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Semiprimitivity of group algebras of locally finite groups. II ☆

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Abstract

Let K be a field of characteristic $p > 0$, let G be a locally finite group, and let $K[G]$ denote the group algebra of G over K. In this paper we study the Jacobson radical $JK[G]$ when G has a finite subnormal series with factors which are either p'-groups, infinite simple, or generated by locally subnormal subgroups. For example, we show that if such a group G has no finite locally subnormal subgroup of order divisible by p, then $JK[G] = 0$. The argument here is a mixture of group ring and group theoretic techniques and requires that we deal more generally with twisted group algebras. Furthermore, the proof ultimately depends upon certain consequences of the classification of the finite simple groups. In particular, we use J.I. Hall's classification of the locally finite finitary simple groups.

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0. Introduction

This paper is a continuation of recent work in [8–10]; it concerns group algebras $K[G]$, where K is a field of characteristic $p > 0$ and where G is a locally finite group. Specifically, we show that if G has a particular global structure, then $K[G]$ is semiprimitive, or equivalently that its Jacobson radical $JK[G]$ is zero. Recall that a finite subgroup A of G is *locally subnormal* if $A \lhd \lhd B$ for all finite subgroups B of G containing A. For example, if G is locally nilpotent, then every finite subgroup of G is locally subnormal. Furthermore, if G is an f.c. group, then G is generated by

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its finite normal subgroups and hence by its locally subnormal subgroups. The goal of this paper is to prove

Main Theorem. *Let $K[G]$ be the group algebra of a locally finite group G over a field K of characteristic $p > 0$. Suppose that G has a finite subnormal series*

$$1 = G_0 \lhd G_1 \lhd \cdots \lhd G_n = G$$

with each quotient G_{i+1}/G_i either
(i) *a p'-group, or*
(ii) *a nonabelian simple group, or*
(iii) *generated by its locally subnormal subgroups.*
Then $K[G]$ is semiprimitive if and only if G has no locally subnormal subgroup of order divisible by p.

A twisted generalization of this result is obtained in Section 3 and a brief outline of its proof is as follows.

First, we can assume that K is algebraically closed. Then we proceed by induction on the number of factors of the subnormal series for G which are infinite simple but not a p'-group. It turns out that we can quickly reduce to the case of just one such factor. Indeed, it suffices to assume that G has a normal subgroup N with $G/N = H$ an infinite simple group containing an element of order p, and with $|N : \mathbb{C}_N(g)| < \infty$ for all $g \in G$. Furthermore, N is a p'-group and we can suppose that G has no nontrivial f.c. homomorphic image. In other words, the pair (G, N) is a *p'-f.c. cover* of H. Now if N is central in G, then G is a central cover of H and $K[G]$ is a subdirect product of various twisted group algebras $K^t[H]$. Thus, the main theorems of [9,10] apply here and yield the result.

On the other hand, if N is not central in G, then we show that $H = G/N$ is a finitary linear group over the Galois field $\mathrm{GF}(q)$ for some prime q involved in the subgroup N. Furthermore, since H cannot be a linear group, the results of [3–5] imply that H is isomorphic to one of the stable finitary groups Alt_∞, $\mathrm{FSL}_\infty(F)$, $\mathrm{FSp}_\infty(F)$, $\mathrm{FSU}_\infty(F)$, or $\mathrm{F}\Omega_\infty(F)$ for some locally finite field F of characteristic q. As will be apparent, the bulk of this paper is concerned with these few special cases.

Let W be any locally finite group. We say that W is *p-insulated* if, for every finite subset $\{x_1, x_2, \ldots, x_t\}$ of nonidentity elements of W, there exists a p-element z of W such that no zx_i is a p-element. For example, any p'-group is p-insulated and, as is shown in [10], if W is p-insulated then $JK[W] = 0$. In fact, the main result of the latter paper is obtained by proving that any nonlinear locally finite simple group is p-insulated. In Section 1 of this paper we define a stronger version of this concept and show that if H is *strongly p-insulated*, then any p'-f.c. cover G of H is p-insulated and hence satisfies $JK[G] = 0$. Thus all that remains is to prove that the stable groups H are strongly p-insulated, and this is done in Section 2. Note that, since N is a p'-group, we have $q \neq p$. Thus we need only consider the stable groups in characteristic $q \neq p$, and this is a great simplification. Nevertheless, for the sake of completeness and for possible later applications, we show in the rather long and unpleasant Section 4 that stable groups in characteristic p are also strongly p-insulated.

1. p-Insulated groups

Let G be a locally finite group and let p be a fixed prime. If $\{x_1, x_2, \ldots, x_t\}$ is a set of nonidentity elements of G, then a *p-insulator* for this set is a p-element $z \in G$ such that no zx_i is a p-element. We say that G is a *p-insulated group* if every finite subset of $G \setminus 1$ has a p-insulator. This concept, with a somewhat different name, was used in [2] and later in [10] to prove that certain group algebras are semiprimitive. Here we need a stronger version to handle larger classes of groups.

We say that G is *strongly p-insulated* if, for any $x_1, x_2, \ldots, x_t \in G\setminus 1$ and any integer $r \geq 1$, there exists a homocyclic p-subgroup P of G having rank r such that every generator z of P is a p-insulator for $\{x_1, x_2, \ldots, x_t\}$. Note that every p'-group G is p-insulated, but that no p'-group is strongly p-insulated. In this section we briefly consider some applications of these concepts.

Let P be a finite abelian p-group and define the *rank* of P to equal the minimal number of generators of the group. In particular, if $\Phi(P)$ denotes the Frattini subgroup of P and if $|P/\Phi(P)| = p^r$, then we know that $r = \operatorname{rank} P$. Furthermore, if $\Omega_1(P) = \{x \in P \mid x^p = 1\}$, then $|\Omega_1(P)| = p^r$ and hence $\operatorname{rank} \Omega_1(P) = r = \operatorname{rank} P$. Of course, z is a generator of P if and only if $z \in P \setminus \Phi(P)$.

We say that P is *homocyclic of type* p^n if $P \cong \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^n} \times \cdots \times \mathbb{Z}_{p^n}$ is a direct product of finitely many copies of the cyclic group $\mathbb{Z}_{p^n}$ of order p^n. Some basic properties of such groups are as follows.

(1) If $A \subseteq P$, then $A \subseteq \tilde{A} \subseteq P$ where $\tilde{A}$ is homocyclic of type p^n and $\operatorname{rank} \tilde{A} = \operatorname{rank} A$. Of course, $\tilde{A}$ is not uniquely determined by A.

(2) If $A \subseteq P$, then A is a direct factor of P if and only if A is homocyclic of type p^n.

These are easy to prove directly. They also follow from the fact that $\mathbb{Z}/p^n\mathbb{Z}$ is a self-injective ring. We first need

Lemma 1.1. *Let P be a homocyclic group of type p^n and let $S \subseteq R$ be subgroups of P. Then there exists a direct factor Q of P with* $\operatorname{rank} P/Q \leq \operatorname{rank} R/S$ *and $Q \cap R \subseteq S$.*

Proof. We proceed by induction on $|P|$, the result being trivial when $|P| = 1$. There are two cases to consider according to the relationship between the rank of S and the rank of P.

Case 1: $\operatorname{rank} S < \operatorname{rank} P$.

Proof. Write $P = \tilde{R} \times Y$ and $\tilde{R} = \tilde{S} \times X$, so that $P = \tilde{S} \times X \times Y$. By assumption, $\operatorname{rank} \tilde{S} = \operatorname{rank} S < \operatorname{rank} P$ and therefore $|\tilde{S}| < |P|$. Now observe that $\tilde{S} \supseteq R \cap \tilde{S} \supseteq S$

and that $R \supseteq (R \cap \tilde{S}) \times (R \cap X)$. Thus $R/S \supseteq (R \cap \tilde{S})/S \times (R \cap X)$ and hence

$$r = \operatorname{rank} R/S \geq \operatorname{rank}(R \cap \tilde{S})/S + \operatorname{rank}(R \cap X) = s + t,$$

where $s = \operatorname{rank}(R \cap \tilde{S})/S$ and $t = \operatorname{rank}(R \cap X)$. Since $\operatorname{rank} \tilde{R} = \operatorname{rank} R$, it follows that $\Omega_1(\tilde{R}) = \Omega_1(R)$. In particular, $R \cap X \supseteq \Omega_1(\tilde{R}) \cap X = \Omega_1(X)$, so $\operatorname{rank} X = \operatorname{rank}(R \cap X) = t$.

By induction applied to the subgroups $S \subseteq R \cap \tilde{S}$ of $\tilde{S}$, there exists a direct factor U of $\tilde{S}$ with $\operatorname{rank} \tilde{S}/U \leq s$ and $U \cap (R \cap \tilde{S}) \subseteq S$. Finally, set $Q = U \times Y$ so that Q is a direct factor of P and $P/Q \cong \tilde{S}/U \times X$ has rank

$$\operatorname{rank} P/Q = \operatorname{rank} \tilde{S}/U + \operatorname{rank} X \leq s + t \leq r.$$

Furthermore, since $Q \subseteq \tilde{S} \times Y$, $R \subseteq \tilde{S} \times X$ and $(\tilde{S} \times Y) \cap (\tilde{S} \times X) = \tilde{S}$, it follows that

$$Q \cap R = (Q \cap \tilde{S}) \cap (R \cap \tilde{S}) = U \cap (R \cap \tilde{S}) \subseteq S$$

and this case is proved. □

Case 2: $\operatorname{rank} S = \operatorname{rank} P$.

Proof. If $n = 1$, then $\operatorname{rank} S = \operatorname{rank} P$ implies that $S = R = P$. Hence $\operatorname{rank} S/R = 0$ and we can take $Q = P$. Thus we may suppose that $n \geq 2$.

If $L = \Omega_1(P)$, then the equality of ranks implies that $P \supseteq R \supseteq S \supseteq L$. Furthermore, $L \subseteq \Phi(P)$ since $n \geq 2$. Let $\bar{\ }: P \to P/L$ be the natural epimorphism and note that $\bar{P} \supseteq \bar{R} \supseteq \bar{S}$, $\bar{R}/\bar{S} \cong R/S$ has rank $r = \operatorname{rank} R/S$, and $\bar{P}$ is homocyclic of type p^{n-1}. By induction, we can write $\bar{P} = \bar{Q} \times \bar{V}$ where $\operatorname{rank} V \leq r$ and $\bar{Q} \cap \bar{R} \subseteq \bar{S}$. Finally, choose $Q, V \subseteq P$ such that $\operatorname{rank} Q \leq \operatorname{rank} \bar{Q}$, $\operatorname{rank} V \leq \operatorname{rank} \bar{V}$ and $QL/L = \bar{Q}$, $VL/L = \bar{V}$. Then $QVL = P$ and, since $L \subseteq \Phi(P)$, we have $QV = P$. By order considerations, it follows that Q and V are homocyclic of type p^n and that $P = Q \times V$. Thus $\operatorname{rank} P/Q = \operatorname{rank} V \leq \operatorname{rank} \bar{V} \leq r$. Furthermore, $\overline{Q \cap R} \subseteq \bar{Q} \cap \bar{R} \subseteq \bar{S}$ so, since $S \supseteq L$, we have $Q \cap R \subseteq S$ and the lemma is proved. □

As a consequence, we obtain

Lemma 1.2. *Let $H = BP$ where $B \triangleleft H$, B is a locally finite p'-group and P is a finite homocyclic p-group. Furthermore, let A be a finite normal subgroup of B. Then there exists a direct factor Q of P, with $\operatorname{rank} P/Q \leq 2\log_2|A|$, such that za is not a p-element for any $z \in Q$ and $1 \neq a \in A$.*

Proof. We proceed by induction on $|A|$, the case $|A| = 1$ being clear. Suppose now that $|A| > 1$.

We can assume that $B = A^P$ is generated by the P-conjugates of A. Thus B is finite and $H = \langle A, P \rangle$. Now choose $L \subseteq B$ maximal with respect to $L \triangleleft H$ and $L \cap A = 1$. If $\bar{\ }: H \to H/L$ is the natural epimorphism, then $\bar{A} \cong A$. Furthermore, if $\bar{z}\bar{a} = \overline{za}$ is not a p-element, then certainly za is not a p-element. Thus it suffices to work in $\bar{H}$, or equivalently we can assume that $L = 1$. In particular, any nonidentity subgroup of B which is normal in H must meet A nontrivially.

Suppose there is a proper subgroup N of B which is normal in H. Since $N \neq 1$, we know that $N \cap A \neq 1$. On the other hand, if $N \cap A = A$ then $N \supseteq A$ so $N \supseteq A^P = B$, a contradiction. Thus $1 \subset N \cap A \subset A$. By induction, P has a direct factor R, with $\operatorname{rank} P/R \leq 2\log_2 |N \cap A|$, such that za is not a p-element for any $z \in R$ and $1 \neq a \in N \cap A$. Furthermore, by induction working in H/N, R has a direct factor Q, with $\operatorname{rank} R/Q \leq 2\log_2 |A/(N \cap A)|$, such that za is not a p-element for any $z \in Q$ and $a \in A \backslash (N \cap A)$. Since

$$\begin{aligned}\operatorname{rank} P/Q &\leq \operatorname{rank} P/R + \operatorname{rank} R/Q \\ &\leq 2\log_2 |N \cap A| + 2\log_2 |A/(N \cap A)| = 2\log_2 |A|,\end{aligned}$$

the result follows in this situation.

We can therefore assume that no such N exists. In particular, B has no proper characteristic subgroup, so B is either an elementary abelian q-group for some prime $q \neq p$ or it is semisimple. We consider the two cases separately.

