider finite nilpotent groups that are not Dedekind.

Theorem 4.3. *Let G be a finite nilpotent group such that G is not Dedekind. Then G has NCN if and only if $G = P \times A$ where P is an NCN p-group for some prime p, P is not Dedekind, and A is a cyclic group of order prime to p.*

Proof. Suppose that G has NCN. Since G is nilpotent and not Dedekind, at least one of the Sylow p-subgroups of G, say P, is not Dedekind. Moreover, $G = P \times A$ for some normal subgroup A of G and $\gcd(p, |A|) = 1$. If A is not cyclic, then $P \cap A = 1$ and Lemma 4.2(ii) imply that P is Dedekind, a contradiction. Thus A is cyclic and $G = P \times A$ has the desired form since P inherits the NCN property from G.

Conversely, suppose that $G = P \times A$ where P is an NCN p-group, P is not Dedekind, and A is cyclic of order prime to p. We want to show that G has NCN. To this end, let Y be any noncyclic subgroup of G. Since G is nilpotent, so is Y, and hence $Y = (Y \cap P) \times (Y \cap A)$.

Note that $Y \cap A$ is a subgroup of A and hence it is cyclic. On the other hand, if $Y \cap P$ is also cyclic, then $Y = (Y \cap P) \times (Y \cap A)$ is cyclic since $Y \cap P$ and $Y \cap A$ have coprime orders, a contradiction. Thus, $Y \cap P$ is not cyclic, and since P has NCN, we see that $Y \cap P$ is normal in P and hence normal in $G = P \times A$. On the other hand, $Y \cap A$ is central and hence normal in G. We conclude that $Y = (Y \cap P) \times (Y \cap A)$ is normal in G, and therefore G has NCN, as required. □

Note that Theorem 4.3 reduces the study of finite nilpotent NCN groups to finite NCN p-groups. However, as we observed before, finite NCN p-groups were previously classified, for example in [BJ]. Therefore, finite nilpotent NCN groups are now well understood.

5. Non-nilpotent groups with NCN

In this section, we discuss non-nilpotent finite groups with NCN. As mentioned earlier, these groups were studied in [ZGQX], but their classification is not quite correct. The error is pointed out in the paragraphs before Proposition 5.5, which actually corrects the work of [ZGQX]. A complete classification of non-nilpotent finite groups with NCN is given in Theorem 5.7. We provide a self-contained detailed proof to avoid any inaccuracies.

We start by considering Sylow subgroups of NCN groups.

Lemma 5.1. *Let G be a finite NCN group and let p and q be distinct primes dividing $|G|$. If there exists a Sylow p-subgroup P and a Sylow q-subgroup Q of G that are both noncyclic, then G is nilpotent.*

Proof. We want to show that for every prime r, a Sylow r-subgroup of G is normal in G. If $r = p$ or q, then P and Q are normal in G since they are both noncyclic and G has NCN. If $r \neq p$ and $r \neq q$, let R be a Sylow r-subgroup of G. Since P is normal in G and noncyclic, PR is a noncyclic subgroup and hence normal in G. Similarly, QR is normal in G. It follows that $R = PR \cap QR$ is normal in G. Therefore, all Sylow subgroups of G are normal and G is nilpotent. □

Lemma 5.2. *Let G be a finite NCN group that is not nilpotent. Then G has a nontrivial normal Sylow p-subgroup P for some prime p, such that G/P is Dedekind. In particular, G is solvable.*

Proof. If G has a noncyclic Sylow p-subgroup P, then the result follows from Lemma 4.2(i).