Case 1: *B is abelian.*

Proof. Notice that any P-stable subgroup of B is normal in H. Thus, by assumption, P acts irreducibly on B. In particular, if $C = \mathbb{C}_P(B)$ then P/C is cyclic. Now, by Lemma 1.1, there exists a direct factor Q of P with $\operatorname{rank} P/Q \leq 1 \leq 2\log_2 |A|$ and $Q \subseteq C$. Since $BQ = B \times Q$ and B is a p'-group, it therefore follows that zb is not a p-element for any $z \in Q$ and $1 \neq b \in B$. $\square$

Case 2: *B is semisimple.*

Proof. Let $\{ S_i \mid 1 \leq i \leq v \}$ be the set of normal simple subgroups of B. Then B is the direct product $B = \prod_1^v S_i$ and, since $A \triangleleft B$, we can assume that $A = \prod_1^r S_i$. Certainly $r \leq \log_2 |A|$. Also P permutes $\{ S_i \}$ and the direct product of the factors in any orbit is a normal subgroup of H. Thus, since H has no normal subgroup N with $1 \subset N \subset B$, it follows that P is transitive on $\{ S_i \}$. Indeed, if L is the kernel of this permutation action then, since P is abelian, P/L acts regularly. For each $1 \leq i \leq r$, choose $x_i \in P$ so that $S_1^{x_i} = S_i$ and let R be the subgroup of P generated by L and $x_1, x_2, \ldots, x_r$. Then $\operatorname{rank} R/L \leq r$ and $S_1^R \supseteq A$. Moreover, since R is a subgroup of P and $R \supseteq L$, it is easy to see that $R = \{ y \in P \mid S_1^{Ry} \cap S_1^R \neq 1 \}$.

Now L stabilizes all S_i, so L normalizes A and we let $C = \mathbb{C}_L(A)$. In particular, L/C is an abelian p-group which embeds in the symmetric group Sym_A. Since the largest elementary abelian p-subgroup of Sym_A has rank $t \leq \log_p |A| \leq \log_2 |A|$, it follows that $\operatorname{rank} L/C \leq t$. Thus

$$\begin{aligned}\operatorname{rank} R/C &\leq \operatorname{rank} R/L + \operatorname{rank} L/C \\ &\leq r + t \leq 2\log_2 |A|.\end{aligned}$$

By Lemma 1.1 applied to $P \supseteq R \supseteq C$, there exists a direct factor Q of P with $\operatorname{rank} P/Q \le 2\log_2 |A|$ and $R \cap Q \subseteq C$. In particular, $L \cap Q \subseteq C$ so $R \cap Q = L \cap Q = C \cap Q$.

Finally, let $z \in Q$ and let $1 \ne a \in A$. Say z has order p^n modulo $R \cap Q = L \cap Q = C \cap Q$ and observe that

$$(za)^{p^n} = z^{p^n} a^{z^{p^n - 1}} \cdots a^{z^2} a^{z} a.$$

Since $z^j \notin R$ for $1 \le j \le p^n - 1$ and $R = \{\, y \in P \mid S_1^{Ry} \cap S_1^R \ne 1 \,\}$, it follows that $a^{z^j} \in S_1^{Rz^j}$ has all S_1^R-components equal to 1 in the direct product $B = \prod_1^v S_i$. In other words, if we write $B = A \times D$, then $(za)^{p^n} = z^{p^n} ad$ for some $d \in D$. Now $z^{p^n} \in L$ so $D \lhd \langle B, z^{p^n} \rangle$. Also $z^{p^n} \in C$, so z^{p^n} centralizes A. Thus $\langle B, z^{p^n} \rangle / D \cong A \times \langle z^{p^n} \rangle$ and, since A is a p'-group, this clearly implies that $(za)^{p^n} D = z^{p^n} a^{p^n} D$ is not a p-element in $\langle B, z^{p^n} \rangle / D$. But then $(za)^{p^n}$ is not a p-element and therefore neither is za. $\square$

Our main application is as follows.

Proposition 1.3. *Let G be a locally finite group and let B be a normal f.c. subgroup of G. If G/B is strongly p-insulated and if B is a p'-group, then G is p-insulated. In particular, if K is a field of characteristic p, then any twisted group algebra $K^t[G]$ is semiprimitive.*

Proof. We can write finitely many nonidentity elements of G as

$$a_1, a_2, \ldots, a_k, x_1, x_2, \ldots, x_n,$$

where $a_1, a_2, \ldots, a_k \in B \setminus 1$ and where $x_1, x_2, \ldots, x_n \in G \setminus B$. Since B is a locally finite f.c. group, there exists a finite normal subgroup A of B with $a_1, a_2, \ldots, a_k \in A \setminus 1$. Let r be an integer larger than $2\log_2 |A|$.

Now let $\bar{\ }: G \to \bar{G} = G/B$ be the natural epimorphism. Since $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n \in \bar{G} \setminus 1$ and since $\bar{G}$ is strongly p-insulated, there exists a homocyclic p-subgroup $\bar{P}$ of $\bar{G}$ with $\operatorname{rank} \bar{P} = r$ such that, for all generators $\bar{z}$ of $\bar{P}$, no $\bar{z}\bar{x}_i$ is a p-element of $\bar{G}$. Furthermore, since B is a p'-group and G is locally finite, there exists a homocyclic p-subgroup P of G such that $P \cong PB/B = \bar{P}$.

Finally, we apply Lemma 1.2 to $H = BP$ with $A \lhd B$ to conclude that there exists a direct factor Q of P, with $\operatorname{rank} P/Q \le 2\log_2 |A|$, such that za is not a p-element for any $z \in Q$ and $1 \ne a \in A$. Furthermore, since $\operatorname{rank} P = r > 2\log_2 |A|$, we have $Q \ne 1$. Thus we can choose $z \in Q$ to be a generator of P. For this p-element, we know that no za_i is a p-element and that no $\bar{z}\bar{x}_j$ is a p-element. Therefore, no zx_j is a p-element and G is indeed p-insulated. The last remark concerning $K^t[G]$ follows from [10, Lemma 7.4]. $\square$

A minor modification of the above proves that G is strongly p-insulated.

2. Finitary linear groups I

The goal of this section is to show that certain finitary simple groups of infinite rank are strongly p-insulated. The techniques used are extensions of those of [10]. For convenience, we let ∞ denote a fixed set of countably infinite size.

Theorem 2.1. *If G is a locally finite group with $\mathrm{Alt}_\infty \subseteq G \subseteq \mathrm{Sym}_\infty$, then G is strongly p-insulated.*

Proof. Suppose we are given nonidentity elements $x_1, x_2, \ldots, x_n$ of $G \subseteq \mathrm{Sym}_\infty$ and a positive integer $r \geq 2$. Say $x_1, x_2, \ldots, x_n \in \mathrm{Sym}_k$ so that these elements move points in the set $\{1, 2, \ldots, k\}$ and fix the remaining ones. Let P be a homocyclic p-group of type p^k and of rank r. We define an embedding of P into $\mathrm{Alt}_\infty \subseteq G$ as follows.

For each $j = 1, 2, \ldots, k$ let $L_j = \mho_j(P) = \{ g^{p^j} \mid g \in P \}$ so that L_j is a characteristic subgroup of P with P/L_j homocyclic of type p^j. Let $\pi_j \colon P \to P/L_j$ denote the natural epimorphism. Under the regular permutation representation, let P/L_j act on the set Γ_j. Then by using the various maps π_j, there is a natural permutation action of P on $\Gamma = \Gamma_1 \dot\cup \Gamma_2 \dot\cup \cdots \dot\cup \Gamma_k$. We now embed Γ into the set $\{1, 2, \ldots\} = \infty$ in such a way that $j \in \Gamma_j$. Since P acts faithfully on Γ_k, we see that $P \subseteq \mathrm{Sym}_\Gamma \subseteq \mathrm{Sym}_\infty$.

Let z be a generator of P so that z has order p^k. Then $\pi_j(z)$ has order p^j in P/L_j and, since P/L_j acts regularly on Γ_j, it follows that the cycle structure of $\pi_j(z)$ on Γ_j consists entirely of cycles of length p^j. But $j \in \Gamma_j$ and thus the cycle structure of z on Γ looks like

$$(1 * \cdots *)(2 * \cdots *) \cdots (k * \cdots *)(* \cdots *) \cdots (* \cdots *),$$

where $(j * \cdots *)$ has length p^j and where the $*$'s indicate distinct elements.

It now follows as in [2] that no zx_i is a p-element. Indeed, if $x_i \in \mathrm{Sym}_k$ contains the nontrivial cycle $(a_1\, a_2 \cdots a_t)$, then zx_i contains the cycle

$$(a_1 * \cdots * a_2 * \cdots * \cdots a_t * \cdots *)$$

of length $\ell = p^{a_1} + p^{a_2} + \cdots + p^{a_t}$. Since the latter exponents are all distinct, ℓ is not a power of p and therefore zx_i is not a p-element.

Finally, if p is odd, then it is clear that $P \subseteq \mathrm{Alt}_\infty \subseteq G$. On the other hand, if $p = 2$ then, since $r \geq 2$, we see that each $\pi_j(z)$ is a product of an even number of cycles of length 2^j. Thus $\pi_j(z) \in \mathrm{Alt}_{\Gamma_j}$ and again $P \subseteq \mathrm{Alt}_\infty \subseteq G$. $\square$

We now move on to consider linear groups over a locally finite field F. Specifically, we study the finitary special linear, unitary, symplectic, and orthogonal groups of infinite degree over F. Fortunately, with one minor exception, these can all be studied simultaneously using sesquilinear forms.

Let V be a vector space over the field F and let $\Phi \colon V \times V \to F$ be a map which is F-linear in the second variable. Then Φ is said to be a *sesquilinear form* if, for all $v, w \in V$, we have $\Phi(w, v) = \varepsilon \Phi(v, w)^\kappa$ for some fixed $\varepsilon = \pm 1$ and some field automorphism $^\kappa$

of order 1 or 2. As usual, we write $\mathrm{FU}(V,\Phi)\subseteq \mathrm{FGL}(V)$ for the group of finitary isometries associated with Φ, and we assume throughout that Φ is nonsingular.

When ${}^{\kappa}$ is nontrivial and $\varepsilon = 1$, then Φ is a unitary form and $\mathrm{FU}(V,\Phi)$ is the usual finitary unitary group. On the other hand, if ${}^{\kappa} = 1$, then Φ is either orthogonal or symplectic according to whether $\varepsilon = 1$ or -1. Of course, when char $F = 2$, then symplectic forms require $\Phi(v,v) = 0$ for all $v \in V$ and orthogonal forms are defined somewhat differently, but are nevertheless based on an auxiliary symplectic form. See [1, Chap. 1] for details.

If $\dim V = n < \infty$, then these forms can also be described matrix theoretically. To this end, let * denote the composition of matrix transpose with ${}^{\kappa}$. Then the $n \times n$ matrix Φ determines the sesquilinear form $v \times w \mapsto v^*\Phi w$ if and only if $\Phi^* = \varepsilon\Phi$, and the form is nonsingular precisely when Φ is a nonsingular matrix. Furthermore, $x \in \mathrm{GL}_n(F)$ is an isometry if and only if $x^*\Phi x = \Phi$, and we let $U_n(F,\Phi)$ denote the set of all such x. We will use this notation even for orthogonal groups in characteristic 2.

If $\dim V = \infty$ and F is a locally finite field, then it is well known that there is a unique nonsingular unitary, symplectic or orthogonal form defined on V. Therefore, we just write $\mathrm{FU}_\infty(F)$ for the group of isometries in this case, and we let $\mathrm{FU}_\infty(F)'$ be its commutator subgroup. Of course, these commutator groups are all contained in the finitary special linear group $\mathrm{FSL}_\infty(F)$. Note that $\mathrm{FSL}_\infty(F)$ is the (unique) stable group since ∞ has countably infinite size.

The goal now is to obtain linear group analogs of Theorem 2.1. As in [10], there are two cases to consider according to whether $p = \mathrm{char}\, F$ or not. Again, the latter case is quite easy while the former requires a good deal of work. Recall that if $\ell < k$ are integers, then we embed the general linear group $\mathrm{GL}_\ell(F)$ into $\mathrm{GL}_k(F)$ via the map $x \mapsto \tilde{x} = \mathrm{diag}(x,e')$ where e' is the $(k-\ell)\times(k-\ell)$ identity matrix. For obvious reasons, we call this the *corner embedding*. It allows us to distinguish different matrix representations for the same element of $\mathrm{FGL}_\infty(F)$.

Theorem 2.2. *Let F be a locally finite field, let $\mathrm{FU}_\infty(F)$ denote the finitary unitary, symplectic or orthogonal group of infinite degree, and let G be a group with*

$$\mathrm{FU}_\infty(F)' \subseteq G \subseteq \mathrm{FGL}_\infty(F).$$

If $p \neq \mathrm{char}\, F$, then G is strongly p-insulated.

Proof. Let $x_1,x_2,\ldots,x_t$ be nonidentity elements of G and let $r \geq 1$ be a given integer. For convenience, assume that $\{x_1,x_2,\ldots,x_t\}$ contains a p-element and, furthermore, if x_i is a p-element, then $x_i^u \in \{x_1,x_2,\ldots,x_t\}$ for all exponents u prime to p. By reordering the subscripts, we can suppose that $x_1,x_2,\ldots,x_{m-1}$ are p-elements and that $x_m,x_{m+1},\ldots,x_t$ are not. Since $G\subseteq \mathrm{FGL}_\infty(F)$ and since $\mathrm{FU}_\infty(F)$ is determined by a nonsingular sesquilinear form, it follows that, for some integer $\ell \geq 1$, we have $\{x_1,x_2,\ldots,x_t\}\subseteq \mathrm{GL}_\ell(F)$ with ϕ, the restriction of the sesquilinear form to this $\ell\times\ell$

upper left-hand corner, nonsingular. Now define the $m\ell \times m\ell$ matrices X and Z by

$$X = \operatorname{diag}(x_1, x_2, \dots, x_{m-1}, d) \quad \text{and} \quad Z = (X^*)^{-1},$$

where $d = \operatorname{diag}(d', 1, \dots, 1)$ is an $\ell \times \ell$ matrix with $d' = \prod_{i=1}^{m-1} (\det x_i)^{-1}$. Since each x_i with $i \le m-1$ is a p-element, it is clear that X and Z are p-elements in $\mathrm{SL}_{m\ell}(F)$ with order $|X| = |Z| = \max\{\, |x_i| \mid 1 \le i \le m-1 \,\}$.

Let E denote the $rm\ell \times rm\ell$ identity matrix and set $k = \ell + 2rm\ell = (1 + 2rm)\ell$. Since $\mathrm{FU}_\infty(F)$ depends only on the nature of the sesquilinear form, we can assume that the $k \times k$ upper left corner of $\mathrm{FU}_\infty(F)$ has the form determined by the matrix

$$\Phi = \begin{pmatrix} \phi & 0 & 0 \\ 0 & 0 & E \\ 0 & \varepsilon E & 0 \end{pmatrix}$$

(or the characteristic 2 quadratic form as described in [10, Lemma 4.2]). Next, let e be the $\ell \times \ell$ identity matrix and define $Q = Q_r$ to be the set of all $k \times k$ block diagonal matrices

$$\operatorname{diag}(e, X^{a_1}, X^{a_2}, \dots, X^{a_r}, Z^{a_1}, Z^{a_2}, \dots, Z^{a_r})$$

with $a_i \in \mathbb{Z}$. It is clear that Q is a homocyclic p-group of type $n = |X|$ and rank r. Furthermore, [10, Lemmas 4.1(i) and 4.2(i)] imply that $Q \subseteq U_k(F, \Phi)'$.

If α is the $\ell \times rm\ell$ matrix $\alpha = (\, e \; e \; \cdots \; e \,)$, then it follows from [10, Lemmas 4.1(ii) and 4.2(ii)] that there exist F-matrices β and γ of suitable size with

$$Y = \begin{pmatrix} e & \alpha & 0 \\ 0 & E & 0 \\ \gamma & \beta & E \end{pmatrix}$$

contained in $U_k(F, \Phi)$. Set $P = P_r = Y^{-1} Q Y \cong Q$. Then

$$P \subseteq U_k(F, \Phi)' \subseteq \mathrm{FU}_\infty(F)' \subseteq G$$

and we claim that if z is a generator of P and if $\tilde{}$ is the corner embedding defined on $\mathrm{GL}_\ell(F)$, then no $z\tilde{x}_i$ is a p-element. To this end, we first note that

$$z = \begin{pmatrix} z_1 & \\ z_2 & z_3 \end{pmatrix},$$

where

$$z_1 = \begin{pmatrix} e & \alpha \\ & E \end{pmatrix}^{-1} \begin{pmatrix} e & \\ & \mathscr{X} \end{pmatrix} \begin{pmatrix} e & \alpha \\ & E \end{pmatrix}$$

and where $\mathscr{X} = \operatorname{diag}(X^{a_1}, X^{a_2}, \dots, X^{a_r})$. Thus, since

$$z\tilde{x}_i = \begin{pmatrix} z_1 \tilde{x}_i & \\ z_2 & z_3 \end{pmatrix}$$

has order divisible by $|z_1 \tilde{x}_i|$, it suffices to show that $z_1 \tilde{x}_i$ is not a p-element.

Suppose first that x_i is not a p-element, and note that z_1 is in upper block triangular form. Specifically, we have

$$z_1 = \begin{pmatrix} e & * \\ & * \end{pmatrix} \quad \text{and} \quad z_1\tilde{x}_i = \begin{pmatrix} x_i & * \\ & * \end{pmatrix}.$$

Thus $z_1\tilde{x}_i$ has order divisible by $|x_i|$ and therefore it is not a p-element.

On the other hand, let x_i be a p-element so that $1 \le i \le m-1$. By assumption, since z is a generator of P, some integer a_j is prime to p. Since the set $\{x_1, x_2, \ldots, x_{m-1}\}$ is closed under p' powers, it follows that x_i occurs as a block diagonal entry of X^{a_j}. Thus, by suitably permuting the rows and columns of $\mathscr{X}$, we can assume that x_i occurs as the first block entry of X^{a_1}. In other words, z_1 is similar to a matrix of the form

$$\begin{pmatrix} A_i & * \\ & * \end{pmatrix}$$

where

$$A_i = \begin{pmatrix} e & e \\ & e \end{pmatrix}^{-1} \begin{pmatrix} e & \\ & x_i \end{pmatrix} \begin{pmatrix} e & e \\ & e \end{pmatrix} = \begin{pmatrix} e & e - x_i \\ & x_i \end{pmatrix}.$$

Therefore, $z_1\tilde{x}_i$ is similar to

$$\begin{pmatrix} B_i & * \\ & * \end{pmatrix},$$

where

$$B_i = A_i \begin{pmatrix} x_i & \\ & e \end{pmatrix} = \begin{pmatrix} x_i & e - x_i \\ & x_i \end{pmatrix} = \begin{pmatrix} x_i & \\ & x_i \end{pmatrix} \begin{pmatrix} e & x_i^{-1} - e \\ & e \end{pmatrix}.$$

Notice that the latter two matrices commute, that

$$\begin{pmatrix} x_i & \\ & x_i \end{pmatrix}$$

is a p-element and that

$$\begin{pmatrix} e & x_i^{-1} - e \\ & e \end{pmatrix}$$

has order $q = \operatorname{char} F \neq p$ since $x_i \neq e$. Thus B_i is not a p-element and therefore neither is $z_1\tilde{x}_i$. Since $r \ge 1$ was chosen arbitrarily, the result follows. $\square$

The characterisitc p analog of Theorem 2.2 will be considered in Section 4.

3. Semiprimitivity

We continue to work in the context of locally finite groups and we recall some notation from [8]. To start with, suppose G is a group and H is a normal subgroup.

Then

$$\mathbb{D}_G(H) = \{x \in G \mid |H : \mathbb{C}_H(x)| < \infty\}$$

is the normal subgroup of G consisting of all those elements which act in a *finitary manner* on H. In particular,

$$\mathbb{D}_G(G) = \Delta(G) = \{x \in G \mid |G : \mathbb{C}_G(x)| < \infty\}$$

is the *f.c. center* of G, and G is said to be an *f.c. group* if $G = \Delta(G)$.

Next, the ordered pair (C,N) is said to be an *f.c. cover* of G if

(i) C is a locally finite group, $N \triangleleft C$ and $C/N \cong G$.

(ii) $C = \mathbb{D}_C(N)$, so that every element of C acts in a finitary manner on N.

(iii) C has no nontrivial f.c. factor group.

Notice that if (C,N) exists, then G itself has no nontrivial f.c. factor group and, in particular, G cannot be a nonidentity finite group. On the other hand, if G satisfies (iii), then G has at least one f.c. cover, namely $(G,1)$. Since any f.c. group is center-by-(residually finite), it is clear that condition (iii) is equivalent to

(iii′) C has no nontrivial finite or abelian factor group.

Furthermore, (ii) certainly implies that N is an f.c. group and hence that it is locally normal.

The f.c. cover is clearly an analog of the usual Schur covering group where N is taken to be central and where C is assumed to be equal to its commutator subgroup. However, unlike the Schur situation, there is no universal f.c. covering group. For example, let $G = \mathrm{Alt}_\infty$ be the infinite simple alternating group and let W be any finite nonabelian simple group. Then the permutation wreath product $C = W \wr G$ is easily seen to yield an f.c. cover of G with N, the base group of the wreath product, being an infinite direct sum of copies of W. If x is any element of $C \setminus N$, then x must move one of the W direct summands of N and therefore $|N : \mathbb{C}_N(x)| \geq |W|$. But $|W|$ can be taken to be arbitrarily large and thus no universal f.c. cover of G could map onto all such $W \wr G$.

The following is a slight sharpening of [8, Lemma 4.3] with essentially the same proof. For convenience we let $\pi(G)$ denote the set of prime divisors of the orders of the elements of G.

Lemma 3.1. *Let (C,N) be an f.c. cover of G and suppose that $\mathbb{C}_V(G) \neq 1$ whenever G acts in a finitary manner on an infinite elementary abelian q-group V with $q \in \pi(N)$. Then N is central in C.*

Proof. Since (C,N) exists, it follows that G has no nontrivial f.c. factor group. In particular, $G = G'$ and G has no nontrivial finite factor groups. We can clearly assume that $N \neq 1$ and we proceed in a series of steps.

Step 1: Suppose that G acts in a finitary manner on an abelian group A with $\pi(A) \subseteq \pi(N)$. Then G centralizes A. Similarly, suppose G acts as permutations on a set Ω and that each element of G moves only finitely many points. Then G acts trivially on Ω.

Proof. Let G act on A. We first observe that if $A \neq 1$ then $Z = \mathbb{C}_A(G) \neq 1$. To this end, let $q \in \pi(A) \subseteq \pi(N)$ and let V be the set of elements of A of order 1 or q. Then V is a nonidentity characteristic subgroup of A, so G acts on V in a finitary manner. If $|V|$ is finite, then $|G : \mathbb{C}_G(V)| < \infty$ and, since G has no nontrivial finite factor group, it follows that G centralizes V. On the other hand, if $|V| = \infty$, then $Z \neq 1$ follows from the hypothesis of the lemma. Suppose now that $Z \neq A$. Then G acts on $\bar{A} = A/Z \neq 1$ and hence, by the above, there exists $W \supset Z$ such that $W/Z = \mathbb{C}_{\bar{A}}(G)$. Notice that G acts on W, stabilizes the chain $W \supseteq Z \supseteq 1$ and acts trivially on each factor. Thus the commutator subgroup G' of G centralizes W. But $G' = G$, so $W \subseteq \mathbb{C}_A(G) = Z$, a contradiction, and therefore $Z = A$.

Finally, suppose that G acts as permutations on Ω. Let $F = \mathrm{GF}(q)$ for some $q \in \pi(N)$ and let $B = F\Omega$ be the permutation G-module determined by Ω. Since $|F| < \infty$, it is clear that G acts in a finitary manner on B. Thus, by the result of the previous paragraph, G centralizes B and hence it acts trivially on Ω. □

Now we consider the f.c. cover (C,N).

Step 2: G centralizes N/N' and $N' = N''$.

Proof. It is clear that $G = C/N$ acts on the abelian group N/N'. Furthermore, since $\mathbb{D}_C(N) = C$, it follows from Step 1 that G centralizes N/N'. In particular, N/N' is central in C/N'.

Next, we show that $N' = N''$. For this it suffices to assume that $N'' = 1$ and then prove that $N' = 1$. Set $A = N'$, so that C/N' acts on the abelian group A and set $Z = \mathbb{C}_A(C/N')$. Note that N is an f.c. group and therefore, if $x \in N/N'$, then $[A,x] \cong A/\mathbb{C}_A(x)$ is finite. But x is central in C/N', so $[A,x]$ is C/N'-stable and therefore $[A,x] \subseteq Z$ since C has no nontrivial f.c. factor group. It follows that N acts trivially on A/Z, so G acts on A/Z and Step 1 implies that G centralizes A/Z. In particular, C also centralizes A/Z and hence C stabilizes the chain $N \supseteq A \supseteq Z \supseteq 1$ and acts trivially on each factor. Thus C'' acts trivially on N. But $C = C''$, so N is central in C and, in particular, N is abelian. Thus $N' = 1$, as required. □

Step 3: *N is locally solvable.*

Proof. Let S be the largest normal locally solvable subgroup of N. Then S is characteristic in N and N/S has no nonidentity solvable normal subgroup. Thus it suffices to assume that $S = 1$ and then prove that $N = 1$.

Suppose by way of contradiction that $N \neq 1$. Since N is an f.c. group, it has a minimal nonidentity finite normal subgroup T. Since T is not solvable, it is semisimple and hence so is $D = T^C$, the normal closure of T in C. In other words, $D = \prod_i D_i$

is the weak direct product of finite nonabelian simple groups. As is well known, any normal subgroup of D is a partial direct product of the D_i's. In particular, N permutes the D_i's by conjugation and the minimal normal subgroups of N contained in D are precisely the products of D_i's over the finite N-orbits under this action. Thus, by combining factors, we see that $D = \prod_j M_j$ is the weak direct product of those (finite) minimal normal subgroups M_j of N which are contained in D.

Now C normalizes both N and D and therefore C permutes the set Ω of all such M_j by conjugation. Furthermore, since each $M_j \lhd N$, we see that N acts trivially on Ω and therefore that $G = C/N$ acts as permutations on this set. Notice that if $x \in C$, then $\mathbb{C}_D(x)$ has finite index in D and therefore contains a normal subgroup of D of finite index. This normal subgroup is clearly the weak direct product of all but finitely many D_i's and therefore it contains the weak direct product of all but finitely many of the M_j's. In other words, every element of G moves only finitely many points in its action on Ω. By Step 1, we conclude that G acts trivially on Ω and hence that each M_j is a finite normal subgroup of C. This means that $|C : \mathbb{C}_C(M_j)| < \infty$ and, since C has no nontrivial f.c. factor group, we see that each M_j is central in C. But this implies that $D \subseteq \mathbb{Z}(C)$, a contradiction since $D \neq 1$ is semisimple. □

Step 4: *N is central in C.*

Proof. Let M be any finite normal subgroup of N. Since $N/\mathbb{C}_N(M)$ is finite and N is locally solvable, it follows that $N/\mathbb{C}_N(M)$ is solvable. Thus $N^{(k)}$, the kth term of the derived series for N, is contained in $\mathbb{C}_N(M)$ for some k. But $N' = N''$, by Step 2, and therefore we see that N' centralizes all such M. Indeed, since N is locally normal, N is generated by its finite normal subgroups and therefore $N' \subseteq \mathbb{Z}(N)$. This implies that N' is abelian and, since $N' = N''$, we conclude that $N' = 1$. Thus N is abelian and, since G centralizes N/N' by Step 2, the lemma is proved. □

As a consequence, we have

Lemma 3.2. *Let G have an f.c. cover (C,N) with N not central in C. Then some nonidentity homomorphic image G/H of G is a finitary linear group over the field* GF(q) *for some prime $q \in \pi(N)$.*

Proof. Since N is not central in C, the previous lemma implies that, for some prime $q \in \pi(N)$, G acts in a finitary manner on an infinite elementary abelian q-group V with $\mathbb{C}_V(G) = 1$. In particular, if $H = \mathbb{C}_G(V)$, then G/H is a finitary linear group over GF(q). Furthermore, $G/H \neq 1$ since $\mathbb{C}_V(G/H) = \mathbb{C}_V(G) = 1$. □

By combining all of our results so far with the work of J. I. Hall and others, we can now prove

Proposition 3.3. *Let K be a field of characteristic $p > 0$ and let $N \lhd C$ be locally finite groups with $G = C/N$ infinite simple. If $C = \mathbb{D}_C(N)$ and if N is a p'-group, then any twisted group algebra $K^t[C]$ is semiprimitive.*

Proof. We begin with several reductions using results described in [8, Section 2]. To start with, if $\tilde{K}$ is the algebraic closure of K, then $\tilde{K}^t[C] = \tilde{K} \otimes_K K^t[C]$ is a twisted group algebra over $\tilde{K}$ with Jacobson radical satisfying $J\tilde{K}^t[C] \supseteq JK^t[C]$. Thus it suffices to show that $J\tilde{K}^t[C] = 0$ or equivalently we can now assume that $K = \tilde{K}$ is algebraically closed.