Thus we can assume that all Sylow subgroups of G are cyclic, that is G is a Z-group. Then [P, Proposition 12.11] implies that $G = U \rtimes X$, where U and X are cyclic groups of coprime order and $G' = U$. Since G is not nilpotent, $G' = U$ cannot be central. Thus some nontrivial Sylow p-subgroup P of U is not central in G. Note that P is characteristic in cyclic U, so $P \triangleleft G$. Also since $\gcd(|U|, |X|) = 1$, it follows that P is a Sylow p-subgroup of G. Since P is not central and U is cyclic, it follows that PX is a nonabelian subgroup of G. Hence PX is normal in the NCN group G, and G/PX is cyclic since U is cyclic. Thus $U = G' \subseteq PX \cap U = P$, so we conclude that $U = P$ is a nontrivial Sylow p-subgroup of G. Since G/U is cyclic, the result follows. □

The following lemma is [ZGQX, Lemma 3.3]. We provide a proof for the sake of completeness. As usual, $\Phi(P)$ denotes the Frattini subgroup of P and C_n denotes the cyclic group of order n. Furthermore, if A and B are subgroups of G, then $[A, B]$ denotes their commutator.

Lemma 5.3. *Let G be a finite NCN group that is not nilpotent. Then $G = P \rtimes X$, where P is a nontrivial Sylow p-subgroup of G for some prime p, and X is a cyclic p'-subgroup of G. Furthermore, we have*

i. X acts nontrivially on P by conjugation.
ii. $\Phi(P)X$ is a cyclic subgroup of G.
iii. $P/\Phi(P)$ is elementary abelian of rank ≤ 2.

Proof. Since G is a finite NCN group that is not nilpotent, Lemma 5.2 implies that G has a nontrivial normal Sylow p-subgroup P for some prime p with G/P Dedekind. Furthermore, G is solvable so we can let X be a Hall p'-subgroup of G. Then $G = P \rtimes X$ and $X \cong G/P$ is Dedekind and hence nilpotent.

If X acts trivially on P by conjugation, then $G = P \times X$, and G is nilpotent, a contradiction. Thus X acts nontrivially on P and (i) is proved.

Since $F = \Phi(P)$ is a characteristic subgroup of P, we see that F is normal in G and hence stable under the action of X. Suppose by way of contradiction that FX is not cyclic. Then FX is normal in the NCN group G. Moreover P is normal in G, so we have $[P, X] \subseteq [P, FX] \subseteq P \cap FX = F$, and X acts trivially on P/F. By [P, Theorem 11.7], X acts trivially on P and this is a contradiction. So FX is cyclic and hence X is also cyclic. Thus (ii) follows.

Finally, if P/F is elementary abelian of rank ≥ 3, then we can find normal subgroups P_1, P_2, P_3 of P containing F such that $P_1 \cap P_2 \cap P_3 = F$ and $P_i/F \cong C_p \times C_p$ for $i = 1, 2, 3$. Obviously, each P_i is noncyclic and hence normal in G. Furthermore, P_iX is also noncyclic and hence normal in G. Thus $[P, X] \subseteq [P, P_iX] \subseteq P \cap P_iX = P_i$ for each $i = 1, 2, 3$. So $[P, X] \subseteq P_1 \cap P_2 \cap P_3 = F$ and X centralizes P/F, again a contradiction by [P, Theorem 11.7]. Therefore, P/F has rank ≤ 2. □

We record one more lemma of a similar nature that will be needed later.

Lemma 5.4. *Let $G = P \rtimes X$, where P is a noncyclic Sylow p-subgroup and X is a cyclic p'-subgroup acting nontrivially on P by conjugation. If the group G has NCN, then*

i. P has at most one X-stable maximal subgroup M such that MX is noncyclic.
ii. P has at most one noncyclic maximal subgroup.

Proof. Since P is noncyclic, Lemma 5.3(iii) implies that $P/\Phi(P)$ has rank 2. If M_1, M_2 are two distinct X-stable maximal subgroups of P such that M_1X and M_2X are noncyclic, then $M_1 \cap M_2 = \Phi(P)$, $M_1X \lhd G$, $M_2X \lhd G$ and we have $[P, X] \subseteq [P, M_iX] \subseteq P \cap M_iX = M_i$ for $i = 1, 2$. It follows that $[P, X] \subseteq M_1 \cap M_2 = \Phi(P)$ and X acts trivially on $P/\Phi(P)$, a contradiction by [P, Theorem 11.7]. This proves (i).

Since G has NCN, any noncyclic maximal subgroup M of P is X-stable, and clearly MX is also a noncyclic subgroup of G. Thus (ii) follows from (i). □

Now we consider the case where $G = P \rtimes X$ is a finite group, with $P \cong C_p \times C_p$ for some prime p, and with X a cyclic p'-subgroup acting nontrivially on P. In [ZGQX, Theorem 3.4] it is claimed that this group is an NCN group if X acts irreducibly on P. However, this condition is not strong enough. To construct a counterexample, let p be an odd prime and let $\mathbb{F}$ be the finite field $\mathbb{F} = \mathrm{GF}(p^2)$. Then the multiplicative group $X = \mathbb{F}^\bullet$ is a cyclic p'-group that acts as automorphisms on the additive group $P = \mathbb{F}^+ \cong C_p \times C_p$ by ordinary field multiplication. Since X is transitive on the nonidentity elements of P, it follows that X acts irreducibly on P. On the other hand, $X \supseteq D = \langle -1 \rangle$, a group of order 2 that acts on P by taking inverses. In particular, if P_1 is any subgroup of P of order p, then the dihedral subgroup $H = P_1 \rtimes D$ is not cyclic and not normal in $G = P \rtimes X$. Thus G does not have NCN.

From this counterexample, we see that assuming X acts irreducibly on P is not sufficient. D is a subgroup of X acting nontrivially but reducibly on P, and this situation must be excluded for G to satisfy NCN. Indeed, we have the following

Proposition 5.5. *Let $G = P \rtimes X$ be a finite group, where p is a prime, $P \cong C_p \times C_p$, and X is a cyclic p'-group acting nontrivially on P. Then the following are equivalent.*

i. G has NCN and P is minimal normal in G.
ii. If S is a subgroup of X and S acts nontrivially on P, then S acts irreducibly on P.
iii. $\gcd(|G : \mathfrak{C}_G(P)|, p - 1) = 1$.

Proof. For (i) ⇒ (ii), let S be any subgroup of X acting nontrivially on P. If P_1 is any nontrivial S-stable subgroup of P, we want to show that $P_1 = P$. To this end, suppose that $P_1 \neq P$ so that $|P_1| = p$. Since S has order prime to p, Maschke's theorem [P, Theorem 15.1] implies that $P = P_1 \times P_2$ for some S-stable subgroup P_2 of P with $|P_1| = |P_2| = p$. If both P_1S and P_2S are cyclic, then S acts trivially on $P_1P_2 = P$, a contradiction. Thus at least one of these two groups, say P_iS, is noncyclic. Then $P_iS \triangleleft G$ since G has NCN. But P_i is the unique Sylow p-subgroup of P_iS, so P_i is characteristic in P_iS and hence normal in G, contradicting the assumption that P is minimal normal in G. Thus $P_1 = P$ and S acts irreducibly on P.

For (ii) $\Rightarrow$ (iii), first note that since $P \subseteq \mathfrak{C}_G(P)$, so $G = \mathfrak{C}_G(P)X$ and $|X : \mathfrak{C}_X(P)| = |G : \mathfrak{C}_G(P)|$. Let q be any prime dividing $|X : \mathfrak{C}_X(P)|$. We want to show that $q \neq 2$ and q divides $p+1$.

Since q divides $|X : \mathfrak{C}_X(P)|$, there exists $s \in X \setminus \mathfrak{C}_X(P)$ such that $s^q \in \mathfrak{C}_X(P)$. Thus $S = \langle s \rangle$ acts nontrivially on P like a group of order q, and hence by assumption S acts irreducibly on P. If $q = 2$, take any $a \in P$ of order p not fixed by s. Then $1 \neq c = a^{-1}a^s \in P$ and, since $a^{s^2} = a$, it follows easily that $c^s = c^{-1}$. Thus $\langle c \rangle$ is S-stable, contradicting the fact that S acts irreducibly on P. Therefore $q \neq 2$.