Next, let $\alpha \in JK^t[C]$ and let $H_1 \supseteq N$ be the subgroup of C generated by N and the support of α. Then H_1/N is a finite subgroup of the infinite simple group $C/N = G$ and hence there exists a countably infinite simple subgroup H/N of G with $H_1 \subseteq H$. But then

$$\alpha \in JK^t[C] \cap K^t[H] \subseteq JK^t[H]$$

so it suffices to prove that $K^t[H]$ is semiprimitive. In other words, we can now assume that $C = H$ or equivalently that G is a countably infinite simple group. We can of course also assume that C is not a p'-group and hence that G is not a p'-group.

Finally, let L be the subgroup of C generated by its p-elements. Then C/L is a locally finite p'-group, so $JK^t[C] = JK^t[L] \cdot K^t[C]$ and it suffices to prove that $K^t[L]$ is semiprimitive. Notice also that L is not contained in N, so LN/N is a nonidentity normal subgroup of G, and thus $L/(L \cap N) \cong LN/N = G$. In other words, L has the same structure as C, and we can therefore assume that $C = L$ is generated by its p-elements. With this, we see that (C,N) is an f.c. cover of G. Indeed, it is clear that (C,N) satisfies the defining conditions (i) and (ii). Furthermore, for (iii), let $M \neq C$ be any normal subgroup of C. Since C/M is generated by its p-elements and since N is a p'-group, it follows that $MN \neq C$. Thus, since $MN/N \lhd G$, we conclude that $MN = N$ and hence that $M \subseteq N$. But then $G = C/N$ is a homomorphic image of C/M, so C/M is not an f.c. group and condition (iii) is proved. There are now two cases to consider.

Case 1: *N is central in C.*

Proof. Let $\bar{C} = \{\bar{c} \mid c \in C\}$ be the group basis for $K^t[C]$ and let $x \in N \subseteq \mathbb{Z}(C)$. Then, $\mathbb{C}^t_C(x) = \{c \in C \mid \bar{c}\bar{x} = \bar{x}\bar{c}\}$ is a subgroup of C and, by [8, Section 2], $C/\mathbb{C}^t_C(x) = \mathbb{C}_C(x)/\mathbb{C}^t_C(x)$ is abelian. But (C,N) is an f.c. cover, so $C = C'$ and therefore $\mathbb{C}^t_C(x) = C$. We conclude that $K^t[N]$ is central in $K^t[C]$.

Now N is a p'-group, so $K^t[N]$ is a semiprimitive commutative algebra. Furthermore, since the group basis $\bar{N}$ is periodic modulo $K^\bullet = K \setminus 0$ and since K is algebraically closed, it follows that all irreducible representations of $K^t[N]$ are homomorphisms into K. Thus there exists a family $\{I_i \mid i \in \mathscr{I}\}$ of ideals of $K^t[N]$ of codimension 1 with $\bigcap_i I_i = 0$. Of course, these ideals are central in $K^t[C]$ and freeness implies that $\bigcap I_i K^t[C] = 0$. Therefore, it suffices to prove that each factor $K^t[C]/I_i K^t[C]$ is a semiprimitive ring. But it is easy to see that

$$K^t[C]/I_i K^t[C] = K^{t_i}[C/N] = K^{t_i}[G]$$

is a suitable twisted group algebra of $C/N = G$. Thus, since the main results of [9,10] imply that $JK^{t_1}[G] = 0$ for any locally finite infinite simple group G, the result follows in this case. □

Case 2: *N is not central in C.*

Proof. Since (C,N) is an f.c. cover of G with $N \nsubseteq \mathbb{Z}(G)$ and since G is infinite simple, it follows from the preceding lemma that G is a finitary linear group over $\mathrm{GF}(q)$ for some prime $q \in \pi(N)$. In particular, since N is a p'-group, it follows that $q \neq p$. At this point, we can apply the recent characterization of locally finite, infinite simple groups which are finitary linear groups.

Suppose first that G can be realized as a finite dimensional linear group over some field F_1. Then [6,11] imply that G is a simple group of Lie type over some infinite, locally finite field F_2, and this contradicts [8, Theorem 4.5]. Thus G cannot be realized in this manner and results of J. I. Hall apply. In particular, since G is countably infinite, it follows from [3–5] that $G \cong \mathrm{Alt}_\infty$, $\mathrm{FSL}_\infty(F)$, $\mathrm{FSp}_\infty(F)$, $\mathrm{FSU}_\infty(F)$ or $\mathrm{F}\Omega_\infty(F)$ for some locally finite field F and, indeed, F must have characteristic q. This latter fact follows from the work in [3, Section 7]. Specifically, if G is one of the latter four groups and if char $F = r$, then G has a classical sectional cover $\mathscr{Q}$ in characteristic r. Furthermore, since G is a finitary linear group in characteristic q, [3, Theorem 7.2] implies that $\mathscr{Q}$ has a subcover which is classical in characteristic q. Thus $r = q$, as claimed.

Finally, since char $F = q \neq p$, it follows that G satisfies the hypothesis of Theorem 2.1 or 2.2 and therefore G is strongly p-insulated. Proposition 1.3 now implies that C is p-insulated and hence that $K^t[C]$ is semiprimitive. This completes the proof of the second case and the result follows. □

The goal now is to generalize [9, Theorem 3.2]. As it turns out, the original argument can be considerably simplified by using the earlier techniques of [7]. We first recall several definitions and then we quote the necessary facts from the latter paper, but in the context of twisted group algebras.

Let G be a locally finite group. A finite subgroup A of G is said to be *locally subnormal*, written A lsn G, if A is subnormal in every finite subgroup B of G with $A \subseteq B$. For example, every finite subnormal subgroup of G is locally subnormal. Furthermore, if G is locally nilpotent, then every finite subgroup is locally subnormal. Note that if $A_1, A_2, \ldots, A_n$ are locally subnormal subgroups of G, then so is $\langle A_1, A_2, \ldots, A_n\rangle$, the group they generate. Consequently, G is generated by locally subnormal subgroups if and only if it is a union of locally subnormals. Furthermore, this property of G is inherited by subgroups and quotient groups.

Lemma 3.4. *Let $K^t[G]$ be a twisted group algebra of the locally finite group G over a field K of characteristic p.*

(i) *If A lsn G, then $JK^t[A] \subseteq JK^t[G]$.*

(ii) *If* $N \triangleleft G$ *with* $JK^t[N]$ *nilpotent, then* $JK^t[G] = JK^t[D] \cdot K^t[G]$ *where* $D = \mathbb{D}_G(N)$.

(iii) *Let* $N \triangleleft\triangleleft\, G$ *with* $JK^t[N] = 0$ *and suppose that*

$$N = N_0 \triangleleft N_1 \triangleleft \cdots \triangleleft N_m = G$$

with each quotient N_{i+1}/N_i *either a* p'*-group or generated by its locally subnormal subgroups. If* G *has no locally subnormal subgroup of order divisible by* p*, then* $JK^t[G] = 0$.

Proof. (i) It suffices to show that the right ideal $JK^t[A] \cdot K^t[G]$ is nil and this can be verified locally. Indeed, we need only show that $JK^t[A] \cdot K^t[B]$ is nil for any finite group B with $A \subseteq B \subseteq G$. But $A \triangleleft\triangleleft\, B$, so $JK^t[A] \cdot K^t[B]$ is contained in the nilpotent ideal $JK^t[B]$ and the result follows.

(ii) This is the twisted version of [7, Lemma 3.7] and it is proved in precisely the same manner.

(iii) By induction on m, it suffices to assume that $N \triangleleft G$ and that G/N is either a p'-group or is generated by its locally subnormal subgroups. In the former case we have $JK^t[G] = JK^t[N] \cdot K^t[G] = 0$, so we need only consider the latter situation. For this, let $\alpha \in JK^t[G]$ and choose $N \subseteq A \subseteq G$ with A/N lsn G/N and with $\alpha \in K^t[A]$. Notice that any locally subnormal subgroup of A is locally subnormal in G. Thus, by assumption, A has no locally subnormal subgroup of order divisible by p and, in particular, A has no finite normal subgroup of order divisible by p. Hence, by [8, Proposition 2.5], $K^t[A]$ is semiprime. Furthermore, since $JK^t[N] = 0$ and $|A : N| < \infty$, it follows from [8, Proposition 2.1] that $JK^t[A]$ is nilpotent, and therefore that $JK^t[A] = 0$ in this semiprime ring. Finally, we have

$$\alpha \in JK^t[G] \cap K^t[A] \subseteq JK^t[A] = 0,$$

so $\alpha = 0$ as required. $\square$

The next result is well known and allows us to better understand the nature of the groups considered in Theorem 3.6. We include a full proof for the sake of completeness.

Lemma 3.5. *Let* G *be an infinite locally finite simple group.*

(i) G *is not locally solvable.*

(ii) G *has no nonidentity locally subnormal subgroup.*

Proof. (i) Certainly G is nonabelian, so we can choose $a, b \in G$ with $1 \neq c = [a, b]$. Furthermore, since G is simple it follows that $G = \langle c \rangle^G = \langle c^g \mid g \in G \rangle$, and hence there exists a finite subgroup H of G containing a, b, c and with $a, b \in \langle c \rangle^H = C$. Now C is a nonidentity normal subgroup of H and $a, b \in C$, so $c = [a, b] \in C'$. But then $\langle c \rangle^H \subseteq C' \triangleleft H$, so $C = C'$ and G is not locally solvable.

(ii) Suppose by way of contradiction that G contains a nonidentity locally subnormal subgroup A. We can assume that A has minimal order and therefore that A is simple.

Furthermore, since G is simple, we have $G = A^G = \langle A^g \mid g \in G \rangle$. Suppose first that A is abelian, so that A has prime order q. If $B = \langle A^{g_1}, A^{g_2}, \ldots, A^{g_t} \rangle$, then each A^{g_i} is a subnormal q-subgroup of B and hence is contained in $\mathbb{O}_q(B)$. Thus B is a q-group, so G is locally nilpotent and this contradicts part (i) above. On the other hand, if A is nonabelian simple, then so is each A^g, and by considering the subgroup $C = \langle A, A^g \rangle$, we see that A^g normalizes A. Thus $A \triangleleft G$ and again we have a contradiction. □

We can now easily prove the main result of this paper.

Theorem 3.6. *Let $K^t[G]$ be a twisted group algebra of the locally finite group G over a field K of characteristic $p > 0$. Suppose that G has a finite subnormal series*

$$1 = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_n = G$$

with each quotient G_{i+1}/G_i either
(i) *a p'-group, or*
(ii) *a nonabelian simple group, or*
(iii) *generated by locally subnormal subgroups.*
If G has no locally subnormal subgroup of order divisible by p, then $K^t[G]$ is semiprimitive.

Proof. For any group G with a subnormal series as above, let $\rho(G)$ denote the number of infinite simple factors of the series which are not p'-groups. Since any two finite subnormal series have equivalent refinements, it follows from the preceding lemma that $\rho(G)$ is finite and well defined. Notice that if $M \triangleleft G$, then M and G/M have subnormal series of the same type, and certainly $\rho(G) = \rho(M) + \rho(G/M)$. Notice also that if $S \triangleleft\triangleleft\, G$, then S has no locally subnormal subgroup of order divisible by p. The proof now proceeds by induction on $\rho(G)$.

If $\rho(G) = 0$, then the result follows from Lemma 3.4(iii) with $N = 1$. Thus we may suppose that $\rho(G) \geq 1$, and we let k be the largest subscript with G_k/G_{k-1} infinite simple and not a p'-group. In view of Lemma 3.4(iii) again, it suffices to prove that $JK[G_k] = 0$. Indeed, since $\rho(G_k) = \rho(G)$, we may assume that $G_k = G$. In other words, G has a normal subgroup H with G/H infinite simple and not a p'-group. Note that $\rho(H) < \rho(G)$ and therefore, by induction, $JK^t[H] = 0$. Thus, Lemma 3.4(ii) implies that $JK^t[G] = JK^t[C] \cdot K^t[G]$ where $C = \mathbb{D}_G(H)$.

If $C \subseteq H$, then $\rho(C) \leq \rho(H) < \rho(G)$, so $JK^t[C] = 0$ and hence $JK^t[G] = 0$. Thus, we may suppose that $C \nsubseteq H$ and therefore, since H is maximal normal, we have $G = HC$. In particular, if $N = H \cap C$, then $C/N \cong G/H$ is infinite simple. Furthermore, since $N \subseteq H$, we have $\mathbb{D}_C(N) = C$. Consequently, N is an f.c. group and, since N has no locally subnormal subgroup of order divisible by p, it follows that N is a p'-group. Proposition 3.3 now implies that $JK^t[C] = 0$ and therefore $JK^t[G] = JK^t[C] \cdot K^t[G] = 0$, as required. □

Notice that if G has a locally subnormal subgroup of order divisible by p and if K is a perfect field, then Lemma 3.4(i) implies that $JK^t[G] \neq 0$. Thus the locally subnormal hypothesis is definitely required in the preceding theorem. In fact, in the case of ordinary group algebras, this observation holds even when K is not perfect. Thus Theorem 3.6 and Lemma 3.4(i) yield the Main Theorem, as stated in the Introduction. Of course, it would be nice to obtain a complete description of $JK^t[G]$ even when locally subnormal subgroups of order divisible by p do exist. If there is a bound on the p-parts of such subgroups, then the techniques of [8, Theorem 6.1] easily solve the problem. But the more general situation seems to require new group theoretic ideas and we leave this for a later project.

4. Finitary linear groups II

The goal of this final section is to obtain the characteristic p analog of Theorem 2.2. While it is not needed for the particular semiprimitivity problem studied in this paper, this result may nevertheless have later applications and it is certainly interesting in its own right. As will be apparent, the proof of Theorem 4.5 pushes the methods of [10] to their limit.

To start with, we need the following extension of [10, Lemma 5.1]. Here we use the notation $\{b,c\}\{a,c\} = 0$ to indicate that $b_j a_{i'} = b_j c_{i',j'} = c_{i,j} a_{i'} = c_{i,j} c_{i',j'} = 0$ for all appropriate subscripts i, j, i', j'.

Lemma 4.1. *Let e be the $\ell \times \ell$ identity matrix, let $g \in \mathrm{GL}_\ell(F)$, and consider the partitioned $n\ell \times n\ell$ matrix $\mathcal{M}$ given by*

$$\begin{pmatrix} \lambda_n e & -g & & & \\ & \lambda_{n-1} e & -g & & \\ & & \ddots & \ddots & \\ & & & \lambda_2 e & -g \\ & & & & \lambda_1 e \end{pmatrix} + \begin{pmatrix} a_{n-1} & c_{n-1,n-1} & c_{n-1,n-2} & \cdots & c_{n-1,1} \\ a_{n-2} & c_{n-2,n-1} & c_{n-2,n-2} & \cdots & c_{n-2,1} \\ \vdots & \vdots & \vdots & & \vdots \\ a_1 & c_{1,n-1} & c_{1,n-2} & \cdots & c_{1,1} \\ d & b_{n-1} & b_{n-2} & \cdots & b_1 \end{pmatrix}$$

where $\lambda_1, \lambda_2, \dots, \lambda_n \in F$. We assume that most products of the a's, b's and c's are zero and write the precise assumption symbolically by $\{b,c\}\{a,c\} = 0$. Furthermore, we suppose that $gc_{i,j} = c_{i,j} = c_{i,j}g$ and $b_j g = b_j$ for all appropriate subscripts $1 \le i,j \le n-1$. If $\mathcal{N}$ is the $\ell \times \ell$ matrix given by

$$\mathcal{N} = \lambda_1\lambda_2\cdots\lambda_n e + g^{n-1}d + \sum_{i=1}^{n-1}\Big(\prod_{u\le i}\lambda_u\Big)g^{n-1-i}a_i$$
$$+\sum_{j=1}^{n-1}\Big(\prod_{v>j}\lambda_v\Big)g^{n-1}b_j + \sum_{i=1}^{n-1}\sum_{j=1}^{n-1}\Big(\prod_{u\le i}\lambda_u\Big)\Big(\prod_{v>j}\lambda_v\Big)c_{i,j}$$

then $\det\mathcal{M} = \det\mathcal{N}$.

Proof. If $w = \operatorname{diag}(e, g, g^2, \ldots, g^{n-1})$, then the multiplicative nature of g implies that $w\mathcal{M}w^{-1}$ is the matrix $\mathcal{M}'$ given by

$$\begin{pmatrix} \lambda_n e & -e & & & \\ & \lambda_{n-1} e & -e & & \\ & & \ddots & \ddots & \\ & & & \lambda_2 e & -e \\ & & & & \lambda_1 e \end{pmatrix} + \begin{pmatrix} a'_{n-1} & c_{n-1,n-1} & c_{n-1,n-2} & \cdots & c_{n-1,1} \\ a'_{n-2} & c_{n-2,n-1} & c_{n-2,n-2} & \cdots & c_{n-2,1} \\ \vdots & \vdots & \vdots & & \vdots \\ a'_1 & c_{1,n-1} & c_{1,n-2} & \cdots & c_{1,1} \\ d' & b'_{n-1} & b'_{n-2} & \cdots & b'_1 \end{pmatrix}$$

where $a'_i = g^{n-1-i}a_i$, $d' = g^{n-1}d$ and $b'_j = g^{n-1}b_j$ for all subscripts i, j. Furthermore, $\{b', c\}\{a', c\} = 0$ by the original product assumption and the multiplicative nature of g. Thus [10, Lemma 5.1] applies to $\mathcal{M}'$ and, since $\det \mathcal{M} = \det \mathcal{M}'$, the lemma is proved. □

The next four results use the following notation. Let F be a field, let ℓ and n be positive integers, and let $\tilde{\ }: \mathrm{GL}_\ell(F) \to \mathrm{GL}_{n\ell}(F)$ be the corner embedding. Fix $x \in \mathrm{GL}_\ell(F)$, suppose e is the $\ell \times \ell$ identity matrix, and define the partitioned $n\ell \times n\ell$ matrices y and z by

$$y = \begin{pmatrix} e & \beta_{n-1} & \cdots & \beta_2 & \beta_1 \\ \alpha_{n-1} & e & & & \\ \vdots & & \ddots & & \\ \alpha_2 & & & e & \\ \alpha_1 & & & & e \end{pmatrix} \quad \text{and} \quad z = \begin{pmatrix} & e & & & \\ & & e & & \\ & & & \ddots & \\ & & & & e \\ e & & & & \end{pmatrix}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}, \beta_1, \beta_2, \ldots, \beta_{n-1}$ are any $\ell \times \ell$ matrices over F with $\alpha_i\beta_j = \beta_j\alpha_i = 0$ for all i, j. Set $z_1 = yzy^{-1}$, let $g \in \mathrm{GL}_\ell(F)$ satisfy $\alpha_i g = \alpha_i = g\alpha_i$ and $\beta_i g = \beta_i = g\beta_i$ for all i, and define $\hat{g} = \operatorname{diag}(g, g, \ldots, g)$. Our goal is to compute the characteristic polynomial of $z_1^u \hat{g}\tilde{x}$ for various integers u and then to show that these polynomials yield information about x. We start with

Lemma 4.2. *Use the above notation and assume that u is relatively prime to n. Then the characteristic polynomial $\psi_x(\lambda) \in F[\lambda]$ of the matrix $z_1^u\hat{g}\tilde{x}$ is equal to $\det N$ where*

$$N = \lambda^n e - g^n x + \sum_{i=1}^{n-1} \lambda^i \alpha_{iu}(x - e) + \sum_{j=1}^{n-1} \lambda^{n-j} g^{n-1}(e - gx)\beta_{ju} + \sum_{i=1}^{n-1}\sum_{j=1}^{n-1} \lambda^{n+i-j}\alpha_{iu} x \beta_{ju}$$

and where the subscripts are computed modulo n.

Proof. For convenience, we consider two separate cases.

Case 1: $u = 1$.

Proof. Since $\alpha_i\beta_j = \beta_j\alpha_i = 0$, we see that y is the identity plus a matrix of square 0 and therefore

$$y^{-1} = \begin{pmatrix} e & -\beta_{n-1} & \cdots & -\beta_2 & -\beta_1 \\ -\alpha_{n-1} & e & & & \\ \vdots & & \ddots & & \\ -\alpha_2 & & & e & \\ -\alpha_1 & & & & e \end{pmatrix}.$$

Thus, since $\tilde{x} = \mathrm{diag}(x, e, \ldots, e)$, the multiplicative property of g implies that

$$y^{-1}\hat{g}\tilde{x}y = \hat{g} + \begin{pmatrix} -g(e-x) & (gx-e)\beta_{n-1} & \cdots & (gx-e)\beta_1 \\ \alpha_{n-1}(e-x) & -\alpha_{n-1}x\beta_{n-1} & \cdots & -\alpha_{n-1}x\beta_1 \\ \vdots & \vdots & & \vdots \\ \alpha_1(e-x) & -\alpha_1 x\beta_{n-1} & \cdots & -\alpha_1 x\beta_1 \end{pmatrix}.$$

Let $E = \mathrm{diag}(e, e, \ldots, e)$ be the $n\ell \times n\ell$ identity matrix.

Now the nature of z implies that the product $z(y^{-1}\hat{g}\tilde{x}y)$ is obtained from $y^{-1}\hat{g}\tilde{x}y$ by cyclically permuting its n rows. Thus $\lambda E - zy^{-1}\hat{g}\tilde{x}y$ is precisely the matrix $\mathcal{M}$ of Lemma 4.1 provided we set $a_i = \alpha_i(x-e)$, $b_j = (e-gx)\beta_j$, $c_{i,j} = \alpha_i x\beta_j$ and $d = -gx$. Of course, $\lambda_i = \lambda$ for all i. Furthermore, note that $\alpha_i\beta_j = \beta_j\alpha_i = 0$ for all i,j implies that $\{b,c\}\{a,c\} = 0$. Thus, by the multiplicative nature of g, we can apply the previous lemma to compute $\psi_x(\lambda) = \det(\lambda E - zy^{-1}\hat{g}\tilde{x}y) = \det\mathcal{M}$, the characteristic polynomial of $zy^{-1}\hat{g}\tilde{x}y$. Indeed, since $\prod_{u\le i}\lambda_u = \lambda^i$ and $\prod_{v>j}\lambda_v = \lambda^{n-j}$, we conclude that $\psi_x(\lambda) = \det\mathcal{M} = \det\mathcal{N}$ where

$$\mathcal{N} = \lambda^n e - g^n x + \sum_{i=1}^{n-1}\lambda^i\alpha_i(x-e) + \sum_{j=1}^{n-1}\lambda^{n-j}g^{n-1}(e-gx)\beta_j + \sum_{i=1}^{n-1}\sum_{j=1}^{n-1}\lambda^{n+i-j}\alpha_i x\beta_j.$$

Finally, since $zy^{-1}\hat{g}\tilde{x}y = y^{-1}(z_1\hat{g}\tilde{x})y$, we see that $\psi_x(\lambda)$ is also the characteristic polynomial of $z_1\hat{g}\tilde{x}$. □

Case 2: $\gcd(u,n) = 1$.

Proof. Since u is relatively prime to n it follows that z and z^u are conjugate in $\mathrm{GL}_{n\ell}(F)$. Specifically, let w be the partitioned $n\ell \times n\ell$ matrix

$$w = \begin{pmatrix} w_{n,n} & w_{n,n-1} & \cdots & w_{n,1} \\ \vdots & \vdots & & \vdots \\ w_{2,n} & w_{2,n-1} & \cdots & w_{2,1} \\ w_{1,n} & w_{1,n-1} & \cdots & w_{1,1} \end{pmatrix},$$

where $w_{i,j} = e$ if $j \equiv iu \mod n$ and $w_{i,j} = 0$ otherwise. Since $\gcd(u,n) = 1$, it is clear that w is a permutation matrix which commutes with $\hat{g}$ and, since $w_{n,n} = e$, it follows that w commutes with $\tilde{x}$. Furthermore, define

$$y_u = \begin{pmatrix} e & \beta_{(n-1)u} & \cdots & \beta_{2u} & \beta_{1u} \\ \alpha_{(n-1)u} & e & & & \\ \vdots & & \ddots & & \\ \alpha_{2u} & & & e & \\ \alpha_{1u} & & & & e \end{pmatrix},$$

where the subscripts are viewed modulo n. Then it is easy to verify that $zw = wz^u$ and $y_u w = wy$, so

$$z_1^u \hat{g}\tilde{x} = yz^u y^{-1}\hat{g}\tilde{x} = w^{-1}(y_u z y_u^{-1}\hat{g}\tilde{x})w$$

and the characteristic polynomial of $z_1^u\hat{g}\tilde{x}$ is the same as that of $y_u z y_u^{-1}\hat{g}\tilde{x}$. In other words, the only difference between this situation and the $u = 1$ case is that y is replaced by y_u or equivalently that each α_i is replaced by α_{iu} and each β_i is replaced by β_{iu}. Case 1 now yields the result. □

For more general exponents u, let $v = \gcd(u,n)$ be the greatest common divisor of u and n, and set $m = n/v$ and $\bar{u} = u/v$.

Lemma 4.3. *We continue with the above notation and assumptions, and we let $\psi_x(\lambda)$ denote the characteristic polynomial of $z_1^u\hat{g}\tilde{x}$. Furthermore, let N be an $\ell \times \ell$ matrix*

$$\begin{aligned} N &= \lambda^m e - g^m x + \sum_{i=1}^{m-1} \lambda^i \alpha_{iu}(x-e) + \sum_{j=1}^{m-1} \lambda^{m-j} g^{m-1}(e-gx)\beta_{ju} \\ &\quad + \sum_{i=1}^{m-1}\sum_{j=1}^{m-1} \lambda^{m+i-j}\alpha_{iu}x\beta_{ju}, \end{aligned}$$

where the subscripts are computed modulo n. Then $\det N$ *divides $\psi_x(\lambda)$ in the polynomial ring $F[\lambda]$. In fact, if $\alpha_i = \beta_i = 0$ for all i not divisible by v, then $\psi_x(\lambda) =$* $\det N \cdot (\det M)^{v-1}$ *where $M = \lambda^m e - g^m$.*

Proof. In view of the preceding lemma, it suffices to assume that $v > 1$. We again permute the rows and columns of y and z. To this end, recall that $n = mv$ and let w be the $n\ell \times n\ell$ partitioned matrix

$$w = \begin{pmatrix} w_{n,n} & w_{n,n-1} & \cdots & w_{n,1} \\ \vdots & \vdots & & \vdots \\ w_{2,n} & w_{2,n-1} & \cdots & w_{2,1} \\ w_{1,n} & w_{1,n-1} & \cdots & w_{1,1} \end{pmatrix},$$

where $w_{i,j} = e$ if there exist integers r, s with

$$\begin{aligned} n - i &\equiv rm + s \mod \mathrm{n} \quad 0 \le s \le m-1, \\ n - j &\equiv sv + r \mod \mathrm{n} \quad 0 \le r \le v-1, \end{aligned}$$

and $w_{i,j} = 0$ otherwise. Since i uniquely determines r,s, it follows that w is a permutation matrix. Specifically, there is a permutation π of the set $\{1,2,\dots,n\}$ such that $w_{i,j} = e$ if and only if $j = \pi i$. Thus w commutes with $\hat{g}$ and, since $w_{n,n} = e$, we see that w commutes with $\tilde{x}$.