To show that q divides $p+1$, let Ω be the set of all subgroups of order p in P. Then S permutes this set and, since S acts like a group of order q on P, it follows that all S-orbits have size 1 or q. If there is an orbit of size 1, then S stabilizes a subgroup of P of order p, a contradiction. Thus all orbits have size q, and q divides $|\Omega| = p+1$. It follows that q does not divide $p-1$ since $q \neq 2$, and therefore $|G : \mathfrak{C}_G(P)|$ is relatively prime to $p-1$.

For (iii) $\Rightarrow$ (i), we first claim that, for any subgroup B of order p in P, we have $\mathfrak{N}_G(B) = \mathfrak{C}_G(B)$. Indeed, since $\mathfrak{C}_G(P) \subseteq \mathfrak{C}_G(B) \subseteq \mathfrak{N}_G(B) \subseteq G$, we see that $|\mathfrak{N}_G(B) : \mathfrak{C}_G(B)|$ divides $|G : \mathfrak{C}_G(P)|$. Moreover, $|\mathfrak{N}_G(B) : \mathfrak{C}_G(B)|$ divides $|\mathrm{Aut}(B)| = p-1$ since B has order p. But, by assumption, $|G : \mathfrak{C}_G(P)|$ is relatively prime to $p-1$. Hence $|\mathfrak{N}_G(B) : \mathfrak{C}_G(B)| = 1$ and the claim is proved.

Now we show that G has NCN. Let Y be any noncyclic subgroup of G. Since $G/P \cong X$ is cyclic, we see that $B = Y \cap P \neq 1$. Note that B is a normal Sylow p-subgroup of Y since P is a normal Sylow p-subgroup of G. Furthermore, G is solvable, so we have $Y = B \rtimes K$ for some p'-subgroup K that is conjugate to a subgroup of X. Suppose by way of contradiction that $B \neq P$. Then B has order p and hence $\mathfrak{N}_G(B) = \mathfrak{C}_G(B)$ by the above claim. Since $K \subseteq \mathfrak{N}_G(B)$, it follows that $[B, K] = 1$ and Y is cyclic, a contradiction. Thus $B = P$, $Y \supseteq B \supseteq G'$ and we have $Y \triangleleft G$.

Finally, we prove that P is minimal normal in G. Indeed, if this is not the case, let N_1 be a subgroup of P of order p such that N_1 is normal in G. Since X is a p'-group, Maschke's theorem [P, Theorem 15.1] implies that $P = N_1 \times N_2$ for some X-stable subgroup N_2 of order p in P. Note that N_1 and N_2 are normal in G, so the above claim implies that $\mathfrak{C}_G(N_1) = \mathfrak{N}_G(N_1) = G$ and $\mathfrak{C}_G(N_2) = \mathfrak{N}_G(N_2) = G$. It follows that $[P, X] = [N_1N_2, X] = 1$, and this is the required contradiction. $\square$

The following proposition is proved in [ZGQX, Theorem 3.4] based on a result of Rèdei characterizing minimal nonabelian p-groups. Here we offer a different proof which avoids this fact.

Proposition 5.6. *Let $G = P \rtimes X$ be a finite group such that P is a p-group, for some prime p, and X is a cyclic p'-group acting nontrivially on P. If G has NCN and P is nonabelian, then $p = 2$ and $P \cong Q_8$, the quaternion group of order 8.*

Proof. Since P is nonabelian, $\Phi(P) \neq 1$. By Lemma 5.3(ii), X is cyclic and $\Phi(P)$ is cyclic with $[\Phi(P), X] = 1$. Note that $\Phi(P)$ has a unique subgroup Z of order p, so Z is characteristic in $\Phi(P)$ and hence normal in G. Suppose, by way of contradiction, that P has a subgroup U of order p with $U \neq Z$. Then the subgroup $T = ZU \cong C_p \times C_p$ is noncyclic, and since G has NCN, we have $T \triangleleft G$ and X acts on T by conjugation. Since Z is X-stable, Maschke's theorem implies that $T = Z \times W$ for some X-stable subgroup W of T of order p. Note that $\Phi(P) \cap W = 1$.