Now let $\bar{y}$ and $\bar{z}$ be the $m\ell \times m\ell$ partitioned matrices

$$\bar{y} = \begin{pmatrix} e & \beta_{(m-1)v} & \cdots & \beta_{2v} & \beta_{1v} \\ \alpha_{(m-1)v} & e & & & \\ \vdots & & \ddots & & \\ \alpha_{2v} & & & e & \\ \alpha_{1v} & & & & e \end{pmatrix} \quad \text{and} \quad \bar{z} = \begin{pmatrix} & e & & & \\ & & e & & \\ & & & \ddots & \\ & & & & e \\ e & & & & \end{pmatrix}.$$

Furthermore, let y_π and z_π denote the $n\ell \times n\ell$ matrices

$$y_\pi = \begin{pmatrix} e & \beta_{\pi(n-1)} & \cdots & \beta_{\pi 2} & \beta_{\pi 1} \\ \alpha_{\pi(n-1)} & e & & & \\ \vdots & & \ddots & & \\ \alpha_{\pi 2} & & & e & \\ \alpha_{\pi 1} & & & & e \end{pmatrix}$$

and $z_\pi = \mathrm{diag}(\bar{z},\bar{z},\dots,\bar{z})$. Then it is easy to verify that $wy = y_\pi w$, $wz^v = z_\pi w$ and that $\bar{y}$ is the upper left-hand corner of y_π.

Set $u = v\bar{u}$ and observe that $w^{-1}z_\pi^{\bar{u}}w = (w^{-1}z_\pi w)^{\bar{u}} = z^{v\bar{u}} = z^u$. Thus

$$z_1^u\hat{g}\tilde{x} = yz^u y^{-1}\hat{g}\tilde{x} = w^{-1}(y_\pi z_\pi^{\bar{u}} y_\pi^{-1}\hat{g}\tilde{x})w$$

and hence $\psi_x(\lambda)$ is equal to the characteristic polynomials of both $y_\pi z_\pi^{\bar{u}} y_\pi^{-1}\hat{g}\tilde{x}$ and $z_\pi^{\bar{u}} y_\pi^{-1}\hat{g}\tilde{x}y_\pi$. Since $v = \gcd(u,n)$, it follows that $\bar{u}$ is relatively prime to m. Thus, using $u = v\bar{u}$, the previous lemma implies that both $\bar{y}\bar{z}^{\bar{u}}\bar{y}^{-1}\hat{g}\tilde{x}$ and $\bar{z}^{\bar{u}}\bar{y}^{-1}\hat{g}\tilde{x}\bar{y}$ have characterisitic polynomials equal to $\det N$, where N is the given $\ell \times \ell$ matrix. Here, of course, $\hat{g}$ and $\tilde{x}$ are suitably truncated $m\ell \times m\ell$ versions of the original matrices. Let us first consider a simple special case.

Case 1: Suppose $\alpha_i = \beta_i = 0$ for all i not divisible by v.

Proof. The additional assumption implies that $y_\pi = \mathrm{diag}(\bar{y},\bar{e},\bar{e},\dots,\bar{e})$ where $\bar{e}$ is the $m\ell \times m\ell$ identity matrix. Thus since $y_\pi, z_\pi, \hat{g}$ and $\tilde{x}$ are all in block diagonal form, it follows that $\psi_x(\lambda) = \sigma(\lambda)\tau(\lambda)^{v-1}$ where $\sigma(\lambda)$ is the characteristic polynomial of the $m\ell \times m\ell$ matrix $S = \bar{y}\bar{z}^{\bar{u}}\bar{y}^{-1}\hat{g}\tilde{x}$ and where $\tau(\lambda)$ is the characteristic polynomial of $T = \bar{z}^{\bar{u}}\hat{g} = \bar{e}\bar{z}^{\bar{u}}\bar{e}^{-1}\hat{g}\bar{e}$. Again, in S and T, we let $\hat{g}$ and $\tilde{x}$ denote suitably truncated $m\ell \times m\ell$ versions of the original matrices. As we observed above, $\sigma(\lambda) = \det N$ and, of course, T is just a special case of the matrix S with $x = e$ and with all $\alpha_{iv} = \beta_{jv} = 0$. Thus $\tau(\lambda) = \det M$ where $M = \lambda^m e - g^m$ is obtained from N by setting $x = e$ and $\beta_{jv} = 0$. With this, the special case is proved. □

Case 2: The general situation.

Proof. Here there are no additional assumptions on the α_i's and β_j's. Write

$$y_\pi = \begin{pmatrix} \bar{y} & B \\ A & \bar{E} \end{pmatrix} \quad \text{and} \quad z_\pi = \begin{pmatrix} \bar{z} & \\ & \bar{Z} \end{pmatrix},$$

where A has first column consisting of various α_i's and B has first row consisting of various β_j's. Furthermore, $\bar{E}$ denotes the $(n-m)\ell \times (n-m)\ell$ identity and $\bar{Z} = \mathrm{diag}(\bar{z}, \bar{z}, \ldots, \bar{z})$. Since $v > 1$, we have $n - m > 0$, and since $\alpha_i\beta_j = \beta_j\alpha_i = 0$, we know that

$$y_\pi^{-1} = \begin{pmatrix} \bar{y}^{-1} & -B \\ -A & \bar{E} \end{pmatrix}.$$

Again, we use suitably truncated versions of $\hat{g}$ and $\tilde{x}$ in the formulas that follow. In fact, $\hat{g}$ exists at three different sizes, but this should cause no confusion.

Note that

$$\begin{aligned} y_\pi^{-1}\hat{g}\tilde{x}y_\pi &= \begin{pmatrix} \bar{y}^{-1} & -B \\ -A & \bar{E} \end{pmatrix} \begin{pmatrix} \hat{g} & \\ & \hat{g} \end{pmatrix} \begin{pmatrix} \tilde{x} & \\ & \bar{E} \end{pmatrix} \begin{pmatrix} \bar{y} & B \\ A & \bar{E} \end{pmatrix} \\ &= \begin{pmatrix} \bar{y}^{-1}\hat{g}\tilde{x}\bar{y} & \bar{y}^{-1}\hat{g}\tilde{x}B - B \\ -A\tilde{x}\bar{y} + A & -A\tilde{x}B + \hat{g} \end{pmatrix} \end{aligned}$$

since $A\hat{g} = A = \hat{g}A$, $B\hat{g} = B$ and $BA = 0$. Thus

$$z_\pi^{\bar{u}} y_\pi^{-1}\hat{g}\tilde{x}y_\pi = \begin{pmatrix} M' & B' \\ A' & C' \end{pmatrix},$$

where

$$\begin{aligned} M' &= \bar{z}^{\bar{u}}\bar{y}^{-1}\hat{g}\tilde{x}\bar{y}, & A' &= \bar{Z}^{\bar{u}}(-A\tilde{x}\bar{y} + A), \\ B' &= \bar{z}^{\bar{u}}(\bar{y}^{-1}\hat{g}\tilde{x}B - B), & C' &= \bar{Z}^{\bar{u}}(-A\tilde{x}B + \hat{g}). \end{aligned}$$

In particular, since we know that $\psi_x(\lambda)$ is equal to the characteristic polynomial of $z_\pi^{\bar{u}} y_\pi^{-1}\hat{g}\tilde{x}y_\pi$, we have

$$\psi_x(\lambda) = \det \begin{pmatrix} \lambda\bar{e} - M' & -B' \\ -A' & \lambda\bar{E} - C' \end{pmatrix},$$

where $\bar{e}$ denotes the $m\ell \times m\ell$ identity.

We now work over the rational function field $F(\lambda)$ and let $\mathscr{A}$ denote the right ideal of the $\ell \times \ell$ matrix ring $M_\ell(F(\lambda))$ generated by $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$. In addition, let $\mathscr{V}$ be the set of all $(n-m)\ell \times m\ell$ partitioned matrices with $\ell \times \ell$ entries in $\mathscr{A}$. Then $\mathscr{V}$ is a finite-dimensional $F(\lambda)$-vector space and $A' \in \mathscr{V}$ since $\bar{Z}^{\bar{u}}$ is a partitioned permutation matrix. Furthermore, since $\hat{g}\mathscr{V} = \mathscr{V}$ and $B\mathscr{V} = 0$, it follows that $(\lambda\bar{E} - C')\mathscr{V} \subseteq \mathscr{V}$. Note that $\lambda\bar{E} - C'$ is an invertible matrix, since its determinant is a monic polynomial in λ, and therefore left multiplication by $\lambda\bar{E} - C'$ affords a one-to-one linear transformation on $\mathscr{V}$. Thus this linear transformation must be onto and there exists $A'' \in \mathscr{V}$ with $(\lambda\bar{E} - C')A'' = A'$. Of course, $B\mathscr{V} = 0$ implies that $B'A'' = 0$.

Finally, note that

$$\begin{pmatrix} \lambda\bar{e} - M' & -B' \\ -A' & \lambda\bar{E} - C' \end{pmatrix} \begin{pmatrix} \bar{e} & \\ A'' & \bar{E} \end{pmatrix} = \begin{pmatrix} \lambda\bar{e} - M' & -B' \\ & \lambda\bar{E} - C' \end{pmatrix}$$

since $B'A'' = 0$ and $(\lambda\bar{E} - C')A'' = A'$. Thus since

$$\det \begin{pmatrix} \bar{e} & \\ A'' & \bar{E} \end{pmatrix} = 1,$$

we conclude that

$$\psi_x(\lambda) = \det \begin{pmatrix} \lambda\bar{e} - M' & -B' \\ & \lambda\bar{E} - C' \end{pmatrix} = \det(\lambda\bar{e} - M') \cdot \det(\lambda\bar{E} - C'),$$

a factorization in the polynomial ring $F[\lambda]$. Furthermore, $\det(\lambda\bar{e} - M')$ is the characteristic polynomial of $M' = \bar{z}^{\bar{u}}\bar{y}^{-1}\hat{g}\tilde{x}\bar{y}$ and thus $\det(\lambda\bar{e} - M') = \det N$, as we observed previously. In other words, we have shown that $\det N$ divides $\psi_x(\lambda)$ in $F[\lambda]$ and the result follows. $\square$

In order to proceed further, we need some additional notation and assumptions. Let γ be a fixed positive integer, define $\Gamma = \{\, i \mid \alpha_i \neq 0 \text{ or } \beta_i \neq 0 \,\}$ and assume that $|\Gamma| \leq \gamma$. Furthermore, suppose that, for any $i_0 \in \Gamma$, the integer equation

$$i_0 \equiv \sum_{i \in \Gamma} \mu(i)i \mod \mathrm{n}$$

with $-\gamma \leq \mu(i) \leq \gamma$ has only the trivial solution $\mu(i_0) = 1$ and $\mu(i) = 0$ otherwise. Finally, if n is even, we assume that $n/2 \notin \Gamma$.

Lemma 4.4. *We continue with the above notation and assumptions. In addition, we suppose that each α_i and β_j has rank ≤ 1, and that $\alpha_i = \beta_i = 0$ if i is not divisible by v. Then u and the characteristic polynomial $\psi_x(\lambda)$ of $z_1^u\hat{g}\tilde{x}$ determine the $\ell \times \ell$ matrix traces* $\operatorname{tr} \alpha_{iv}(x^{-1} - e)$ *for all $1 \leq i \leq m - 1$.*

Proof. Let Γ_u be the set of all integers $1 \leq i \leq m - 1$ with $iu \in \Gamma$ modulo n. Then certainly $|\Gamma_u| \leq \gamma$. Furthermore, let $i_0 \in \Gamma_u$ and suppose that

$$i_0 \equiv \sum_{i \in \Gamma_u} \mu(i)i \mod m$$

with $-\gamma \leq \mu(i) \leq \gamma$ for all i. Then

$$i_0 u \equiv \sum_{i \in \Gamma_u} \mu(i)iu \mod n$$

and, of course, each $iu \in \Gamma$ modulo n. Moreover, if $i_1 u \equiv i_2 u \mod \mathrm{n}$, then since v divides u and n, we have $i_1\bar{u} \equiv i_2\bar{u} \mod \mathrm{m}$. But $\gcd(\bar{u}, m) = 1$, so $i_1 \equiv i_2 \mod \mathrm{m}$

and hence $i_1 = i_2$. In other words, the preceding congruence describing i_0u involves distinct elements of Γ on the right and therefore we conclude from our assumption that $\mu(i_0) = 1$ and that $\mu(i) = 0$ otherwise. Finally, if $i_0 = m/2$, then n is even and $i_0u \in \Gamma$ modulo n is divisible by $n/2$. But $0, n/2 \notin \Gamma$, so this cannot occur.