We claim that $[W, X] = 1$. If $p = 2$, then W has order 2 and $\mathrm{Aut}(W) = 1$, so X acts trivially on W, as required. If p is odd, note that WX is a subgroup since W is X-stable. If WX is noncyclic, then $WX \triangleleft G$ since G has NCN, and G/WX is Dedekind, by Lemma 4.2. Indeed, G/WX is abelian since G/WX is a p-group and p is odd. Thus $G' \subseteq WX$ and $P' \subseteq \Phi(P) \cap WX = 1$. In other words, P is abelian, contradicting the hypothesis. It follows that WX is cyclic and $[W, X] = 1$ in all cases.

Since P is nonabelian, $P/\Phi(P)$ has rank 2 by Lemma 5.3(iii). Now let $M_1 = \Phi(P)W$. Then $M_1 \supseteq T$ is a noncyclic maximal subgroup of P with $[M_1, X] = 1$. Since X acts on P by conjugation, this induces an action of X on $P/\Phi(P) \cong C_p \times C_p$ and $M_1/\Phi(P)$ is X-stable. By Maschke's theorem there is an X-stable maximal subgroup M_2 of P such that $P/\Phi(P) = M_1/\Phi(P) \times M_2/\Phi(P)$. Since M_1 is noncyclic, M_1X is also noncyclic, and hence M_2X must be cyclic by Lemma 5.4(i). Thus X centralizes both M_1 and M_2, so X centralizes $P = M_1M_2$, a contradiction.

It follows that P has a unique subgroup of order p, and since P is not cyclic, it must be a generalized quaternion group Q_{2^n} for some $n \geq 3$. Note that Q_{2^n} has two noncyclic maximal subgroups if $n > 3$, and this contradicts Lemma 5.4(ii). Thus $n = 3$, $p = 2$ and $P \cong Q_8$. □

Finally, we prove the main theorem of this section.

Theorem 5.7. *Let G be a finite group that is not nilpotent. Then G has NCN if and only if $G = P \rtimes X$ where P is a p-group for some prime p, X is a cyclic group acting nontrivially on P, and one of the following holds.*

i. $P \cong C_p$ *and* $X \cong C_n$.

ii. $P \cong C_p \times C_p$, $X \cong C_n$, *with* $\gcd(|G : \mathfrak{C}_G(P)|, p-1) = 1$ *and with n prime to p.*

iii. $P \cong Q_8$, $X \cong C_n$, *with n odd.*

Proof. First we show that groups of type (i), (ii) or (iii) have the NCN property. For groups of type (i), let Y be any noncyclic subgroup of G. Then $Y \cap P \neq 1$ since $G/P \cong X$ is cyclic. It follows that $Y \supseteq P \supseteq G'$ and hence $Y \triangleleft G$. So G has NCN. Next, Proposition 5.5 shows that groups of type (ii) satisfy NCN. Finally, for groups of type (iii), let Y be any noncyclic subgroup of G. Then $B = Y \cap P \neq 1$ since G/P is cyclic, and B is a normal Sylow 2-subgroup of Y since P is a normal Sylow 2-subgroup of G. Now G is solvable, so $Y = B \rtimes K$ for some Hall $2'$-subgroup K of Y. Furthermore, K is conjugate to a subgroup of X, so K is cyclic. Suppose, by way of contradiction, that $B \neq P \cong Q_8$. Then $B \cong C_2$ or C_4 and $|\mathrm{Aut}(B)| = 1$ or 2. Thus K centralizes B and $Y \cong B \times K$ is cyclic, a contradiction. Therefore, $B = P$ and $Y \supseteq B = P \supseteq G'$. It follows that $Y \triangleleft G$, and hence G has NCN.

Now assume that G is a finite NCN group that is not nilpotent. We want to show that G is a group of type (i), (ii), or (iii). By Lemma 5.3, $G = P \rtimes X$ where P is a p-subgroup of G for some prime p, and X is a cyclic p'-subgroup of G acting nontrivially on P. Furthermore, $\Phi(P)X$ is cyclic and $P/\Phi(P)$ is elementary abelian of rank ≤ 2.