Since $\alpha_i = \beta_i = 0$ if i is not divisible by v, the previous lemma implies that $\psi_x(\lambda) = \det N \cdot (\det M)^{v-1}$ where $M = \lambda^m e - g^m$ and

$$\begin{aligned} N &= \lambda^m e - g^m x + \sum_{i=1}^{m-1} \lambda^i \alpha_{iu}(x-e) \\ &\quad + \sum_{j=1}^{m-1} \lambda^{m-j} g^{m-1}(e-gx)\beta_{ju} + \sum_{i=1}^{m-1}\sum_{j=1}^{m-1} \lambda^{m+i-j}\alpha_{iu} x \beta_{ju} \end{aligned}$$

with subscripts computed modulo n.

Now let $i_0 \in \Gamma_u$ and let $f(i_0)$ denote the coefficient of λ^{i_0} in $\psi_x(\lambda)$. Since

$$\begin{aligned} \psi_x(0) &= \det(-g^m x) \cdot \big(\det(-g^m)\big)^{v-1} \\ &= \det(g^m) \cdot \det(-x) \cdot \big(\det(-g^m)\big)^{v-1} \neq 0, \end{aligned}$$

it follows that $f(i_0)/\psi_x(0)$ is the coefficient of λ^{i_0} in $(\det N') \cdot (\det M')^{v-1}$ where $M' = -g^{-m}M = e - \lambda^m g^{-m}$ and

$$\begin{aligned} N' &= -g^{-m} N x^{-1} \\ &= e - \lambda^m g^{-m} x^{-1} + \sum_{i=1}^{m-1} \lambda^i \alpha_{iu}(x^{-1} - e) \\ &\quad + \sum_{j=1}^{m-1} \lambda^{m-j}(x-e)\beta_{ju}x^{-1} - \sum_{i=1}^{m-1}\sum_{j=1}^{m-1} \lambda^{m+i-j}\alpha_{iu} x \beta_{ju} x^{-1} \end{aligned}$$

since $g\alpha_i = \alpha_i$ and $g\beta_j = \beta_j$.

Note that $i_0 < m$, so the matrices $\lambda^m g^{-m} x^{-1}$ and $\lambda^m g^{-m}$ do not contribute to the λ^{i_0} coefficient. Thus, by definition of Γ_u, we see that $f(i_0)/\psi_x(0)$ is the coefficient of λ^{i_0} in $\det N''$ where

$$\begin{aligned} N'' &= e + \sum_{i \in \Gamma_u} \lambda^i \alpha_{iu}(x^{-1} - e) + \sum_{j \in \Gamma_u} \lambda^{m-j}(x-e)\beta_{ju}x^{-1} \\ &\quad - \sum_{i \in \Gamma_u}\sum_{j \in \Gamma_u} \lambda^{m+i-j}\alpha_{iu} x \beta_{ju} x^{-1}. \end{aligned}$$

If $\Gamma_u^{\#} = \{0\} \cup \Gamma_u$, then the multilinearity of the determinant function implies that

$$\det N'' = \sum_{\omega \in \Omega} \det C_\omega,$$

where Ω is the set of all functions

$$\omega \colon \{1, 2, \ldots, \ell\} \to \Gamma_u^{\#} \times \Gamma_u^{\#}$$

and where the kth row of C_ω is equal to the kth row of

$$\begin{array}{ll} e, & \text{if } \omega(k)=(0,0),\\ \lambda^i\alpha_{iu}(x^{-1}-e), & \text{if } \omega(k)=(i,0),\\ \lambda^{m-j}(x-e)\beta_{ju}x^{-1}, & \text{if } \omega(k)=(0,j),\\ -\lambda^{m+i-j}\alpha_{iu}x\beta_{iu}x^{-1}, & \text{if } \omega(k)=(i,j). \end{array}$$

Observe that the ranks of the latter three matrices are all ≤ 1 since, by assumption, the α_i's and β_j's have rank at most 1. Thus if two rows of C_ω come from the same one of these three matrices, then $\det C_\omega = 0$. Thus we can restrict our attention to those $\omega \in \Omega$ with the property that each element of $\Gamma_u^\# \times \Gamma_u^\#$ not $(0,0)$ occurs at most once as an image point. Furthermore, notice that $\det C_\omega$ is a scalar times a power of λ with the exponent of λ equal to the sum of the contributions from each of the ℓ rows. Specifically, the degree of the kth row of C_ω is equal to

$$\begin{array}{ll} 0, & \text{if } \omega(k)=(0,0),\\ i, & \text{if } \omega(k)=(i,0),\\ m-j, & \text{if } \omega(k)=(0,j),\\ i+(m-j), & \text{if } \omega(k)=(i,j). \end{array}$$

If $\omega(k)=(i,j)$, let us call i a left image of ω and j a right image. It follows from the above that if $t\in\Gamma_u$ is both a left and a right image of ω, then $\det C_\omega$ has degree $\geq t+(m-t)=m$.

Suppose that $\det C_\omega$ has degree i_0. For convenience, let $\mathscr{L}_\omega$ denote the set of left images of ω which are contained in Γ_u and let $\mathscr{R}_\omega$ be the set of right images of ω in Γ_u. Since $i_0<m$, the previous observation implies that $\mathscr{L}_\omega$ and $\mathscr{R}_\omega$ are disjoint. In particular, since each element of $\Gamma_u^\#\times\Gamma_u^\#$ not equal to $(0,0)$ can occur at most once as an image of ω, we see that each element t of $\mathscr{L}_\omega$ or $\mathscr{R}_\omega$ occurs with multiplicity $\mu(t)\leq|\Gamma_u^\#|-1\leq\gamma$. Now $\det C_\omega$ has degree

$$\sum_{i\in\mathscr{L}_\omega}\mu(i)i+\sum_{j\in\mathscr{R}_\omega}\mu(j)(m-j)\equiv\sum_{i\in\mathscr{L}_\omega}\mu(i)i-\sum_{j\in\mathscr{R}_\omega}\mu(j)j \mod \mathrm{m}.$$

Thus, since $\mathscr{L}_\omega\cap\mathscr{R}_\omega=\emptyset$ and $0\leq\mu(t)\leq\gamma$, this degree can equal i_0 modulo m only in the trivial situation where $\mu(i_0)=1$ and $\mu(t)=0$ otherwise. Finally, if $i_0\in\mathscr{R}_\omega$, then $\det C_\omega$ has degree $m-i_0\neq i_0$, since $i_0\neq m/2$. Thus $i_0\in\mathscr{L}_\omega$ and we conclude that those $\det C_\omega$ which contribute to the λ^{i_0} coefficient of $\det N''$ consist precisely of $\ell-1$ distinct rows of e and the complementary row from $\lambda^{i_0}\alpha_{i_0u}(x^{-1}-e)$. It follows from the nature of e that the sum of these determinants is equal to $\lambda^{i_0}\operatorname{tr}\alpha_{i_0u}(x^{-1}-e)$. Therefore, $f(i_0)/\psi_x(0)=\operatorname{tr}\alpha_{i_0u}(x^{-1}-e)$ and $\psi_x(\lambda)$ determines the various matrix traces $\operatorname{tr}\alpha_{i_0u}(x^{-1}-e)$ for all $i_0\in\Gamma_u$.

To summarize, we have shown that the characteristic polynomial $\psi_x(\lambda)$ of $z_1^u\hat{g}\tilde{x}$ determines the matrix traces $\operatorname{tr}\alpha_{i_0u}(x^{-1}-e)$ for all $i_0\in\Gamma_u$. On the other hand, if $1\leq i_0\leq m-1$ and $i_0\notin\Gamma_u$, then $\alpha_{i_0u}=0$ and again the trace is known. Thus $\psi_x(\lambda)$ determines $\operatorname{tr} a_{iu}(x^{-1}-e)$ for all $1\leq i\leq m$. Finally, if u' is a multiplicative inverse for $\bar{u}$ modulo m, then $iv \equiv iuu' \bmod n$. Thus $\operatorname{tr} \alpha_{iv}(x^{-1} - e) = \operatorname{tr} \alpha_{ju}(x^{-1} - e)$ where $j \equiv iu' \bmod m$, and the lemma is proved. □

Let k and ℓ be integers with $k \mid \ell$ and, for any $a \in M_k(F)$, define $\mathscr{B}(a) = \mathscr{B}_\ell(a)$ to be the $\ell \times \ell$ partitioned matrix given by

$$\mathscr{B}(a) = \mathscr{B}_\ell(a) = \begin{pmatrix} a & a & \cdots & a \\ a & a & \cdots & a \\ \vdots & \vdots & & \vdots \\ a & a & \cdots & a \end{pmatrix} \in M_\ell(F).$$

If $a, b, c \in M_k(F)$ and if $\hat{c} = \operatorname{diag}(c, c, \dots, c) \in M_\ell(F)$, then clearly

$$\mathscr{B}(a)\mathscr{B}(b) = (\ell/k)\mathscr{B}(ab), \qquad \mathscr{B}(a)\hat{c} = \mathscr{B}(ac), \qquad \hat{c}\mathscr{B}(a) = \mathscr{B}(ca).$$

In particular, if $\operatorname{char} F = p > 0$ and if $p \mid (\ell/k)$, then $\mathscr{B}(a)\mathscr{B}(b) = 0$. We can now prove the following result.

Theorem 4.5. *Let F be a locally finite field of characteristic $p > 0$ and let $\mathrm{FU}_\infty(F)$ denote the finitary unitary, symplectic or orthogonal group of infinite degree. If G is a group with*

$$\mathrm{FU}_\infty(F)' \subseteq G \subseteq \mathrm{FGL}_\infty(F),$$

then G is strongly p-insulated.

Proof. Let $x_1, x_2, \dots, x_t$ be nonidentity elements of $G \subseteq \mathrm{FGL}_\infty(F)$ and choose an integer $k \geq 1$ with $x_1, x_2, \dots, x_t \in \mathrm{GL}_k(F)$ and such that the restriction θ of the sesquilinear form to this $k \times k$ upper left-hand corner is nonsingular. The goal is to find a homocyclic subgroup of G with appropriate properties. We actually prove a somewhat stronger result.

Claim. *If n and s are fixed powers of p with $n \geq ps \geq (k^2+3)^{k^2}$, then for all integers $r \geq 0$ there exists a group P_r satisfying:*

(i) *$P_r = Z_1 \times Z_2 \times \cdots \times Z_r$ where each Z_i is cyclic of order n.*

(ii) *$P_r \subseteq U_\ell(F, \phi)'$ where $\ell = \ell(r)$ is an integer divisible by pk and where $\phi =$ $\operatorname{diag}(\theta, \theta, \dots, \theta)$ is the direct sum of ℓ/k copies of θ or of its corresponding quadratic form in characteristic 2.*

(iii) *$g\mathscr{B}_\ell(c) = \mathscr{B}_\ell(c) = \mathscr{B}_\ell(c)g$ for all $g \in P_r$ and $c \in \mathrm{M}_k(F)$.*

(iv) *If $h = h_1 h_2 \cdots h_r \in P_r$ with $h_j \in Z_j$ for all j and if $|h_j| > s^j$ for some j, then no hx_i is a p-element.*

Proof. We proceed by induction on r. The case $r = 0$ is trivially satisfied by taking $P_r = 1$ and $\ell = pk$, using the fact that the infinite-dimensional sesquilinear form contains the direct sum of p copies of θ.

Assume the result holds for some $r \geq 0$ and let P_r be given satisfying the above four conditions. For convenience, we modify the notation somewhat and write $P_r = Z_2 \times Z_3 \times \cdots \times Z_{r+1}$. With this change of subscripts, statement (iv) translates to the assertion that if $h = h_2 h_3 \cdots h_{r+1} \in P_r$ and if $|h_j| > s^{j-1}$ for some j, then no hx_i is a p-element. Since $\mathrm{FU}_\infty(F)$ depends only on the nature of the sesquilinear form, we can assume that the $n\ell \times n\ell$ upper left-hand corner of $\mathrm{FU}_\infty(F)$ has the form determined by the matrix $\Phi = \mathrm{diag}(\phi, \phi, \ldots, \phi)$. In other words, Φ is the direct sum of n copies of ϕ or of its corresponding quadratic form in characteristic 2. Now let y and z be the $n\ell \times n\ell$ partitioned matrices

$$y = \begin{pmatrix} e & \beta_{n-1} & \cdots & \beta_2 & \beta_1 \\ \alpha_{n-1} & e & & & \\ \vdots & & \ddots & & \\ \alpha_2 & & & e & \\ \alpha_1 & & & & e \end{pmatrix} \quad \text{and} \quad z = \begin{pmatrix} & e & & & \\ & & e & & \\ & & & \ddots & \\ & & & & e \\ e & & & & \end{pmatrix},$$

where $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ and $\beta_1, \beta_2, \ldots, \beta_{n-1}$ are $\ell \times \ell$ matrices to be described later and where $e \in M_\ell(F)$ is the $\ell \times \ell$ identity matrix. Set $z_1 = yzy^{-1}$ and, for each $g \in P_r \subseteq U_\ell(F, \phi)'$, define

$$\hat{g} = \mathrm{diag}(g, g, \ldots, g) \in U_{n\ell}(F, \Phi)'.$$

Then certainly $\hat{P}_r \cong P_r$ and we set $P_{r+1} = \langle z_1, \hat{P}_r \rangle$. We will show, with appropriate choices for the α_i's and β_j's, that P_{r+1} has all the necessary properties.

To start with, let each $\alpha_i = \mathscr{B}_\ell(a_i)$ for a suitable $a_i \in M_k(F)$. Furthermore, if * denotes the composition of matrix transpose and the field automorphism $^\kappa$, then we set $\beta_i = \mathscr{B}_\ell(b_i)$ where $b_i = -\theta^{-1} a_i^* \theta$. Thus

$$\beta_i = \mathscr{B}_\ell(-\theta^{-1} a_i^* \theta) = -\phi^{-1} \mathscr{B}_\ell(a_i)^* \phi = -\phi^{-1} \alpha_i^* \phi$$

by the definition of ϕ in (ii). Similarly, using $p \mid (\ell/k)$, it follows that $\alpha_i^* \phi \alpha_i = \beta_i^* \phi \beta_i = 0$ and therefore $y^* \Phi y = \Phi$, since $\alpha_i^* \phi + \phi \beta_i = 0$. In other words, $y \in U_{n\ell}(F, \Phi)$ except possibly when we are dealing with a quadratic form Q in characteristic 2. In the latter case, we follow the argument of [10, Lemma 5.4(ii)]. Specifically, we already know that y is an isometry for the symplectic form corresponding to Q and therefore it suffices to show that $Q(yv) = Q(v)$ for all v in a generating set for $\mathrm{col}_{n\ell}(F)$, the $n\ell$-dimensional column space over F. From the nature of y, this will follow if we can show that $Q(\mathscr{B}_\ell(c)w) = 0$ for all $c \in \mathrm{M}_k(F)$ and all $w \in \mathrm{col}_\ell(F)$. But $Q(\mathscr{B}_\ell(c)w) = (\ell/k)Q(w')$ for some column $w' \in \mathrm{col}_k(F)$ and therefore, since $p \mid (\ell/k)$, this fact is proved. Of course, z is clearly also contained in $U_{n\ell}(F, \Phi)$, so certainly $z_1 = yzy^{-1} \in U_{n\ell}(F, \Phi)$. In fact, since $z_1^n = 1$ and n is a power of $p = \mathrm{char}\, F$, we conclude that $z_1 \in U_{n\ell}(F, \Phi)'$.

Next, condition (iii) applied to P_r implies that each $\hat{g} \in \hat{P}_r$ commutes with both y and z. Thus $\hat{P}_r$ commutes with z_1 and, since P_r is abelian, it follows that P_{r+1} is an abelian p-group. Indeed, since $z_1^n = 1$ and since no smaller power of z_1 is contained in $\hat{P}_r$, we see that

$$P_{r+1} = \langle z_1 \rangle \times \hat{P}_r = Z_1 \times \hat{Z}_2 \times \hat{Z}_3 \times \cdots \times \hat{Z}_{r+1},$$

where $Z_1 = \langle z_1 \rangle$. In other words, P_{r+1} is homocyclic of type n and rank $r+1$, and it satisfies conditions (i) and (ii) with $\ell(r+1) = \ell(r)n$. Furthermore, if $c \in \mathrm{M}_k(F)$, then $\mathscr{B}_{n\ell}(c)$ is the partitioned matrix

$$\mathscr{B}_{n\ell}(c) = \begin{pmatrix} \mathscr{B}_\ell(c) & \mathscr{B}_\ell(c) & \cdots & \mathscr{B}_\ell(c) \\ \mathscr{B}_\ell(c) & \mathscr{B}_\ell(c) & \cdots & \mathscr{B}_\ell(c) \\ \vdots & \vdots & & \vdots \\ \mathscr{B}_\ell(c) & \mathscr{B}_\ell(c) & \cdots & \mathscr{B}_\ell(c) \end{pmatrix}$$

and thus condition (iii) applied to P_r implies that $\hat{g}\mathscr{B}_{n\ell}(c) = \mathscr{B}_{n\ell}(c) = \mathscr{B}_{n\ell}(c)\hat{g}$ for all $g \in P_r$. Furthermore, since $p \mid (\ell/k)$, it follows that $\alpha_i = \mathscr{B}_\ell(a_i)$ and $\beta_i = \mathscr{B}_\ell(b_i)$ annihilate $\mathscr{B}_\ell(c)$ and thus multiplication by both y and z fix $\mathscr{B}_{n\ell}(c)$. In other words, $z_1\mathscr{B}_{n\ell}(c) = \mathscr{B}_{n\ell}(c) = \mathscr{B}_{n\ell}(c)z_1$ and P_{r+1} also satisfies condition (iii).

Notice that n and s are powers of p with $n \geq ps \geq (k^2+3)^{k^2}$, and thus n/ps is an integer with $(n/ps)(k^2+3)^{k^2} \leq n$. At this point, we specify our choices for the matrices α_i and β_i. First, choose $q = k^2+2$ or k^2+3, so that p does not divide q. Since $(n/ps)q^{k^2-1} < n$, we can define Γ to be the subset of $\{1, 2, \ldots, n-1\}$ of size $\gamma = k^2$ given by

$$\Gamma = \{(n/ps), (n/ps)q, (n/ps)q^2, \ldots, (n/ps)q^{k^2-1}\}.$$

Finally, let $a_i = 0$ if $i \notin \Gamma$ and let a_i run through the k^2 matrix units of $\mathrm{M}_k(F)$ for $i \in \Gamma$. Of course, a_i now determines $\alpha_i = \mathscr{B}_\ell(a_i)$ and $\beta_i = -\phi^{-1}\alpha_i^*\phi$.

It remains to prove that condition (iv) holds for P_{r+1}. To this end, let $h \in P_{r+1}$ satisfy the hypothesis of (iv) and write $h = z_1^u\hat{g}$ where $g = g_2g_3\cdots g_{r+1} \in P_r$ with $g_j \in Z_j$. We use the notation of the preceding three lemmas and, in particular, we set $v = \gcd(u,n)$ and $m = n/v = |z_1^u|$. Notice that all the basic hypotheses are satisfied. In particular, $\alpha_i\beta_j = \beta_j\alpha_i = 0$ for all i,j and $\Gamma = \{i \mid \alpha_i \neq 0 \text{ or } \beta_i \neq 0\}$. Furthermore, since $\alpha_i = \mathscr{B}_\ell(a_i)$ and $\beta_i = \mathscr{B}_\ell(b_i)$, it follows that $\alpha_i g = \alpha_i = g\alpha_i$ and $\beta_i g = \beta_i = g\beta_i$ by condition (iii). We will discuss the hypothesis preceding Lemma 4.4 when it is required. If $x \in \mathrm{GL}_k(F) \subseteq \mathrm{GL}_\ell(F) \subseteq \mathrm{GL}_{n\ell}(F)$, then we will use the corner embeddings $\tilde{}$ to distinguish the various containments. Note that, since char $F = p > 0$, an element w of $\mathrm{GL}_{n\ell}(F)$ is a p-element if and only if all its eigenvalues are equal to 1 and hence if and only if its characteristic polynomial is equal to $(\lambda - 1)^{n\ell}$. There are two cases to consider according to whether $|z_1^u| \leq s^1 = s$ or not.

Case 1: $|z_1^u| \leq s$.

Proof. Here $m = |z_1^u| \leq s$ and let $x \in \{x_1, x_2, \ldots, x_t\}$ be viewed as an element of $\mathrm{GL}_\ell(F)$. By Lemma 4.3, the characteristic polynomial $\psi_x(\lambda)$ of $h\tilde{x} = z_1^u\hat{g}\tilde{x}$ is divisible by $\det N$ when N is the $\ell \times \ell$ matrix

$$N = \lambda^m e - g^m x + \sum_{i=1}^{m-1} \lambda^i \alpha_{iu}(x - e)$$

$$+ \sum_{j=1}^{m-1} \lambda^{m-j} g^{m-1}(e - gx)\beta_{ju} + \sum_{i=1}^{m-1} \sum_{j=1}^{m-1} \lambda^{m+i-j} \alpha_{iu} x \beta_{ju}$$

with the subscripts computed modulo n. Now $m \le s$, so $v \ge n/s$ and hence v is divisible by n/s. But, by construction, the only nonzero α_i's and β_i's have subscript $i = (n/ps)q^t$ with q not divisible by p. Thus these subscripts are not divisible by v and hence not divisible by u modulo n. In particular, $N = \lambda^m e - g^m x$.

Since $|z_1^u| \le s$, it follows by assumption that $|g_j| > s^j$ for some $j \ge 2$. Thus $|g_j^m| > s^j/m \ge s^{j-1}$ and, by applying (iv) to the element $g^m = g_2^m g_3^m \cdots g_{r+1}^m \in P_r$, we see that $g^m x$ is not a p-element. Hence $g^m x$ has an eigenvalue distinct from 1, and therefore $\det N = \det(\lambda^m e - g^m x)$ has a root distinct from 1. In other words, $\psi_x(\lambda) \ne (\lambda - 1)^{n\ell}$ and $z_1^u \hat{g} \tilde{x}$ is not a p-element. $\square$

Case 2: $|z_1^u| > s$.

Proof. Here $m > s$, let $\boldsymbol{x}$ be any element of $\mathrm{GL}_k(F)$ and let $x = \tilde{\boldsymbol{x}} \in \mathrm{GL}_\ell(F)$. Note that $\alpha_i = \mathscr{B}_\ell(a_i)$ and a_i is either 0 or a matrix unit in $M_k(F)$. Thus $\operatorname{rank} \alpha_i = \operatorname{rank} a_i \le 1$ and $\operatorname{rank} \beta_i = \operatorname{rank}(-\phi^{-1}\alpha_i^* \phi) \le 1$. Furthermore,

$$\Gamma = \{\, i \mid \alpha_i \ne 0 \text{ or } \beta_i \ne 0 \,\}$$

and $|\Gamma| = \gamma = k^2$. Now suppose we are given the integer equation

$$i_0 \equiv \sum_{i \in \Gamma} \mu(i) i \mod \mathrm{n}$$

with $|\mu(i)| \le \gamma$ for all i. Then, by writing $i = (n/ps)q^j$ in the above and setting $\mu(i) = \bar{\mu}(j)$, we have

$$(n/ps)q^{j_0} \equiv \sum_{j=0}^{k^2-1} \bar{\mu}(j)(n/ps)q^j \mod \mathrm{n}$$

or

$$q^{j_0} \equiv \sum_{j=0}^{k^2-1} \bar{\mu}(j) q^j \mod \mathrm{ps}.$$

But $q = k^2+2$ or k^2+3, so $1+|\bar{\mu}(j)| \le k^2+1 < q$. Thus, since $ps \ge q^{k^2}$, uniqueness of expression in the q-adic expansion implies that $\bar{\mu}(j_0) = 1$ and $\bar{\mu}(j) = 0$ otherwise. In other words, $\mu(i_0) = 1$ and $\mu(i) = 0$ otherwise, so the hypothesis preceding Lemma 4.4 is satisfied. Notice also that if n is even, then the elements of Γ are all odd multiples of $(n/2s)$ and therefore $n/2 \notin \Gamma$ since $s > 1$.

Now $m > s$, so $v < n/s$ and hence $v \mid (n/ps)$. Thus the multiples of v include all multiples of n/ps, and therefore $\alpha_i = \beta_i = 0$ if i is not divisible by v. Thus, by Lemma 4.4, $\psi_x(\lambda)$ determines all matrix traces tr $\alpha_i(x^{-1}-e)$ with $i \in \Gamma$. Let e' denote the $k \times k$ identity matrix and observe that $x = \tilde{\boldsymbol{x}} = \operatorname{diag}(\boldsymbol{x}, e', \dots, e')$ and that $e = \operatorname{diag}(e', e', \dots, e')$. Since

$$\alpha_i(x^{-1}-e) = \mathscr{B}_\ell(a_i) \cdot \operatorname{diag}(\boldsymbol{x}^{-1}-e', 0, \dots, 0)$$
$$= \begin{pmatrix} a_i(\boldsymbol{x}^{-1}-e') & 0 & \cdots & 0 \\ a_i(\boldsymbol{x}^{-1}-e') & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ a_i(\boldsymbol{x}^{-1}-e') & 0 & \cdots & 0 \end{pmatrix},$$

we see that tr $\alpha_i(x^{-1}-e) = $ tr $a_i(\boldsymbol{x}^{-1}-e')$. In other words, $\psi_x(\lambda)$ uniquely determines tr $a_i(\boldsymbol{x}^{-1}-e')$ for all $i \in \Gamma$. But if a_i is the matrix unit $e'_{c,d}$, then tr $a_i(\boldsymbol{x}^{-1}-e')$ is precisely equal to the (d,c)-entry of $\boldsymbol{x}^{-1}-e'$. We conclude therefore, from the choice of the a_i's, that $\psi_x(\lambda)$ determines all entries of $\boldsymbol{x}^{-1}-e'$. Hence it determines the matrices $\boldsymbol{x}^{-1}-e'$, $\boldsymbol{x}^{-1}$ and then $\boldsymbol{x}$.

We have therefore shown that the map $\boldsymbol{x} \mapsto \psi_x(\lambda)$ is one-to-one and, since $z_1^u\hat{g} = z_1^u\hat{g}\tilde{e}$ is a p-element, we see that $e' \mapsto \psi_e(\lambda) = (\lambda-1)^{n\ell}$. In particular, if $\boldsymbol{x}$ is not the identity then $\psi_x(\lambda)$, the characteristic polynomial of $z_1^u\hat{g}\tilde{x} = z_1^u\hat{g}\tilde{\boldsymbol{x}}$, is not equal to $(\lambda-1)^{n\ell}$ and therefore $z_1^u\hat{g}\tilde{\boldsymbol{x}}$ is not a p-element. Since this applies to all $\boldsymbol{x} \in \{x_1, x_2, \dots, x_t\}$, Case 2 is proved and hence so is the claim. □

Proof of Theorem 4.5 (*Conclusion*). The remainder of the proof now follows easily. Let $\{x_1, x_2, \dots, x_t\} \subseteq G \setminus 1$ be given and again choose k so that these elements are contained in $\mathrm{GL}_k(F)$ and such that the restriction of the sesquilinear form to this $k \times k$ upper left-hand corner is nonsingular. Suppose the integer $r \geq 1$ is given and let n and s be powers of p with $ps \geq (k^2+3)^{k^2}$ and $n > s^r$. By the preceding claim, there exists a homocyclic p-group P_r of type n and rank r satisfying condition (i), (ii) and (iv). In particular, $P_r \subseteq \mathrm{FU}_\infty(F)' \subseteq G$. Finally, suppose $h = h_1h_2\cdots h_r$ is a generator of P_r with $h_j \in Z_j$. Then $|h| = n$, so $|h_j| = n$ for some j. But then $|h_j| = n > s^r \geq s^j$, so (iv) implies that no hx_i is a p-element, and G is indeed strongly p-insulated. □

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