If P is abelian, we follow the approach used in the proof of [ZGQX, Theorem 3.4] to show that $[P,X] \cong C_p$ or $C_p \times C_p$. To start with, Fitting's Theorem ([I, Theorem 4.34]) implies $P = \mathfrak{C}_P(X) \times [P,X]$, since P is abelian and X has order prime to p. If N is a nonidentity normal subgroup of G contained in $[P,X]$, then $N \cap \mathfrak{C}_P(X) = 1$, so NX is nonabelian and hence NX is normal in the NCN group G. Thus $[P,X] \subseteq [P,NX] \subseteq P \cap NX = N$ and $[P,X]$ is minimal normal in G. Since $[P,X]$ is a minimal normal p-subgroup, it follows that $[P,X]$ is elementary abelian. Furthermore, since $P/\Phi(P)$ is elementary abelian of rank ≤ 2, we conclude that $[P,X] \cong C_p$ or $C_p \times C_p$.

If $[P,X] \cong C_p$, then $\mathfrak{C}_P(X)$ is cyclic since $P/\Phi(P)$ has rank at most 2. It follows that $G = ([P,X] \times \mathfrak{C}_P(X)) \rtimes X \cong C_p \rtimes (\mathfrak{C}_P(X) \times X)$, where $\mathfrak{C}_P(X) \times X$ is cyclic since $\mathfrak{C}_P(X)$ and X are cyclic of coprime orders. Therefore, G is a group of type (i).

On the other hand, if $[P,X] \cong C_p \times C_p$, then $\mathfrak{C}_P(X) = 1$ since $P/\Phi(P)$ has rank at most 2 and $P = \mathfrak{C}_P(X) \times [P,X]$. Thus $P = [P,X] \cong C_p \times C_p$ and P is minimal normal in G. Proposition 5.5 now implies that $\gcd(|G : \mathfrak{C}_G(P)|, p-1) = 1$ and G is a group of type (ii).

Finally, if P is nonabelian, then $P \cong Q_8$ by Proposition 5.6, and hence G is a group of type (iii). □

We remark that in (i), p and n need not be relatively prime. In (iii), since $P \cong Q_8$, for any subgroup S of X, it follows that $|\mathfrak{N}_S(P) : \mathfrak{C}_S(P)|$ divides $|\mathrm{Aut}(P)| = |\mathrm{Sym}_4| = 24$. Thus $|\mathfrak{N}_X(P) : \mathfrak{C}_X(P)| = 3$ and hence 3 divides n. Moreover, if S has order prime to 3, then $\mathfrak{N}_S(P) = \mathfrak{C}_S(P)$. Thus, for some $k \geq 1$, we see that $G \cong (Q_8 \rtimes C_{3^k}) \times C_m$, where $\gcd(2,m) = \gcd(3,m) = 1$, and this is the group described in [ZGQX, Theorem 3.4(iii)].

After our paper was completed, we became aware of reference [SZS], written by the same authors who considered nilpotent groups having property NCN. This new paper studies non-nilpotent groups with NCN and contains a classification given in [SZS, Theorem 4.4]. While we have not checked the proof, the result listed there is certainly consistent with our Theorem 5.7 when one looks very closely. Unfortunately, [SZS, Theorem 4.4] obscures the simple structure of the groups by studying $\mathfrak{Z}(G)$ and $G/\mathfrak{Z}(G)$ separately, and by considering the latter as a "strong" Frobenius group. This "strong" condition is evident in our Proposition 5.5(ii), but is quickly replaced by the simpler condition (iii) of that result. On the other hand, the "strong" Frobenius condition is automatically satisfied by the other two families of groups described in Theorem 5.7 and, in our opinion, its mention only serves to confuse the situation.

Even more recently, we discovered reference [BCZ], where non-nilpotent NCN groups, called $\mathfrak{Z}$-Dedekind groups, are also characterized. This is a special case of the paper's more general work on so-called $\mathfrak{C}$-Dedekind groups, for various classes $\mathfrak{C}$ of nilpotent groups. Again [BCZ, Theorem 4] is consistent with our Theorem 5.7, but it does not describe all the groups that occur as semi-direct products. Rather it uses notation and descriptions that fit into the more general $\mathfrak{C}$-Dedekind group theory that is developed. In particular, the condition on the order of $H/\mathfrak{C}_H(M)$ in groups of type (i) is certainly unnecessary, and the assumption that $|M| = 2m$ in groups of type (iii) obscures the simple semi-direct product structure of the group G.

References

[BCZ] Adolfo Ballester-Bolinches, John Cossey and Liangcai Zhang, *Generalised norms in finite soluble groups*, J. Algebra **402** (2014), 392-405. MR 3160429

[B] W. Burnside, *On a class of groups of finite order*, Trans. Cambridge Phil. Soc. **18** (1899), 269–276.

[BJ] Zdravka Božikov and Zvonimir Janko, *A complete classification of finite p-groups all of whose noncyclic subgroups are normal*, Glas. Mat. Ser. III **44(64)** (2009), no. 1, 177–185. MR 2525662 (2010g:20029)

[G] David M. Goldschmidt, *Elements of order two in finite groups*, Delta (Waukesha) **4** (1974/75), 45–58. MR 0352256 (50 #4743)

[I] I. Martin Isaacs, *Finite Group Theory*, Graduate Studies in Mathematics, vol. 92, Amer. Math. Soc., Providence, RI, 2008. MR 2426855 (2009e:20029)

[L] Chia-Hsin Liu, *Multiplicative Jordan decomposition in group rings and p-groups with all noncyclic subgroups normal*, J. Algebra **371** (2012), 300–313. MR 2975398

[Lim] F. N. Liman, 2-*groups with normal noncyclic subgroups*, Mathematical Notes **4** (1968), 535–539, 10.1007/BF01429816.

[LP1] Chia-Hsin Liu and D. S. Passman, *Multiplicative Jordan decomposition in group rings of* 3-*groups*, J. Algebra Appl. **8** (2009), no. 4, 505–519. MR 2555517 (2010m:16034)

[LP2] ———, *Multiplicative Jordan decomposition in group rings of* $2,3$-*groups*, J. Algebra Appl. **9** (2010), no. 3, 483–492. MR 2659731 (2011j:16046)

[P] Donald S. Passman, *Permutation Groups*, Dover Publications, Inc., Mineola, NY, 2012, Revised reprint of the 1968 original. MR 2963408

[P1] ———, *Nonnormal subgroups of p-groups*, J. Algebra **15** (1970), 352–370. MR 0258959 (41 #3604)

[SSZ] Zhencai Shen, Wujie Shi, and Jinshan Zhang, *Finite non-nilpotent generalizations of Hamiltonian groups*, Bull. Korean Math. Soc. **48** (2011), no. 6, 1147–1155. MR 2894883

[SZS] Zhencai Shen, Jinshan Zhang, and Wujie Shi, *On a generalization of Hamiltonian groups and a dualization of PN-groups*, Comm. Algebra **41** (2013), no. 5, 1608–1618. MR 3062813

[Suz] Michio Suzuki, *The nonexistence of a certain type of simple groups of odd order*, Proc. Amer. Math. Soc. **8** (1957), 686–695. MR 0086818 (19,248f)

[ZGQX] Qinhai Zhang, Xiaoqiang Guo, Haipeng Qu, and Mingyao Xu, *Finite groups which have many normal subgroups*, J. Korean Math. Soc. **46** (2009), no. 6, 1165–1178. MR 2572507 (2010j:20038)

Department of Mathematics, National Taiwan Normal University, Taipei, Taiwan, R.O.C.

E-mail address: `chliu@math.ntnu.edu.tw`

Department of Mathematics, University of Wisconsin-Madison, Madison, Wisconsin 53706, USA

E-mail address: `passman@math.wisc.edu`